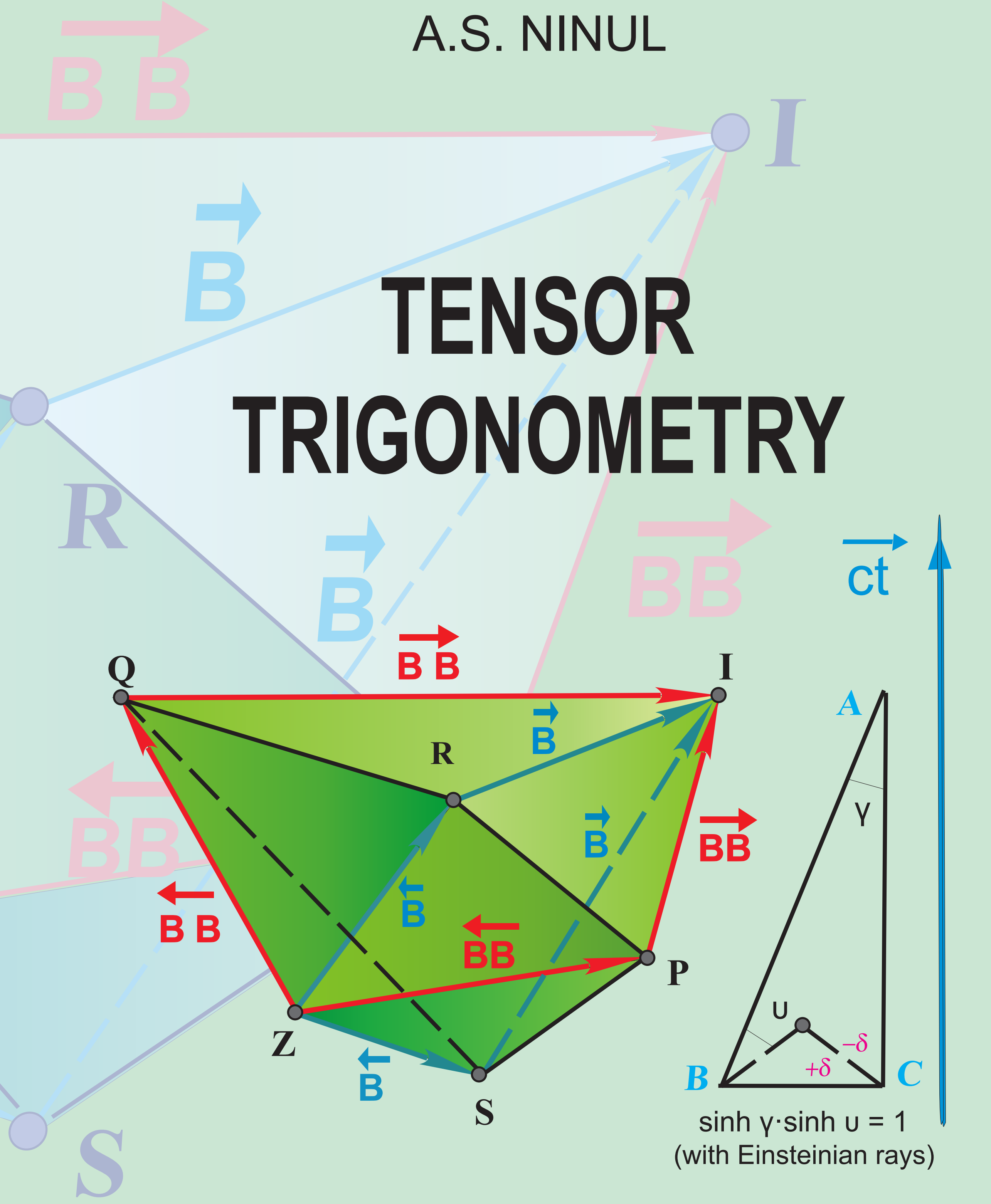
A.S. NINUL

TENSOR TRIGONOMETRY



sinh γ·sinh υ = 1
(with Einsteinian rays)

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Planimetry includes metric part and trigonometry. In geometries of metric spaces from the end of XIX age their tensor forms are widely used. However the trigonometry is remained only in its scalar form in a plane. The tensor trigonometry is development of the flat scalar trigonometry from Leonard Euler classic forms into general multi-dimensional tensor forms with vector and scalar orthoprojections and with step by step increasing a complexity and opportunities. Described in the book are fundamentals of this new mathematical subject with many initial examples of its applications.

In theoretic plan, the tensor trigonometry complements naturally Analytic Geometry and Linear Algebra. In practical plan, it gives the clear instrument for solutions of various geometric and physical problems in homogeneous isotropic spaces, such as Euclidean, quasi- and pseudo-Euclidean ones. So in these spaces, the tensor trigonometry gives very clear general laws of motions in complete forms and with polar decompositions into principal and secondary motions, their descriptive trigonometric vector models, which are applicable also to n-dimensional non-Euclidean geometries in subspaces of constant radius embedded into enveloping metric spaces, and in the theory of relativity. In STR, the applications were considered till a trigonometric pseudo-analog of the classic theory by Frenet–Serret with absolute differentially-geometric, kinematic and dynamic characteristics in the current points of a world line.

New methods of the tensor trigonometry can be also useful in other domains of mathematics and physics. The book is intended for researchers in the fields of multi-dimensional spaces, analytic geometry, linear algebra with theory of matrices, non-Euclidean geometries, theory of relativity and to all those who is interested in new knowledges and applications, given by exact sciences. It may be useful for educational purposes with this new math subject in the university departments of algebra, geometry and physics.

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The first edition of the book (ISBN-10 : 5-03-003717-9 and ISBN-13 : 978-5-03-003717-2) was reviewed by professor of Moscow State University, Dr.Sc. M. M. Postnikov and by professor of Moscow Regional University, Dr.Sc. O. V. Manturov.

The first edition in Russian of the book in September 2004 was devoted by the author to the 175ys anniversary of the first publications on non-Euclidean Geometry, to the 100ys anniversary of the first publications on Theory of Relativity and to their great creators – Lobachevsky, Bolyai, Lorentz, Poincaré, Einstein

To the readers

Seldom, what division of mathematical science is so well-known and understandable yet since school years as the Trigonometry. Originated in antiquity it practically completed own development and obtained its modern form at the end of the 18th century in the works of great Leonard Euler. Meanwhile Geometry, from the historically initial Euclidean forms, passed far ahead for the last two centuries. Furthermore, its various multi-dimensional and non-Euclidean tensor forms were discovered and studied.

In the monograph, we undertake constructing general forms of the TensorTrigonometry in multi-dimensional homogeneous and isotropic spaces with quadratic metrics (as Euclidean, quasi- and pseudo-Euclidean ones). The classic Scalar Trigonometry acts on eigenplanes of the binary trigonometric subspace of a tensor angle. The angle between two lines (or vectors), between two subspaces (or lineors) in multi-dimensional linear spaces has accordingly the nature of bivalent tensors, determined by the set reflector tensor of the binary space. However, its kind is determined by the concrete quadratic metric. In these metric spaces, a tensor angle and its trigonometric functions are respectively either orthogonal, or quasi-orthogonal, or pseudo-orthogonal tensors. (In particular, for Euclidean spaces, the simplest reflector tensor is a unity matrix, and we can deal only with the middle reflector of the concrete tensor angle.)

These tensor angles and all their trigonometric functions can be defined in the two forms: (1) projective one by a pair of eigenprojectors or eigenreflectors; (2) motive one by the given rotational or deformational matrix. Projective and motive angles are one-to-one connected.

In order to obtain the tensor construction, it was necessary to consider highly thoroughly a number of related questions in the Theory of Exact Matrices, what is a part of Linear Algebra. In addition to this, our efforts were rewarded by attainments of interesting and unexpected results in Algebra, Geometry and Theoretical Physics.

Tensor Trigonometry point of view gives such advantages, that some rather difficult and not easily perceivable mathematical or physical theories became quietly transparent and natural for understanding. We exposed this on more elementary examples of trigonometric modelling different motions with the use of their polar representations in quasi-, pseudo-Euclidean and non-Euclidean geometries and in the Theory of Relativity. So, the hyperbolic tensor of motion with the certain scalar multipliers produces all the kinematic and dynamic scalar, vector and tensor physical relativistic characteristics of a moving material body, and gives the general law of summing motions and relativistic velocities. The hyperbolic tensor of deformation produces all the relativistic seeming geometric parameters of a moving object.

Main content of the book are at the joint of problems studied in multi-dimensional Geometry and Linear Algebra. Since the exposition of the theory required many of additional notations and terms, the author tried to give them the most convenient and logical forms. So, this relates to the matrix alphabet based on wide-spreading examples. The author will be grateful to readers who will express opinion, remarks or proposals concerning the book on my web-site.

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Contents

Introduction

In Theory of Matrices such usual notions as a singular matrix, its rank, eigenvalues, eigenvectors or eigensubspaces, annuling polynomial, and so on, have a sense only for exact matrices and at exact computations. We distinguish the exact theory of notions and the approximating theory of notions estimates. Each of them places its own part. The notions connected with exact numerical characteristics relate to the exact theories. These theories are used not only for constructing and analysis of abstractions, but they are also important for analyzable objects from applied problems because the numerical characteristics of objects are always exact and only their estimates are approximate.

The main two parts of the monograph, in twelve chapters, contain both the results of our investigations in Theory of Exact Matrices (Part I, chapters $1 \div 4$) and developed on this platform Tensor Trigonometry (Part II, chapters $5 \div 12$). The latter is a constituent division of the corresponding to it Geometry with a certain quadratic metric.

The historical roots of Scalar Trigonometry, as a constituent part of two-dimensional Geometry, refer to far-away times. Yet in the Euclidean "Elements" some trigonometric formulations were be found. Much later, in II age Claudius Ptolemy of Alexandria widely used in "Almagest" sine-cosine formulae with his trigonometric equivalent of the Pythagorean Theorem. Some spherical functions were used also in IX–X ages in the works of Arabian mathematicians. It is of interest that the Trigonometry on a sphere became developed much earlier than the one on a plane. It was, due to the fact, that it was needed in the practical astronomy. So, in 1603, Th. Harriot connected the angular excess of a spherical triangle with area and radius. Though some trigonometric elements were introduced into the European science by R. Wallingford in the beginning of XIY age. Thus, in particular, he used it in solving of a right triangle.

Hyperbolic functions were discovered by A. Moivre (1722) and obtained in complete set by V. Riccati from a unity hyperbola (1757). First these functions were used really in hyperbolic Scalar Trigonometry and in geometric investigations by J. Lambert and F. Taurinus. So, in 1763, J. Lambert connected the angular defect of a hyperbolic triangle with area and radius. In 1825, F. Taurinus discovered the first (cosine) formula for summing two segments in the hyperbolic geometry. Creators of the hyperbolic non-Euclidean geometry N. Lobachevsky and J. Bolyai used it analogy in the small with the spherical geometry as a main mathematical instrument for inferences of its metric formulae. These geometries have such distinction: their geodesic arcs–segments are hyperbolic and spherical. In pseudo-Euclidean or so called *quasi-Euclidean* geometries, there are straight segments, but with hyperbolic or spherical angles between them!

The modern perfect form of the Scalar Trigonometry was given by L. Euler, who realized also its complexification. On the other hand, Geometry continues to develop and essentially violently according to the appeared idea of a multi-dimensional space.

Multi-dimensional space was arisen apparently at the middle of XIX age in classical work of H. Grassmann "Die lineale Ausdehnungslehre" [1]. H. Grassmann and, independently of him, W. Hamilton laid the foundation of Vector Analysis in the spaces. Before (in 1808) J.-G. Garnier emits Analytical Geometry as the whole division of Geometry. The outstanding contribution in justification of the algebraic approach to the geometry of any objects in multi-dimensional arithmetic spaces was realized by the famous "Cantor–Dedekind Axiom about Continuum".

About of that time appearance of Linear Algebra and its following development in the works of F. Frobenius, G. Cramer, L. Kronecker, A. Capelli, J. Silvester, L. Hesse, C. Jordan, Ch. Hermite and other mathematicians led, with time, to its larger filling by geometric content. That is why, Linear Algebra found effective applications in the theory of vector Euclidean spaces and also, after the well-known works of H. Poincare and H. Minkowski, in the theory of new pseudo-Euclidean spaces. This process was activated thanks to algebraic definitions of notions connected with metric properties of arithmetic spaces and of their geometric objects (the lengths of vectors and the values of scalar angles between them). As well-known for the basic algebraic definitions of measures mathematicians used the Pythagorean Theorem and the algebraic cosine Inequality of Cauchy or sine Inequality of Hadamard.

Besides, for the strict algebraic approach to the geometry in arithmetic spaces, it is impossible to realize it completely without Theory of Exact Matrices. For example, E. Moore and later R. Penrose proposed the general methods of quasi-inversion of singular matrices. R. Courant developed the large parameter optimization method with penalty functions, useful in such algebraic applications too. A. Tichonov gave the small parameter method of regularization with the limit method for normal solving degenerated systems of linear equations. Results of these investigations had also a big geometric importance and, to some degree, served for initiating the present work.

The main aims of this monograph were (as 1st) to develop with further applications a number of algebraic and geometric notions in Theory of Exact Matrices (Part I, chapters $1\div4$), and then (as 2nd) on the platform to work out the basic aspects of Tensor Trigonometry for binary tensor angles formed by two linear subspaces or formed by rotation of a linear subspace in the superspace (Part II, chapters $5\div12$). Since Tensor Trigonometry has a lot of applications in other mathematical and some physical domains, the largest examples of which are exposed in the book's Appendix.

First of all, the structure of matrix characteristic coefficients for $n \times n$-matrices in the explicit form is installed by special differential method. They appeared in Theory of Exact Matrices in middle of XX age in the works of J.-M. Souriau and D. K. Faddeev in addition to scalar characteristic coefficients with the well-known structure. These last were used yet in XIX age by U. Le Verrier at his famous prediction of Neptune. We express all eigenprojectors and quasi-inverse matrices in explicit form, in terms of the scalar and matrix coefficients. And the minimal annulling polynomial for $n \times n$-matrix in explicit form is identified with the connections of all matrix singularity parameters.

In passing, the general inequality for all average values is inferred, and hierarchical invariants for the spectrally positive matrix are installed for the justification of the stated geometric norms. The new global limit method for step by step calculating all roots of a real algebraic equation is proposed, and the more strict necessary condition for all its roots reality and positivity, than the classical Descartes condition, is gotten.

The particular (of order t) and general (of order r) quadratic norms (measures) are introduced for the linear geometric objects *lineors*, determined by $n \times r$-matrices A, where $1 \leq r < n$ (at $r = 1$ they are vectors), and for the *tensor angles* between them or between their images in n-dimensional arithmetic spaces. In particular, at $t = 1$ they are the Euclidean and Frobenius norms (measures). The theoretical basis for these particular and general norms is the hierarchical general inequality for all average positive values. Also the specific multiplications of cosine and sine types are defined for a pair of these lineors with inferring the general cosine and sine inequalities through the special matrix trigonometric spectra. Their elementary algebraic and trigonometric cases are the cosine Inequality of Cauchy and the sine Inequality of Hadamard.

Tensor Trigonometry, as the main new content of this monograph, is exposed then in its three kinds: projective, reflective and motive, which naturally complement each others. Two types of motive trigonometric transformations, as rotational (sine-cosine) and deformational (tangent-secant), are defined. Besides, the general homogeneous transformations, due to the former polar representations, are divided as either principal spherical and orthospherical rotations or principal hyperbolic and orthospherical ones.

The special dual relations are established between primary spherical and hyperbolic notions on the basis of spherical-hyperbolic analogy in abstract and specific senses. This is widely used in developing Tensor Trigonometry and in its applications. So, similar definitions of quasi-Euclidean and pseudo-Euclidean metric spaces and their Tensor Trigonometries are exposed, in terms of the initially given *reflector tensor of a binary space* with the kind of quadratic metrics (Euclidean or pseudo-Euclidean).

In the pairs (spherical, orthospherical), (hyperbolic, orthospherical) all rotations in these binary spaces, with an identical reflector tensor, form two noncommutative groups. The first one is the homogeneous group of quasi-Euclidean motions. The second one is the homogeneous pseudo-Euclidean group of Lorentz. In the so-called *universal bases*, the intersection of these two groups is a proper subgroup of secondary orthospherical rotations. This reflector tensor and both quadratic metrics define in the same spaces two sets of reflective transformations. Obviously, reflections do not form groups.

Most importantly, the quasi-Euclidean space and geometry filled in this monograph a previously unnoticed gap that existed in the theory of homogeneous isotropic spaces. This can be explained by the fact that the pseudo Euclidean space with the Lorentz group was introduced back in 1905 by H. Poincaré as the mathematical apparatus of the theory of relativity. This name was given to it later by M. Planck. The geometry of this space was developed by H. Minkowski in 1907. But mathematically, they are pseudoanalogues namely of this original quasi-Euclidean space and its geometry!

In Appendix (chapters 1A÷10A) – see more in the Preface to it, as the rather important case, we considered the tensor trigonometric transformations in elementary forms, with single principal and orthospherical eigen angles of motions and therefore with one frame axis for them. In the case, the new interesting possibilities are discovered for the very clear study of various types of motions in pseudo-Euclidean Geometry of Minkowski, in quasi-Euclidean Geometry with the same reflector tensor but with Euclidean quadratic metric, in both non-Euclidean Geometries of constant radius, and also in the Theory of Relativity. So, the rotations defined by Tensor Trigonometry in these metric spaces with quadratic metrics are equivalent to the motions defined by geometry in the small for subspaces of constant radius (hyperspheroid or hyperboloids) embedded into them. The general law of summing non-collinear principal motions or velocities in STR is established in the trigonometric matrix, vector and scalar forms with identification of the secondary orthospherical rotation. In all these non-Euclidean geometries and STR, we represented the law in noncommutative biorthogonal form with Big and Small Pythagorean Theorems. As bright novelty we gave the solution of a pseudo-Euclidean right triangle in a pseudoplane with connections of its angles, and we proposed an updated concept of the parallel angle for all non-Euclidean geometries.

In the *Kunstkammer* in the book end, the readers may test themselves in solving a number of the suggested by the author questions and tasks near to the work's topics.

In conclusion, it is necessary to clarify the name of the new subject: Why tensor?

Usual angles are binary as between two linear geometric objects. They and their functions are completely determined by square matrices how for any bivalent tensors. In the presence of some of two quadratic metrics, the tensors are orthogonal; in the absence of any metric, they are affine. The new subject mainly deals with orthogonal tensors, their projections, and all scalar invariants. What is more. On a quasiplane, these tensors are spherically orthogonal, in a quasi-Euclidean space they are quasi-Euclidean orthogonal. On a pseudoplane, these tensors are hyperbolically orthogonal, in a pseudo-Euclidean space they are pseudo-Euclidean orthogonal. In addition, they may be symmetric and anti-symmetric, real, imaginary and complex, and so one. For trigonometric functions of the binary tensor angles we use by analogy with scalar ones, as most convenient here, the classical notations of J. Lagrange and K. Scherffer.

Tensor trigonometry in Russian was issued in 2004 by scientific Publisher "MIR" thanks to a bright review of encyclopedically versatile eminent Russian mathematician Postnikov M.M., widely known as author of a large number of valuable monographs and textbooks in various algebraic and geometric fields. In the updated English version, all theorems and formulae under 750 numbers and others of the 2004 edition are saved.

New methods of Tensor Trigonometry can be used further more widely in the various domains of mathematics and physics. The author hopes that readers will find a lot of interesting contents and new knowledges. He will welcome, if somebody wishes to dare in this direction for its following development with new surprising results.

Notations

1. Notations of matrices (Matrices alphabet)

A – rectangular $n \times m$- or $m \times n$-matrix, or $n \times r$-*lineor* in a space (at $r = 1$ – $n \times 1$-vector $\mathbf{a}$),

$\{\mathrm{lig}(t)A\}$ – rows submatrix of A of order t,

$\{\mathrm{col}(t)A\}$ – columns submatrix of A of order t,

A^+ – spherically orthogonal quasi-inverse matrix of Moore–Penrose,

B – quadratic $n \times n$-matrix or external multiplication $B = A_1 A_2'$ of $n \times r$-*lineors* A_1, A_2;

B^V – adjoint matrix for nonsingular B ($B^{-1} = B^V / detB$),

$B_i = B - \mu_i I$ – i-th singular eigenmatrix for B,

$\{D\text{-minor } (t)B\}$ – $t \times t$-submatrix for the diagonal (or principal) minor of B of order t,

$\{Dh\text{-minor } (t)B\}$ – $t \times t$-submatrix for the *hypodiagonal* minor of B of order t,

B (or Bp) – *null-prime* singular matrix: $\langle ker\ B\rangle \cap \langle im\ B\rangle \equiv \langle \mathbf{0}\rangle$,

B (or Bm and Bn) – *adequately and Hermitian null-normal* matrices: $\langle ker\ B\rangle \perp \langle im\ B\rangle$,

B^- (or Bp^-) – affine (or oblique, or hyperbolically orthogonal) quasi-inverse matrix,

B (or Bc) – *null-cell (two-block-diagonal) form* of Bp, Bm, Bn,

$\overrightarrow{B}$ (or $\overrightarrow{Bp}$) – affine or oblique eigenprojector into $\langle ker\ B\rangle$ parallel to $\langle im\ B\rangle$,

$\overleftarrow{B}$ (or $\overleftarrow{Bp}$) – affine or oblique eigenprojector into $\langle im\ B\rangle$ parallel to $\langle ker\ B\rangle$,

$\overrightarrow{B}$ (or $\overrightarrow{Bm}$) – spherically orthogonal eigenprojector into $\langle ker\ B\rangle \equiv \langle ker\ B'\rangle$,

$\overleftarrow{B}$ (or $\overleftarrow{Bm}$) – spherically orthogonal eigenprojector into $\langle im\ B\rangle \equiv \langle im\ B'\rangle$,

C – free cellular matrix multiplier or internal multiplication $C = A_1' A_2$ of $n \times r$-*lineors* A_1 and A_2,

$C_\mu(B)$ – basic (q-block-diagonal) form of the matrix B (q – quantity of the eigenvalues of B),

D – diagonal matrix,

$\tilde{E}_k$ – certain base (frame of reference),

$\tilde{E}_1$ – unity base of the diagonal cosine or *universal base* with the spherical-hyperbolic analogy,

$F(\dots)$ – matrix function of $(\dots)$,

$\{G^+\}(\mathbf{x})$, $\{G^\pm\}(\mathbf{u})$ and $\hat{G}$, – metric tensors (positive, sign-indefinite and mutual with G),

H – Hermitean complex matrix, $H^\oplus$ – positively definite Hermitean complex matrix,

I – unity matrix, $\{I^+\}$ and $\{I^-\}$ – reflector tensor of Euclidean and anti-Euclidean spaces,

$\{I^\pm\}$ and $\{R'_W\ I^\pm R_W\} = \{\sqrt{I}\}_S$ – reflector tensors of *quasi-Euclidean* and pseudo-Euclidean spaces,

It– *totally-unity* matrix: all the elements of which are equal to 1,

$J_\mu\ (B)$ – canonic Jordan form of a matrix B,

K – anti-symmetric real or complex matrix,

$K_B(\epsilon)$ – matrix characteristic polynomial of the parameter ε for a matrix B,

$K_1(B,t)$ and $K_2(B,t)$ – matrix characteristic coefficients for a matrix B of order t,

$\pm Ref\{Bm\}$ – eigenreflectors for matrices Bm (spherically orthogonal),

$\pm Ref\{Bp\}$ – eigenreflectors for matrices Bp (affine or oblique or hyperbolically orthogonal),

$L_\mu(B)$ – q-block-triangular form of a matrix B (q - quantity of eigenvalues of B),

M – normal (real and adequately complex) normal matrix,

N – Hermitean complex normal matrix,

O – nilpotent matrix,

P – prime matrix,

Q – anti-Hermitean complex matrix,

$Q_B(\epsilon)$ – reduced matrix characteristic polynomial of the parameter ε for a matrix B,

$Q_1(B,t)$ and $Q_2(B,t)$ – reduced matrix characteristic coefficients for a matrix B of order t

R – orthogonal (real and adequately complex) orthogonal matrix, Rq – *quasi-orthogonal* $n \times r$-matrix,

R_W – orthogonal modal matrix for transformation of a prime matrix P into its W-form,

S – symmetric real or complex matrix, $S^\oplus$ – positively definite symmetric real matrix,

T – matrix of the rotational trigonometric modal transformation (active or passive),

U – unitary (*Hermitean orthogonal*) complex matrix,

V – matrix of the general linear modal transformation (active or passive),

$W(P)$ – mono-binary form of a prime matrix P,

X – matrix argument,

Y – matrix function, connected one-to-one two spaces in their direct sum in a certain basis space,

Z – zero matrix.

2. Notations of binary tensor angles and their functions

$\tilde{\Phi} = \tilde{\Phi}'$ – principal tensor spherical projective angle between two planars and in reflectors,

$\Phi = -\Phi'$ – principal tensor spherical motive angle in rotations and deformations,

$\tilde{\Xi}$ and Ξ – complementary tensor spherical angles till the tensor spherical right angle $\Pi/2$,

$\tilde{\Gamma} = -\tilde{\Gamma}'$ – principal tensor hyperbolic projective angle between two planars and in reflectors,

$\Gamma = \Gamma'$ – principal tensor hyperbolic motive angle in rotations and deformations,

$\tilde{\Upsilon}$ and Υ – complementary tensor hyperbolic angles till the tensor hyperbolic infinite right angle Δ,

$\tilde{\Theta} = \tilde{\Theta}'$ – secondary tensor orthospherical projective angle (additional to the angle $\tilde{\Phi}$ or the angle $\tilde{\Gamma}$),

$\Theta = -\Theta'$ – secondary tensor orthospherical motive angle (additional to the angle Φ or the angle Γ),

$\tilde{\Psi} = \tilde{\Phi} + i\tilde{\Gamma}$, $\Psi = \Phi + i\Gamma$ – complex adequate tensor projective and motive spherical angles,

$\tilde{\mathcal{H}} = \tilde{\Phi} + i\tilde{\Gamma} = \tilde{\mathcal{H}}^*$ – complex Hermitean tensor projective spherical angle ($\tilde{\Phi} = \tilde{\Phi}^*$, $\tilde{\Gamma} = -\tilde{\Gamma}^*$),

$\mathcal{H} = \Phi + i\Gamma = -\mathcal{H}^*$ – complex skew-Hermitean tensor motive spherical angle $\Phi = -\Phi^*$, $\Gamma = \Gamma^*$).

(all the tensor angles correspond to the set reflector tensor of the space – see in item 2),

$Rot\ \Phi$ and $rot\ \Phi$ – principal spherical rotation at the angle Φ (and elementary one),

$Roth\ \Gamma$ and $roth\ \Gamma$ – principal hyperbolic rotation at the angle Γ (and elementary one),

$Rot\ \Theta$ and $rot\ \Theta$ – secondary orthospherical rotation at the angle Θ (and elementary one),

$Def\ \Phi$ and $def\ \Phi$ – spherical deformation at the angle Φ (and elementary one),

$Defh\ \Gamma$ and $defh\ \Gamma$ – hyperbolic deformation at the angle Γ (and elementary one),

3. Notations of spaces and sub-spaces

$\langle \mathcal{A}^n \rangle$ – arithmetic affine n-dimensional space,

$\langle im \ldots \rangle$ and $\langle ker \ldots \rangle$ – image and kernel of the matrix $\ldots$,

$\langle \mathcal{E}^n \rangle$ – Euclidean n-dimensional space, $\langle \mathcal{C}^n \rangle$ – Euclidean cylindrical n-dimensional space,

$\langle \mathcal{E}^{n+q} \rangle$ – complex binary Euclidean $(n+q)$-dimensional space of the index q (q $\leq$ n),

$\langle \mathcal{P}^{n+q} \rangle$ – real binary pseudo-Euclidean $(n+q)$-dimensional space of the index q (q $\leq$ n),

$\langle \mathcal{Q}^{n+q} \rangle$ – real binary quasi-Euclidean $(n+q)$-dimensional space of the index q (q $\leq$ n),

(last two spaces at $q = 1$ are over-spaces for n-dimensional hyperbolic and spherical geometries, and for this case: $\langle\langle \mathcal{E}^n \rangle\rangle$ – projective flat hyperplane, $\langle\langle \mathcal{C}^n \rangle\rangle$ – projective cylindrical hyperplane),

$\langle \mathcal{E}^n \rangle^{(k)}$, $\langle \mathcal{E}^q \rangle^{(k)}$ – Euclidean subspaces in $\langle \mathcal{Q}^{n+q} \rangle$ or $\langle \mathcal{P}^{n+q} \rangle$ with respect to the base $\tilde{E}_k$,

$\langle \mathcal{P}_i \rangle$, $\langle \mathcal{P}_{ij} \rangle$ – trigonometric subspaces of the tensor angle.

4. Other notations

$a, b, ...$ and $\mathbf{a}$, $\mathbf{b}$, ... – scalar and vectorial number-elements,

$||\mathbf{a}||_E$ – Euclidean norm for the $n \times 1$-vector $\mathbf{a}$,

$||A||_F = ||A||_1$ – Frobenius norm (first order quadratic norm) for the $n \times m$-matrix A,

$||A||_t$ – particular quadratic order t norms for the $n \times m$-matrix or $n \times r$-lineor A,

$\overline{||A||_t}$ – trimmed particular quadratic order t or algebraic norms (algebraic medians of order t),

$||A||_r$ – general quadratic order r or geometric norm (geometric median),

C_n^t – binomial coefficients of Newton,

$det\ B$ – determinant of the matrix B,

$\mathbf{d(x)}$ – residual of the linear algebraic equation of $\mathbf{x}$,

$\mathcal{D}l(r)B$ – *dianal* of the singular $n \times n$-matrix B, i. e. the full sum of its basis principal minors,

$\langle im\ A\rangle$ or $\langle im\ B\rangle$ – image of the matrix A or of the matrix B,

$\langle ker\ A'\rangle$ and $\langle ker\ B\rangle$ – kernel of the matrix A' or of the matrix B,

$k_B(\epsilon) = det(B + \epsilon I)$ – scalar characteristic polynomial of parameter ϵ for the matrix B,

$k_B(-\mu) = det(B - \mu_i I) = 0$ – secular equation for the matrix B,

$k(B, t)$ – scalar characteristic coefficient for the matrix B of order t,

l – Euclidean and quasi-Euclidean length, λ – pseudo-Euclidean length,

$\overline{m_t}$ – algebraic mean (small median) of order t, $\overline{M_\theta}$ – power mean (large median) of order θ,

$\mathcal{M}t(r)A$ or $\mathcal{M}t(r)B$ – *minorant* of the singular matrix, i. e., the square root from the full sum of its quadric basis minors,

n – dimension of the space,

q – index of the quasi- or pseudo-Euclidean space,

$\mathrm{q}_B(\epsilon)$ – reduced scalar characteristic polynomial of parameter ϵ for the matrix B,

$q_B(-\mu) = 0$ – reduced secular equation for the matrix B,

$q(B, t)$ – reduced scalar characteristic coefficient of the matrix B of order t,

$r = rankB$ ($r = rankA$) – rank of the matrix,

r' – 1st *rock* of the singular matrix B, i. e. maximal order of non-zero $k(B, t)$,

r'' – 2sd *rock* of the singular matrix B, i. e. maximal order of non-zero $K(B, t)$,

s and s' – geometric and algebraic multiplicities of the zero eigenvalue of a singular matrix B,

$s_i^0 = r_i'' - r_i' + 1$ – annulling multiplicity of the i-th eigenvalue of a quadratic matrix B,

t – order of matrices characteristics, dimension of submatrices and minors,

trB – trace of the matrix B,

$\overline{v_t}$ – reversive algebraic mean (reversive small median) of order t,

$\overline{V_\theta}$ – reversive power mean (reversive large median) of order θ,

$\mathbf{x}$, $\mathbf{y}$ – real-number vectorial arguments (variables),

$\mathbf{z}$ and $\overline{\mathbf{z}}$ – complex-number vectorial arguments (conjugate variables),

sin, sinh, cos, cosh, tan, tanh, sec, sech, cot, coth, cosec, csch – trigonometric functions,
arcsin, arsinh, arccos, arcosh, arctan, artanh, arcsec, arsech – reverse trigonometric functions.

Greek some notations :

φ – principal scalar spherical angle,

γ – principal scalar hyperbolic angle,

θ – secondary scalar orthospherical angle (respectively to the principal angles φ or γ).

ξ – complementary spherical angle at φ (till right spherical angle $\pi/2$),

υ – complementary hyperbolic angle at γ in a rectangular pseudo-Euclidean triangle,

δ – infinite hyperbolic angle in a rectangular pseudo-Euclidean triangle,

η – scalar Hermitean spherical angle,

π – Archimedes Number and an open spherical angle,

$\omega = arsh\ 1$ – especial hyperbolic angle (and number) as analog of the spherical angle $\pi/4$

μ_i – i-th eigenvalue of a quadratic matrix with its quantity q_i,

σ_j – j-th eigenvalue of multiplicative matrices AA' and $A'A$,

2τ – trigonometric rank of the binary tensor angle,

ν' – dimension of the sub-space of the intersection $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ (i. e., of zero sine),

ν'' – dimension of the sub-space of the intersection $\langle im\ A_1\rangle$ and $\langle ker\ A_2'\rangle$ (i. e., of zero cosine).

5. Using symbols

$'$ – mark of simple transposing, $*$ – mark of Hermitean transposing,

$\ldots \subset \ldots$ – the set ... belong to the set ... , $\ldots \subseteq \ldots$ – the set ... belong or is identical to the set ... ,

$\ldots \in \ldots$ – element ... belong to the set ... , $\ldots \notin \ldots$ – element ... no belong to the set ... ,

$\ldots \cup \ldots$ – mark of summing (joining) the two sets, $\ldots \cap \ldots$ – mark of intersecting the two sets,

$\ldots \equiv \ldots$ – mark for the identity of the two sets,

$\ldots \oplus \ldots$ – mark of the direct summing the two sets,

$\ldots \boxplus \ldots$ and $\ldots \boxtimes \ldots$ – marks of the spherical and hyperbolic orthogonal direct summing the two sets,

$\ldots \uplus \ldots$ – mark of the geometric summing the two angles,

$\overset{\angle}{\Phi}$ and $\overset{\angle}{\Gamma}$ – mark over the summarized tensor angles in the case of reverse order of two- or multistep rotations (particular motions), and in the case of reverse angular shifting.

Part I

Theory of Exact Matrices: some of general questions

The developed further Tensor Trigonometry (in Part II), at the beginning in its projective version, is based on the use of eigenprojectors for a *null-prime* singular $n \times n$-matrix B, whose image and kernel form a direct sum. Besides, any prime matrix P has a full trigonometric spectrum formed by mono-binary spectrums of its null-prime eigenmatrices $P_i = \{P - \mu_i I\}$. The essential role in a strict foundation of the eigenprojectors and the trigonometric spectrum for a null-prime matrix is played by the coefficients of its characteristic polynomials – scalar and matrix. The most logic way for introducing the coefficients is the use of a resolvent for the given square matrix B.

In Chapter 1, respectively the structure and properties of these scalar and matrix characteristic coefficients are found and studied in details. The fundamental inequality for basic parameters of singularity for matrix B is established. As additional result, from the highest orders r', r'' of these scalar and matrix characteristic coefficients for eigenmatrices B_i a minimal annulling polynomial of the matrix B is identified in the explicit form. The general inequality for average values (means) is formulated and proved in a whole form including a chain of particular inequalities for algebraic means as a basis of hierarchical algebraic norms entered subsequently. Its opportunities are shown in the theory and technique for solutions of real algebraic equations, in that number of secular ones. In the case of positive equation roots (e. g., of the eigenvalues for positively definite matrices), the limit method and formulae for calculating of maximal and minimal roots are gotten in terms of the equation coefficients (with following sequential calculating of all the roots).

In Chapter 2, the explicit formulae for two characteristic eigenprojectors and the quasi-inverse matrix for a null-prime singular $n \times n$-matrix B in terms of its matrix and scalar characteristic coefficients of the highest order $r = rankB$ are established. (The simplest case of null-prime matrices is a $n \times n$-matrix B consisting from r of basis columns and $n - r$ of zero columns.) As very important especial case, the *null-normal* singular $n \times n$-matrices B, whose image and kernel form a direct orthogonal sum, are entered and studied. (Their considered separately important particular cases are symmetric S and multiplicative matrices $AA', A'A$.) Besides, the modal matrices for transformations of these null-prime and null-normal matrices into the two-cell block-diagonal canonic form are gotten. As additional applications of the eigenprojectors and quasi-inverse matrices, the general formulae of solutions for vector and matrix linear equations are gotten.

In Chapter 3, the more general linear geometric objects in linear spaces than $n \times 1$-vectors and lines are entered additionally into consideration, as $n \times m$-lineors A and planars $\langle im\ A \rangle$ and $\langle ker\ A \rangle$, i. e., set by the matrix A, where $1 \leq \mathrm{m} \leq \mathrm{n}$ (in particular, at $\mathrm{m} = 1$ they are vector $\mathbf{a}$, lines $\langle im\ \mathbf{a} \rangle$ and hyperspace $\langle ker\ \mathbf{a} \rangle$). The scalar invariant relations for matrices and corresponding to them inequalities having cosine or sine nature (generalizing well-known algebraic norms for a cosine and a sine of the angle between vectors or lines in Euclidean arithmetic space) are defined. As additional result, the limit explicit formulae for the eigenprojectors and quasi-inverse matrices are gotten by algebraic and functional manners.

In last Chapter 4 of the Part, the main alternative complexification variants of algebraic and geometric characteristics are considered upon transition from real arithmetic spaces into different complex ones. It is important, in particular, for following constructing similar complex Tensor Trigonometry variants. A number of the concrete examples of complexification in different mathematical fields, including arithmetic, algebraic, geometric and functional ones, are given.

Chapter 1
Coefficients of characteristic polynomials

1.1 Simultaneous definition of scalar and matrix coefficients

In Theory of Exact Matrices, especial attention is paid to characteristic polynomials. They are studied from algebraic and geometric points of view. Detailed analysis of the question is necessary for further construction of Tensor Trigonometry foundation.

As it is known, for each $n\times n$-matrix there is its own secular equation determined by the *scalar characteristic polynomial* (a polynomial with scalar coefficients) depending on a certain parameter μ. The roots μ_i of this polynomial (the roots of the secular equation) for a given square matrix B are the eigenvalues of the matrix. The matrix B has also the *matrix characteristic polynomial* (a polynomial with matrix coefficients).

Introduce simultaneously two kinds of the characteristic polynomials and their coefficients, we follow mainly to D. K. Faddeev [19, p. 311–316]. Consider a nonzero $n \times n$-matrix B of rank r and the unity matrix I. *The resolvent* of the matrix B is its following transformation:

$$(B+\epsilon I)^{-1} = \frac{(B+\epsilon I)^V}{det(B+\epsilon I)} = \frac{K_B(\epsilon)}{k_B(\epsilon)}. \qquad (1)$$

In fact, it is the usual formula of the inverse matrix for $B+\epsilon I$: the numerator is the adjoint matrix, the denominator is its determinant, ϵ is an arbitrary scalar parameter. This operation determines two characteristic polynomials: scalar one of order n as the denominator and matrix one of order $n-1$ as the numerator of the fraction:

$$k_B(\epsilon) = \sum_{t=0}^{n} k(B,t)\epsilon^{n-t} = \epsilon^n + tr\ B\cdot\epsilon^{n-1} + \cdots + det\ B,$$

$$K_B(\epsilon) = \sum_{t=0}^{n-1} K_1(B,t)\epsilon^{n-t-1}.$$

The formulae of the polynomials contain so-called the *scalar characteristic coefficients* $k(B,t)$ and the *matrix characteristic coefficients of the* 1-*st kind* $K_1(B,t)$ (ones of the 2-nd kind $K_2(B,t)$ will be defined later). The sequential-increasing number t is the order of the scalar and matrix coefficients.

In this book, we consider characteristic polynomials for square matrices in (1), as a rule, in the *sign-constant form* as polynomials in the scalar parameter $\epsilon = -\mu$. The opposite parameter $\mu = -\epsilon$ represents the eigenvalues of the matrix B. The similar scalar polynomial in μ is zero and determines the *sign-alternating* secular equation for the matrix B:

$$k_B(-\mu) = (-\mu)^n + tr\ B\cdot(-\mu)^{n-1} + \cdots + det\ B = 0.$$

Thus the scalar coefficients of order t are the Viète sums of μ_i and the sums of all principal $t \times t$-minors, but with the summands of constant sign. They may be computed by Le Verrier's method [19, 20] with use of the known recurrent Waring formula, where the Viète sums are changed by the scalar characteristic coefficients, and the Waring sums are replaced by the characteristic traces (of the same order t):

$$k(B,t) = \frac{1}{t} \cdot \sum_{\theta=1}^{t} (-1)^{\theta-1} k(B, t-\theta) \cdot tr\ B^{\theta}. \tag{2}$$

It is the recurrent Waring–Le Verrier *direct* formula. Note, that the equivalent explicit expressions

$$k(B,t) = \frac{1}{t!} \cdot det \begin{bmatrix} tr\ B & 1 & 0 & \cdots & 0 \\ tr\ B^2 & tr\ B & 2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ tr\ B^{t-1} & tr\ B^{t-2} & tr\ B^{t-3} & \cdots & t-1 \\ tr\ B^{t} & tr\ B^{t-1} & tr\ B^{t-2} & \cdots & tr\ B \end{bmatrix} \tag{3}$$

are of more theoretical interest [16, p. 38]. Formulae (2) and (3) are obtained from the Newton system of linear equations for n unknown coefficients with n given roots as the result of the change described above. The sequence of the scalar coefficients (the Viète sums) is, due to the Newton system of equations, in the one-to-one correspondence with the sequence of the characteristic traces (the Waring sums) up to the special order, what has the following property

$$t = r' = \min\{rankB^h\} \leq r$$

and all the scalar coefficients of greater orders are equal to 0. Here the number r' is called *the* 1*-st rock* of the matrix B (*the* 2*-nd rock* r'' is the greatest order of the nonzero matrix characteristic coefficients). All problems concerning the scalar coefficients for equations may be expressed in terms of the Waring sums, and ones for the matrices may be analyzed in terms of the characteristic traces.

1.2 The general inequality of means

In main part II, we often deal with positively (semi)definite symmetric and Hermitian matrices of fixed rank and their scalar invariants. Suppose that B is such a matrix. Consider the secular equation for B in the usual sign-alternating form and its scalar coefficients. All these coefficients of orders up to $r' = r = rankB$ are positive real numbers. Moreover, all the roots μ_i of the secular equation (the eigenvalues of the matrix B) are nonnegative real numbers.

Let μ_i be n nonnegative numbers and exactly r of them ($r \le n$) are nonzero. Special characteristics of the set $\langle\mu_i\rangle$, *the small medians* $\overline{m_1}, \overline{m_t}$ (*the algebraic means*) and *the large medians* $\overline{M_1}, \overline{M_\theta}$ (*the power means*), are defined as follows:

$$\overline{m_1} = \overline{M_1} = \sum_{i=1}^{n} \frac{\mu_i}{n}, \tag{4}$$

$$\overline{m_t} = \sqrt[t]{s_t(\mu_i)/C_n^t} = \sqrt[t]{k(B,t)/C_n^t}, \tag{5}$$

$$\overline{M_\theta} = \sqrt[\theta]{S_\theta(\mu_i)/n} = \sqrt[\theta]{tr\ B^\theta/n}, \tag{6}$$

where $s_t(\mu_i)$ are the Viète sums, $S_\theta(\mu_i)$ are the Waring sums, n is the size of the set $\langle\mu_i\rangle$ or of the quadratic matrix, t and θ are orders of the corresponding means, C_n^t are the Newton binomial coefficients. (The arithmetic mean $\overline{m_1} = \overline{M_1}$ is the intersection of the set of all small medians and the set of all large ones.) Therefore formulae (5) express the algebraic medians not only in terms of the Viète sums, but also in terms of the equation coefficients, and formula (6) represents the power medians in terms of the Waring sums as well as in terms of the matrix traces. If there are zeroes among μ_i and $t > r$, then $\overline{m_t} = 0$.

Otherwise the analogous reverse medians are defined as follows:

$$\overline{v_1} = \overline{V_1} = \left(\sum_{i=1}^{n} \frac{\mu_i^{-1}}{n}\right)^{-1}, \tag{7}$$

$$\overline{v_t} = \sqrt[-t]{s_t(\mu_i^{-1})/C_n^t} = \sqrt[-t]{k(B^{-1},t)/C_n^t}, \tag{8}$$

$$\overline{V_\theta} = \sqrt[-\theta]{S_\theta(\mu_i^{-1})/n} = \sqrt[-\theta]{tr\ B^{-\theta})/n}. \tag{9}$$

They too play the role of average values, i. e., the reverse means of the numbers $1/\mu_i$. Notice that the geometric mean $\overline{m_n} = \overline{v_n}$ is the intersection of the set of all small medians and the set of all their reverse analogs; but $\overline{v_1} = \overline{V_1}$ is the harmonic mean.

For a set of n positive real numbers $\langle\mu_i\rangle$ containing at least two distinct ones, the following general inequality of means does hold on all the interval in $\mathbb{R}$ containing $\langle\mu_i\rangle$:

$$\max\langle\mu_i\rangle = \overline{M_\infty} > \cdots > \overline{M_\theta} > \cdots > \overline{M_1} = \tag{10}$$

$$= \overline{m_1} > \cdots > \overline{m_t} > \cdots > \overline{m_n} = \tag{11}$$

$$= \overline{v_n} > \cdots > \overline{v_t} > \cdots > \overline{v_1} = \tag{12}$$

$$= \overline{V_1} > \cdots > \overline{V_\theta} > \cdots > \overline{V_\infty} = \min\langle\mu_i\rangle \tag{13}$$

$$(t = 1, \ldots, n; \quad \theta = 1, \ldots, \infty).$$

The equality for all the means simultaneously does hold iff $\mu_1 = \cdots = \mu_n$. If there are exactly $n - r$ zeroes among μ_i, then $\overline{m_1} \cdots \overline{m_r} \ne 0$ and $\overline{m_t} = 0$ for all $t > r$.

Moreover, if under this condition all nonzero μ_i are equal, then the medians are expressed as the functions

$$\overline{m_t} = \mu \cdot \sqrt[t]{C_r^t / C_n^t}, \quad \overline{M_\theta} = \mu \cdot \sqrt[\theta]{r/n}.$$

Note also, that in the general inequality middle chains (11) and (12) of means are connected by one-to-one functional bound. The same relates to any continuous chains of it from n means iff all the original n numbers are different. This bond is interpreted obviously as direct and back n-vector-function of n-vector-argument. The fact will be used in the next section.

Special cases of the general inequality are the Cauchy inequality for arithmetic and geometric means and its reverse analog for harmonic and geometric means, the Maclaurin inequality for algebraic means and its reverse analog, the Hölder inequality for power means and its reverse analog [22]. Suppose B is a spectrally positive (all $\mu_i > 0$) matrix. The arithmetic, geometric, and harmonic medians are defined as follows:

$$\overline{m_1} = tr\ B/n = \overline{M_1}, \tag{14}$$

$$\overline{m_n} = \sqrt[n]{det\ B} = \overline{v_n}, \tag{15}$$

$$\overline{v_1} = \left(tr\ B^{-1}/n\right)^{-1} = \overline{V_1}. \tag{16}$$

Let A be an $m \times n$-matrix (in particular, $A = \mathbf{a}$ may be an $n \times 1$-vector), $B = AA'$. Then the arithmetic median is expressed in terms of the Frobenius and Euclidean norms:

$$n \cdot \overline{m_1(B)} = tr\ B = \begin{cases} ||A||_F^2, \\ ||\mathbf{a}||_E^2. \end{cases}$$

Since B is a spectral-positive matrix, the chain of simplest inequalities–estimations

$$\max\langle \mu_i^n \rangle \geq tr\ B^n/n \geq (tr\ B/n)^n \geq det\ B \geq$$

$$\geq \left(tr\ (B^{-1})/n\right)^{-n} \geq \left(tr\ (B^{-n})/n\right)^{-1} \geq \min\langle \mu_i^n \rangle \tag{17}$$

follows from (10)–(13). Closer to each other are the eigenvalues, less are all the defects in (17). The equality holds iff the matrix B is proportional to the unit matrix I.

Clearly, the limit medians for B in the general inequality are the extremal eigenvalues of B:

$$\max\langle \mu_i^n \rangle = \lim_{n\to\infty} \overline{M_n}, \tag{18}$$

$$\min\langle \mu_i^n \rangle = \lim_{n\to\infty} \overline{V_n}. \tag{19}$$

Now we prove the general inequality and analyze it with use of differentiation to explore extrema.

Consider n positive numbers x_i as the vector $\mathbf{x} = (x_1, \ldots, x_n)$ in the 1-st quadrant (the basis is standard) and the scalar functions expressing the differences and the ratios of the corresponding means:

$$r\begin{bmatrix} t \\ t+1 \end{bmatrix}(\mathbf{x}) = \overline{m_t(\mathbf{x})} - \overline{m_{t+1}(\mathbf{x})},$$

$$r\begin{bmatrix} 1 \\ n \end{bmatrix}(\mathbf{x}) = \overline{m_1(\mathbf{x})} - \overline{m_n(\mathbf{x})},$$

$$f\begin{bmatrix} t \\ t+1 \end{bmatrix}(\mathbf{x}) = \overline{m_t(\mathbf{x})}/\overline{m_{t+1}(\mathbf{x})},$$

$$f\begin{bmatrix} 1 \\ n \end{bmatrix}(\mathbf{x}) = \overline{m_1(\mathbf{x})}/\overline{m_n(\mathbf{x})},$$

$$R\begin{bmatrix} \theta+1 \\ \theta \end{bmatrix}(\mathbf{x}) = \overline{M_{\theta+1}(\mathbf{x})} - \overline{M_\theta(\mathbf{x})},$$

$$R\begin{bmatrix} \theta \\ 1 \end{bmatrix}(\mathbf{x}) = \overline{M_\theta(\mathbf{x})} - \overline{M_1(\mathbf{x})},$$

$$F\begin{bmatrix} \theta+1 \\ \theta \end{bmatrix}(\mathbf{x}) = \overline{M_{\theta+1}(\mathbf{x})}/\overline{M_\theta(\mathbf{x})},$$

$$F\begin{bmatrix} \theta \\ 1 \end{bmatrix}(\mathbf{x}) = \overline{M_\theta(\mathbf{x})}/\overline{M_1(\mathbf{x})}.$$

Each of the functions $r, R,$ and f, F has the only and common stationary value corresponding to $\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is the bisectrix of the 1-st quadrant. These functions have the zero gradients at all points of $\mathbf{b}$. Therefore,

$$r'(\mathbf{b}) = R'(\mathbf{b}) = f'(\mathbf{b}) = F'(\mathbf{b}) = \mathbf{0}, \quad x_1 = \cdots x_n = b,$$

$$r(\mathbf{b}) = R(\mathbf{b}) = 0, \quad f(\mathbf{b}) = F(\mathbf{b}) = 1,$$

and $\mathbf{b}$ is the region of minimum as the corresponding Hesse matrices are positively semi-definite (their rank is $n-1$):

$$r''\begin{bmatrix} 1 \\ n \end{bmatrix}(\mathbf{b}) = (n-1)r''\begin{bmatrix} t \\ t+1 \end{bmatrix}(\mathbf{b}) =$$

$$= bf''\begin{bmatrix} 1 \\ n \end{bmatrix}(\mathbf{b}) = b(n-1)f''\begin{bmatrix} t \\ t+1 \end{bmatrix}(\mathbf{b}) =$$

$$= R''\begin{bmatrix} \theta+1 \\ \theta \end{bmatrix}(\mathbf{b}) = \frac{1}{\theta-1}R''\begin{bmatrix} \theta \\ 1 \end{bmatrix}(\mathbf{b}) =$$

$$= bF''\begin{bmatrix} \theta+1 \\ \theta \end{bmatrix}(\mathbf{b}) = \frac{b}{\theta-1}F''\begin{bmatrix} \theta \\ 1 \end{bmatrix}(\mathbf{b}) = \frac{nI - It}{n^2 b} = G,$$

where It is the *totally-unity* matrix, all its elements are equal to 1. The matrix G has the positive principal minors of orders r, $r < n$, they are equal to

$$\left(\frac{1}{nb}\right)^r \cdot \frac{n-r}{n}.$$

The Hesse matrix is degenerated at all points of the bisectrix, the one-dimensional linear subspace. The stationary values computed above lead to the following equalities

$$r''\begin{bmatrix} t \\ t+m \end{bmatrix}(\mathbf{b}) = mr''\begin{bmatrix} t \\ t+1 \end{bmatrix}(\mathbf{b}),$$

$$f''\begin{bmatrix} t \\ t+m \end{bmatrix}(\mathbf{b}) = mf''\begin{bmatrix} t \\ t+1 \end{bmatrix}(\mathbf{b}),$$

$$R''\begin{bmatrix} \theta+m \\ \theta \end{bmatrix}(\mathbf{b}) = mR''\begin{bmatrix} \theta+1 \\ \theta \end{bmatrix}(\mathbf{b}),$$

$$F''\begin{bmatrix} \theta+m \\ \theta \end{bmatrix}(\mathbf{b}) = mF''\begin{bmatrix} \theta+1 \\ \theta \end{bmatrix}(\mathbf{b}).$$

Therefore, on the bisectrix $\mathbf{b}$, these facts give us the following logical corollaries.

1. *The Hesse matrices of the adjacent means ratio do not depend on their orders.*

2. *These matrices vary as additive functions of the difference between the orders.*

3. *The Hesse matrices for all adjacent power means ratios are equal to the Hesse matrix for the ratio of the arithmetic and geometric means.*

4. *The Hesse matrices for all adjacent algebraic means ratios consist of* $n-1$ *identical parts of the matrix from Corollary 3.*

But two next corollaries seem surprising and paradoxical. Namely:

5. *The Hesse matrix for the ratio of the power and arithmetic means is unlimited at all points of the bisectrix, it increases as proportional to* θ. *Though the same function* F, *in accordance to* (18), *tends to* $x_{max}/\overline{M_1}$ *as* $\theta \to \infty$, *it is continuous and takes the minimal value* 1 *at all points of the bisectrix.*

6. *The Hesse matrix for the adjacent power means ratio is constant at all points of the bisectrix even as* $\theta \to \infty$. *Though, according to* (18), *the same function* F *tends to* 1 *at all points of the bisectrix, its limit value is the constant for which the gradient and the Hesse matrix are zero.*

These conclusions seem contradictory, but they can be explained by correlation between the infinitely small deviation of $\mathbf{x}$ from the bisectrix and the infinitely large parameter θ. That is why the Hesse matrix is discontinuous and becomes zero in the neighborhood of the bisectrix. The function $F\begin{bmatrix} \theta \\ 1 \end{bmatrix}(\mathbf{x})$, in its turn, tends to 1 as $\theta \to \infty$, but it depends up to infinitesimal on $\mathbf{x}$ and takes the minimal value 1 at points of $\mathbf{b}$. Contrary, the function $F\begin{bmatrix} \theta+1 \\ \theta \end{bmatrix}(\mathbf{x})$ takes the value 1 there at once.

Interpret these facts on the model functions of one scalar variable:

$$F_1\begin{bmatrix}\theta+1\\ \theta\end{bmatrix}(x) = \sqrt[\theta+1]{\frac{1+x^{\theta+1}}{2}} \Big/ \sqrt[\theta]{\frac{1+x^{\theta}}{2}},$$

$$F_2\begin{bmatrix}\theta\\ 1\end{bmatrix}(x) = \sqrt[\theta]{\frac{1+x^{\theta}}{2}} \Big/ \frac{1+x}{2}, \quad (x>0, \qquad \theta \ge 2).$$

Suppose, for certain conditions of the task, that $x \ge 1$, then it is the greatest element of the model set $\langle 1, x\rangle$.

If θ is *finite*, then

$$F_1(1) = F_2(1) = 1 = \min, \quad 1 < F_1(x) < F_2(x);$$

$$\frac{dF_1}{dx}(1) = \frac{dF_2}{dx}(1) = 0;$$

$$\frac{d^2F_1}{dx^2}(1) = \frac{1}{4}, \quad \frac{d^2F_2}{dx^2}(1) = \frac{\theta-1}{4}, \quad \frac{d^2F_2}{dx^2}(x) \ge \frac{d^2F_1}{dx^2}(x) > 0.$$

If θ is *infinite*, then

$$F_1(x) = 1+\beta(x),\ \beta(x)\to 0,\ \beta(1)=0,\ F_2(1) = 1 = \min,$$

$$F_2(x) = \begin{cases} 2x/(1+x) & \text{if } x>1,\\ 2/(1+x) & \text{if } x<1,\end{cases}$$

$$\frac{dF_1}{dx}(x) = \frac{dF_2}{dx}(1) = 0,\ \frac{dF_2}{dx}(1\pm\alpha) = \pm\frac{1}{2} \ \ (\alpha\to 0);$$

$$\frac{d^2F_1}{dx^2}(1) = \frac{1}{4}, \qquad \frac{d^2F_1}{dx^2}(x) = 0 \text{ provided that } x \ne 1,$$

$$\frac{d^2F_2}{dx^2}(1) = \frac{\theta-1}{4} \to \infty, \quad \frac{d^2F_2}{dx^2}(1\pm\alpha) = 0 \ \ (\alpha\to 0).$$

The Hesse matrix is also discontinuous in the neighborhood of $\langle \mathbf{b}\rangle$, that is why the trivalent symmetric matrix of third derivatives tends to infinite one as $\theta\to\infty$ and is negatively semi-definite at all points of the bisectrix. Notice that for the analogous functions of the reverse means, all these facts do hold, the only difference is that the Hesse matrix changes the sign. The same transformation of the Hesse matrix takes place under inverting the ratios.

These arguments as well as limit formulae (18) and (19) complete our proof and analysis of the general inequality of means. Now we consider some applications of the general inequality in the theory and techniques for solving algebraic equations, particularly, secular ones.

1.3 The serial method for solving an algebraic equation with real roots

The small and large medians are connected by the system of modified Newton equations and the modified Waring–Le Verrier formulae, for example, of the *direct type*. These *direct* formulae are similar to (2) provided that $t > r$ and $\overline{m_t} = 0$:

$$C_{n-1}^{t-1}(\overline{m_t})^t = C_n^{t-1}(\overline{m_{t-1}})^{t-1}(\overline{M_1})^1 - C_n^{t-2}(\overline{m_{t-2}})^{t-2}(\overline{M_2})^2 + \cdots +$$
$$+(-1)^{t-2}C_n^1(\overline{m_1})^1(\overline{M_{t-1}})^{t-1} + (-1)^{t-1}(\overline{M_t})^t.$$

If all the coefficients of a secular equation are the same, then the well known particular formula for binomial coefficients

$$C_{n-1}^{t-1} = C_n^{t-1} - C_n^{t-2} + \cdots + (-1)^{t-2}C_n^1 + (-1)^{t-1}$$

follows from one above.

Limit formulae (18) and (19) allow one to compute consequently all the roots of an algebraic equation provided that all its roots are real numbers. Multiplicity of the roots may be found in the process of reducing, but it is worth to separate the roots before solving with use of the 1-st derivative and Euclidean algorithm. Sturm's method [7, p. 225–229] and the prior boundaries of the roots ($\mp\infty$) ensure one that the roots are real numbers. Other useful criterions for identification of the roots reality follow from the inequalities for the real roots of an algebraic equation represented here in its sign-alternating form [16, p. 40]:

$$-1 - \sqrt[h_1]{-\min k_j} = \Delta^{(-)} < \mu_i < \Delta^{(+)} = 1 + \sqrt[h_2]{-\min(-1)^j k_j},$$

where $\Delta^{(-)}$ and $\Delta^{(+)}$ are the boundaries of the negative and positive real roots, h_1 and h_2 are the indexes of the first negative coefficients, respectively k_j and $(-1)^j k_j$. Maclaurin's Theorem is used for inferring these inequalities [7, p. 223].

The serial method for solving an algebraic (secular) equation is the following.

It is supposed to be already known that all the equation roots are real nonnegative numbers, in particular, they may be the eigenvalues of a nonnegatively definite matrix AA' or $A'A$.

The first step is computing the Viète sums and the Waring sums up to order r. For example, the Waring–Le Verrier recurrent formula of the direct type (such as (2)) is used for matrices, and the following Waring–Le Verrier recurrent formula of the *reverse type* [16, p. 38] is used for an arbitrary algebraic (polynomial) equation:

$$S_\theta = s_1 S_{\theta-1} - s_2 S_{\theta-2} + \cdots + (-1)^{r-2} s_{r-1} S_{\theta-r+1} + (-1)^{r-1} s_r S_{\theta-r} =$$
$$= F_\theta(S_1, \ldots, S_r) = f_\theta(s_1, \ldots, s_r), \qquad \theta = r+1, r+2, \ldots.$$

Next step is consequent computing the power medians

$$\overline{M_\theta} = \sqrt[\theta]{s_\theta / r}.$$

Due to (10), the sequence of the fixed root approximations increases. Clearly, more different are the roots, faster is the process. The recurrent formula with limit value (18), being divided by $x^{\theta-n}$, is the original equation as an identity. That is why on a certain iteration computing should be finished in order to avoid a round-off error. This results in the maximal root. The minimal root may be found according to (19) by the similar way with use of the equation inverse form in $(-1/x)$ obtained by dividing original one by $(-x)^n$ and by the highest coefficient k_n. (For matrices, $k_n = det\ B$.)

Approximate computing a *rational root* induces a periodic sequence starting with some significant digit, that is why the precise value of this root should be checked in the original equation. *Irrational roots* are computed up to a given precision. Thus the algorithm results in all the real roots of an algebraic equation. This method has the common limit idea with classical Lobachevsky–Greffe's one (1834) (see detailed comparison of both the methods in other our monograph [18, p. 162–163]).

If all the equation roots are real numbers of arbitrary signs, then the variable x of this equation should be substituted for $x + C$, where the positive constant C shifts the variable into the positive semiaxis. In order to faster convergence, this shift should be as small as possible.

It is known that all the eigenvalues of real symmetric matrices $S = S'$ and imaginary anti-symmetric ones $(iK)' = -iK$, where $K = -K'$ is a real matrix, are real valued numbers. In particular, these matrices are characteristic ones of a real matrix B:

$$S_B = (B + B')/2, \quad K_B = (B - B')/2, \quad B = S_B + K_B.$$

Condition $S_B K_B = K_B S_B$ means that $B \in \langle M \rangle$ is a normal matrix. These matrices may be transformed into diagonal ones simultaneously. The eigenvalues of a normal matrix M are the sums of the summand matrices eigenvalues. Thus separated solving the secular equations for S_M and $-iK_M$ (the secular equation for $-iK_M$ is biquadratic) result in the real and imaginary parts of the matrix M complex eigenvalues. Further, the values obtained should be paired by checking in the secular equation for M.

This approach may be extended on complex matrices by use of the Hermitean and skew-Hermitean conjugations. All eigenvalues of Hermitean matrices are real numbers. Take advantage of the following complex Hermitean normal matrix decomposition:

$$H_B = (B + B^*)/2, \quad Q_B = (B - B^*)/2 \qquad (B = H_B + Q_B = H_B + iH_Q),$$

$$H_B Q_B = Q_B H_B \Leftrightarrow H_B H_Q = H_Q H_B \Leftrightarrow B \in \langle N \rangle, \text{ where } NN^* = N^* N,$$

and so on.

Thus the serial method represented here is also applicable to real normal matrices and complex Hermitean normal ones.

Suppose that all the roots of the secular equation for a some matrix are real numbers and shifting described above is used. Then, for the equation in alternating-sign form, the lower boundary of the negative roots satisfies the following inequality:

$$\min\langle\mu_i\rangle > \Delta^{(-)} = -1 - \sqrt[h_1]{-\min k_j}.$$

Substitution $x = y + \Delta^{(-)}$ results in the equation with the positive coefficients and roots, this may be checked by Sturm's method on $(0; +\infty)$. This shift leads to the matrix transformation $B \to (B - \Delta^{(-)} I)$.

There exists another way as alternative to shifting. If all the eigenvalues of a some matrix B are real numbers of arbitrary signs, then the following sequence of actions may be performed instead of shifting:
1) squaring B,
2) computing the squared eigenvalues,
3) choosing the signs of the eigenvalues by checking in the equation.

If all the roots of an algebraic equation are real positive numbers, then the theoretical value of its greatest root is in the explicit form

$$\max\langle\mu_i\rangle = \lim_{\theta\to\infty} \sqrt[\theta]{det\ K^{(1)}/r}, \tag{20}$$

where $K^{(1)}$ is the following $(r+\theta)\times(r+\theta)$-matrix of the equation coefficients:

$$K^{(1)} = \begin{bmatrix} k_1 & -1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ -2k_2 & k_1 & -1 & \dots & 0 & 0 & \dots & 0 & 0 \\ 3k_3 & -k_2 & k_1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{r-1}rk_r & (-1)^{r-2}k_{r-1} & (-1)^{r-3}k_{r-2} & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & (-1)^{r-1}k_r & (-1)^{r-2}k_{r-1} & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & (-1)^{r-1}k_r & \dots & \dots & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (-1)^{r-2}k_{r-1} & (-1)^{r-3}k_{r-2} & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & (-1)^{r-1}k_r & (-1)^{r-2}k_{r-1} & \dots & k_1 & -1 \\ 0 & 0 & 0 & \dots & 0 & (-1)^{r-1}k_r & \dots & -k_2 & k_1 \end{bmatrix}.$$

All zero elements of the matrix are only in the two triangles of sizes θ and $n+\theta-2$, i. e., for lower and upper ones, other elements are nonzero. Here $det\ K^{(1)} = S_\theta$ is the Waring sum of order θ (see above), according Waring–Le Verrier *reverse explicit formula* [16, p. 38].

By similar arguments and due to (9),

$$\min\langle\mu_i\rangle = \lim_{\theta\to\infty} \sqrt[-\theta]{det\ (K^{(2)}/k_n)/r},$$

where $K^{(2)}$ is the following $(r+\theta)\times(r+\theta)$-matrix of the same equation coefficients considered in the inverse form:

$$K^{(2)} = \begin{bmatrix} k_{r-1} & -k_r & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ -2k_{r-2} & k_{r-1} & -k_r & \dots & 0 & 0 & \dots & 0 & 0 \\ 3k_{r-3} & -k_{r-2} & k_{r-1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{r-1}r & (-1)^{r-2}k_1 & (-1)^{r-3}k_2 & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & (-1)^{r-1} & (-1)^{r-2}k_1 & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & (-1)^{r-1} & \dots & \dots & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (-1)^{r-2}k_1 & (-1)^{r-3}k_2 & \dots & -k_r & 0 \\ 0 & 0 & 0 & \dots & (-1)^{r-1} & (-1)^{r-2}k_1 & \dots & k_{r-1} & -k_r \\ 0 & 0 & 0 & \dots & 0 & (-1)^{r-1} & \dots & -k_{r-2} & k_{r-1} \end{bmatrix}.$$

By Sylvester's criterion, a symmetric or Hermitian matrix is positively definite iff all its principal minors are positive. The minor of the highest order is the determinant, so Sylvester's condition also means that the matrix is nonsingular. Besides, a singular symmetric or Hermitian matrix is positively semi-definite iff all its sign-alternating secular equation's coefficients up to order r are positive, and ones of orders $t > r$ are equal to 0, as all the roots here are real numbers. Thus the elements of normal matrices contain sufficient information for finding all the eigenvalues provided that all the roots of the secular equation are real numbers, and then the serial method is applicable.

Solvability of the same problem for more general matrices as well as the similar one for an arbitrary algebraic equation of degree $n > 4$ depends on the answer to the question: *whether a given algebraic equation has complex conjugate roots?* We showed above that the answer can be found by Sturm's method. However this method does not give necessary and sufficient conditions on the equation coefficients under which all the roots are real numbers and, due to shifting, positive.

One well known necessary condition follows from the Descartes sign Rule: all the coefficients of an equation in the sign-alternating form must be positive. Unfortunately even under this condition pairs of conjugate complex roots are possible. If the shift parameter is greater than noted above, for example, it is equal to $1 + \max |k_j|$, then only the real parts of the roots are necessarily positive [16, p. 39].

Inequalities (11) have the following corollary.

If all the roots of an algebraic equation are real positive numbers, then all its medians in (10) – (13) *are equal to each other iff the equation has the binomial form*

$$(x - \mu)^n = 0.$$

This means also that $\overline{m_t} = \mu$.

If an equation in the sign-alternating form has at least two distinct roots, then its coefficients do not form the binomial sequence and then inequalities (11) do hold. For example, if there exist two adjacent medians equal to each other or some of the equation coefficients of order less than r are equal to zero, or the median hierarchy is violated, then there exist complex conjugate roots.

The following conditions are necessary and stronger than Descartes' one given above.

For all n the roots of an algebraic equation of degree n represented in the sign-alternating form to be positive real numbers it is necessary that all the equation coefficients-medians (5) *satisfy the following two conditions*:

(i) *they are positive real numbers*
(according to the Descartes sign Rule),

(ii) *all of n inequalities* (11) *do hold.*

For an $n \times n$-matrix to be positively definite it is necessary that all the matrix traces-medians (6) *of orders* $1, 2, \ldots, n$ *satisfy two conditions*:

(i) *they are positive real numbers,*

(ii') *first n inequalities* (10) *do hold.*

For any real algebraic equation and any real quadratic matrix condition (i) may be satisfied by use of shifting. For real symmetric or complex Hermitian matrix Sylvester's criterion gives the necessary and sufficient condition for all the roots of the secular equation (its eigenvalues) to be positive real numbers. If a real matrix is of the form AA', then all its eigenvalues are a priori real and nonnegative. The *necessary and sufficient conditions* for all the roots of an algebraic (polynomial) equation of degree n to be positive real numbers are inferred in our next monograph [18, p. 165–191] with the use of the Special diagrams.

Note that for any algebraic median,

$$\sqrt[p]{\overline{m_i}(x_1^p + \cdots + x_n^p)} < \sqrt[q]{\overline{m_i}(x_1^q + \cdots + x_n^q)}$$

provided that

$$1 \leq p < q, \qquad i = 1, \ldots, n-1,$$

there exist at least two distinct elements, and the quantity of the nonzero elements is greater than i. This follows from (10).

1.4 Structures of scalar and matrix characteristic coefficients

For a given square matrix B, its scalar characteristic coefficients of any order t may be represented according to (5) as the Viète sums of the eigenvalues μ_i. The eigenvalues are invariant under all linear transformations of the matrix and the bases; therefore, the scalar coefficients are invariant under such transformations too.

For any matrix B there exists a unique pair of matrices (P_B, O_B) such that P_B is a prime matrix, O_B is a nilpotent matrix, and

$$B = P_B + O_B. \tag{21}$$

The matrices P_B and O_B are determined by the Jordan form J_B or the triangle form of B.

As it is known, a matrix O is nilpotent iff all its scalar characteristic coefficients are equal to zero. Evaluate the nilpotency degree j of the matrix O_B. Let $j(i)+1$ be the maximal size of the Jordan subcell in J_B with the eigenvalue μ_i at the diagonal. Then

$$j = \max_{\langle \mu_i \rangle} \{j(i)\}.$$

Not only O_B but also $O_B P_B$ and $P_B O_B$ are nilpotent matrices, and the matrices B and P_B have the same secular equation as well as the same eigenvalues with the same algebraic multiplicities. Thus the scalar coefficients for the matrix B possess the following additional properties:

$$k(P_B + O_B, t) = k(P_B, t) = k(B, t). \tag{22}$$

$$k(P_B \cdot O_B, t) = k(O_B \cdot P_B, t) = k(O_B, t) = 0. \tag{23}$$

From the structural point of view, any scalar coefficient $k(B,t)$ is the sum of all diagonal (principal) $t \times t$-minors of B [3, p. 78].

Further, consider most important properties of matrix characteristic coefficients, establish their structure and connection with scalar ones.

At first, formula (1) is equivalent to each of the following identities:

$$det\ (B + \epsilon I) I = (B + \epsilon I)(B + \epsilon I)^V.$$

$$k_B(\epsilon) I = (B + \epsilon I) K_B(\epsilon), \tag{24}$$

$$\sum_{t=0}^{n} \epsilon^{n-t} \left[k(B,t) I - B K_1(B, t-1) - K_1(B,t) \right] = Z,$$

where Z is the zero matrix (all the polynomials are here in the constant-sign form). These formulae have in particular the following corollaries.

1. *The scalar parameter* ϵ *in* (24) *may be changed for a matrix parameter* E *commuting with* B:

$$k_B(E) = (B + E) K_B(E).$$

2. *The Hamilton–Cayley Theorem* is proved **in one-line** with (24) at $E = -B$:

$$k_B(-B) = Z.$$

Contrary, if $E = +B$, then $k_B(B) = 2 B K_B(B)$.

3. *The recurrent matrix formula of Jean-Marie Souriau* [20] (a pioneer in symplectic geometry)

$$K_1(B,t) = -B K_1(B, t-1) + k(B,t) I \tag{25}$$

is valid because the parameter ϵ in (24) is arbitrary. *The initial values*

$$k(B, 0) = 1, \qquad K_1(B, 0) = I$$

follow from (1). Note that $k(B,1) = tr\ B,\ k(B,n) = det\ B$.

4. Define additionally *the matrix characteristic coefficients* $K_2(B,t)$ *of the* 2*-nd kind* as

$$K_2(B,t) = BK_1(B,t-1).$$

The initial value is $K_2(B,0) = Z$. Clearly, $K_2(B,1) = B$. Taking this into account, one may transform (25) into

$$K_1(B,t) + K_2(B,t) = k(B,t)I. \tag{26}$$

Repeating application of the recurrent formula (25) with the initial values leads to the following *representation of the matrix characteristic coefficients as polynomials in* B:

$$\left.\begin{array}{rcl} K_1(B,t) & = & \sum_{\theta=0}^{t} k(B,t-\theta)(-B)^{\theta}, \\ K_2(B,t) & = & -\sum_{\theta=1}^{t} k(B,t-\theta)(-B)^{\theta}. \end{array}\right\} \tag{27}$$

Therefore, the matrix coefficients $K_1(B,t)$ and $K_2(B,t)$ commute with each other and with B.

5. *The Jean-Marie Souriau scalar binding formula* [20]

$$k(B,t) = \frac{1}{t}\cdot trK_2(B,t), \quad \{k(B,t) = \frac{1}{n-t}\cdot trK_1(B,t)\} \tag{28}$$

follows from (27) and (2), i. e., using Le Verrier method (see above).

6. In order to compute B^{-1}, J.-M. Souriau suggested in 1948 the algorithm with also successive calculating all characteristic coefficients of order $t \geq 1$. This algorithm was based on his formulae (25) and (28). Unfortunately, his paper [20] in the Proceedings of the French Academy of Sciences was very brief, without details. The same results were repeated later, probably independently, by other authors. So, the approach of D. K. Faddeev [19], with reference onto Souriau' work, was based on (1) with definition of characteristic coefficients in terms of the matrix resolvent. A year after the Souriau' publication, analogous article by Frame J. S., with the same algorithm, was published in "AMS Bulletin", 1949, v. 55, n. 11, p. 1045, without reference onto [20].

Further the first formula in (27) and Hamilton–Cayley Theorem lead to equalities

$$K_1(B,n) = k_B(-B) = Z,$$

and from (25) we infer

$$BK_1(B,n-1) = k(B.n)I = (detB)I = BB^V = K_2(B,n).$$

If the matrix B is nonsingular, then multiplying these equalities by B^{-1} gives us the following:

$$B^{-1} = \frac{K_1(B,n-1)}{k(B,n)} = \frac{B^V}{det\ B}.$$

This is *the Souriau algorithmic method for inverting a matrix and joint computing all the coefficients* $k(B,t)$ *and* $K_1(B,t)$, $t = 1,\ldots,n$.

7. Therefore, *all the values of the matrix coefficients* computed above are the following:

$$\left.\begin{array}{ll} K_1(B,0) = I, & K_2(B,0) = Z, \\ K_1(B,1) = (trB)I - B, & K_2(B,1) = B, \\ \dots & \dots \\ K_1(B,n-1) = B^V, & K_2(B,n-1) = (tr\ B^V)I - B^V, \\ K_1(B,n) = Z, & K_2(B,n) = (det\ B)I. \end{array}\right\} \quad (29)$$

The formulae of $K_1(B,n-1)$ and $K_2(B,n-1)$ are yet inferred only for a nonsingular matrix B, but they are true.

Further, find *the greatest order r'' of the nonzero matrix coefficients* in (29). Due to (28) it is equal to the terminating order of the Souriau algorithm in (25). It does exist, due to (26) and (28), and is called here *the 2-nd rock of the matrix B*. Moreover, $r'' \geq r'$, where r' is the greatest order of the nonzero scalar coefficients, the 1-st rock (see sect. 1.1). If B is a nonsingular $n \times n$ matrix, then $r' = r'' = n$.

Inequality $r' < r$, where $r = rank\ B$, may be inferred only from *the structure of scalar coefficients*: they are *the sums of all diagonal minors of order t*. Similarly, only *the structure of matrix coefficients* determine the 2-nd rock and its connection with other numerical parameters of the square matrix (in particular, the annulling eigenvalues multiplicities in its minimal annulling polynomial) as well as its matrix characteristics, such as eigenprojectors, quasi-inverse matrices and modal matrices.

In order to clear *the structure of matrix characteristic coefficients* of the 1-st and 2-nd kinds, we apply the special differential method for establishing the structures of scalar and matrix coefficients simultaneously.

Although, for the scalar coefficients, the standard (direct) method for exploring their structure is well known (see, for example, in [3, p. 78]).

Consider an $n \times n$-matrix B and an arbitrary set of its m *generating elements* $\{b_{i_k,j_k},\ k = 1,\dots,m,\ 1 \leq m \leq n\}$, i. e., if $p \neq q$, then $i_p \neq i_q$ and $j_p \neq j_q$. The coefficient at $\prod_{k=1}^{m} b_{i_k,j_k}$ in expansion of $det\ B$ is

$$\frac{\partial^m det\ B}{\partial b_{i_1,j_1} \cdots \partial b_{i_m,j_m}} (-1)^{\sum_{k=1}^{m}(i_k+j_k)} \left\{ \frac{\begin{array}{c} i_1,\dots,i_m \notin \\ j_1,\dots,j_m \notin B \end{array}}{\text{minor(n-m)}} \right\} \quad (30)$$

(in the partial differentiation, the variables for all the elements of B are supposed to be distinct). The minors of order $t = n - m$ in (30) is the adjunct of the minor, determined by the set of m generating elements; i_k, j_k are all their indexes of rows and columns. Formula (30) is the result of successive partial differentiating $det\ B$ with respect to $b_{i_1,j_1} \dots, b_{i_m,j_m}$. We will remind that the order of partial differentiation executions doesn't influence the end result.

Further, apply formula (30) for evaluating the resolvent of B in (1), i. e.,

$$(B+\epsilon I)^{-1} = \frac{(B+\epsilon I)^V}{det(B+\epsilon I)} = \frac{K_B(\epsilon)}{k_B(\epsilon)}.$$

Expand the numerator and the denominator in powers of ϵ.

The denominator is the scalar polynomial in ϵ of order n. According to (30) with $m = n-t$, the coefficient at $\prod_{k=1}^{n-t}(b_{i_k,I_k}+\epsilon)$ is

$$\left\{ \frac{(i_1,i_1)\ldots,(i_{n-t},i_{n-t}) \notin}{\text{D-minor(t)}}(B+\epsilon I) \right\}.$$

It is the diagonal t-minor of $B+\epsilon I$ no containing indicated generating elements, the quantity of such minors (and multiplications) is C_n^t. Only diagonal entries of the minor contain ϵ. Put $\epsilon = 0$ in all these minors. We obtain the expression of the coefficient at ϵ^{n-t} in the scalar polynomial *det* $(B+\epsilon I)$ as *the sum of all its diagonal minors of order* t and its initial mean as $k(B,0) = 1$.

The numerator is the following matrix. Its diagonal entries are polynomials in ϵ of degree $n-1$, other entries are polynomials of degree $n-2$. The matrix is represented by the following polynomial in ϵ:

$$(B+\epsilon I)^V = \sum_{t=0}^{n} K_1(B,t)\epsilon^{n-1-t}, \qquad K_1(B,0) = I.$$

We wish to compute $K_1(B,t)$. For this aim it is necessary to consider the (p,p)- and (p,q)-entries of $(B+\epsilon I)^V$. Find the (p,p)-entry. It is equal to

$$\frac{\partial det\ (B+\epsilon I)}{\partial(b_{p,p}+\epsilon)} = Ad_{p,p}(B+\epsilon I) = \left\{ \frac{(p,p)\notin}{\text{D-minor(n-1)}}(B+\epsilon I) \right\},$$

where $Ad_{p,p}$ is the adjunct of the (p,p)-entry $b_{p,p}+\epsilon$. Similarly to arguments above, the coefficient at ϵ^{n-t-1} (as $n-t-1 = (n-1)-t = (n-(t+1))$ in expansion of this determinant is the (p,p)-entry of the matrix $K_1(B,t)$:

$$(p,p)K_1(B,t) = \sum_{(C_{n-1}^t \text{ terms})} \left\{ \frac{(p,p)\notin}{\text{D-minor(t)}} B \right\} =$$

$$= \sum_{(C_{n-1}^t \text{ terms})} Ad_{p',p'} \left\{ \frac{(p,p)\in}{\text{D-sub(t+1)}} B \right\}$$

(here D-sub stands for a diagonal $(t+1)\times(t+1)$-submatrix of B).

These are sums of D-minors. Both the sums consist of C_{n-1}^t terms, as one generating element, $b_{p,p}$, among n ones takes part in the first differentiation, i. e., in forming the first (main) adjunct. Here p' are the new indexes of the rows and the columns in D-minors of order $t+1$.

Then find the (p,q)-entry of $(B+\epsilon I)^V$. It is equal to

$$\frac{\partial det\ (B+\epsilon I)}{\partial b_{q,p}} = Ad_{q,p}(B+\epsilon I) =$$

$$= (-1)^{p+q}\left\{\frac{(p,q)\in}{\text{Dh-minor(n-t)}}(B+\epsilon I)\right\},$$

(here Dh-minor stands for *hypodiagonal minor*).

It contains only one nondiagonal generating element, $b_{p,q}$, and thus, after the first partial differentiation with respect to $b_{q,p}$ (although the order of partial differentiation executions is of no importance) does not contain $b_{q,p}$, $b_{p,p}+\epsilon$, and $b_{q,q}+\epsilon$. Due to (30), the coefficient at $\prod_{k=1}^{n-t-1}(b_{i_k,i_k}+\epsilon)$ in expansion of the determinant is

$$\frac{\partial^{n-t-1}\left\{\frac{(p,q)\in}{\text{Dh-minor(n-t)}}(B+\epsilon I)\right\}}{\partial(b_{i_1,i_1}+\epsilon)\cdots\partial(b_{i_{n-t-1},i_{n-t-1}}+\epsilon)} =$$

$$= \frac{\partial\left[\frac{\partial^{n-t-1}det(B+\epsilon I)}{\partial(b_{i_1,i_1}+\epsilon)\cdots\partial(b_{i_{n-t-1},i_{n-t-1}}+\epsilon)}\right]}{\partial b_{q,p}} =$$

$$= Ad_{q',p'}\left\{\frac{(i_1,i_1),\ldots,(i_{n-t-1},i_{n-t-1})\notin}{\text{D-sub(t+1)}}(B+\epsilon I)\right\}.$$

Put here $\epsilon = 0$, obtain the coefficient at ϵ^{n-t-1}, i. e. the (p,q)-entry of $K_1(B,t)$:

$$(p,q)K_1(B,t) = \sum_{(C_{n-2}^{t-1}\text{ terms})}(-1)^{p''+q''+1}\left\{\frac{(p,q)\in}{\text{Dh-minor(t)}}B\right\} =$$

$$= \sum_{(C_{n-2}^{t-1}\text{ terms})} Ad_{q',p'}\left\{\frac{(p,q)\in}{\text{D-sub(t+1)}}B\right\}.$$

Here D-sub stands for a diagonal $(t+1)\times(t+1)$-submatrix of B. Both the sums consist of C_{n-2}^{t-1} terms, as two generating elements $b_{p,q}$ and $b_{q,p}$ are used in forming the first (main) adjunct. The (p,q)-element has indexes p',q' in the diagonal minor and p'',q'' in the hypodiagonal one, $p'+q' = p''+q''+1$.

The two parts are the full formula for $K_1(B,t)$. From it and formula (26) expressions for $K_2(B,t)$ follow. *The structure of matrix coefficients is completely specified.* These structural properties of all the characteristic coefficients confirms formulae (29), (28), and, taking (27) into account, the Waring–Le Verrier recurrent formula (2).

Note the *corollary* of these transformations: for a quadratic matrix B, the adjunct of $b_{p,p}$ or $b_{p,q}$ may be interpreted as the partial derivative of *det* B with respect to $b_{p,p}$ or $b_{p,q}$ according to (30), and conversely, the reverse operation, convolution of given adjuncts into *det* B, may be interpreted as their partial integrating on $b_{p,p}$ or $b_{p,q}$.

Compare the scalar and matrix coefficients structures. Both kinds of the coefficients are expressed with the use of minors sums. For scalar coefficients the summands are exactly all diagonal minors. Unlike them, the summands of matrix coefficients are diagonal minors and hypodiagonal ones, other minors cannot be the summands, moreover, other r-minors can exist only under condition $1 < r < n-1$. These facts specify *relationship between the* 1*-st and* 2*-nd rocks* and the rank of a matrix:
(1) $r' \le r''$ (see (28);
(2) $r' < r'' \le r$ *if there exists a unique nonzero hypodiagonal minor of order* r'';
(3) $r' < r'' < r$ *if there exists a unique nonzero minor of order* r'' *and this minor is not diagonal, nor hypodiagonal.*

Thus the structure of scalar and matrix characteristic coefficients specifies the following *fundamental inequalities for basic singularity parameters*:

$$0 \le r' \le r'' \le r \le n. \tag{31}$$

Note the following special cases.

1. $r' = 0 \;\Leftrightarrow\;$ matrix B is nilpotent.
2. $r'' = 0 \;\Leftrightarrow\; B = Z$. (As well $r'' > 0$ iff $K_2(B,1) = B \ne Z$.)
3. $r = 1 \;\Leftrightarrow\; r'' = 1$. (By the same argument).
4. $r = n-1 \;\Leftrightarrow\; r'' = n-1$. ($K_1(B, n-1) = B^V$ contains all minors of rank $n-1$).

The value $t = r''$ is final in the Souriau algorithm. The 1-st and the 2-nd rocks are extremely important singularity parameters of a matrix. They are invariant under linear transformations as well as others.

1.5 The minimal annulling polynomial of a matrix in its explicit form

The results obtained enable us to express the minimal annulling polynomial explicitly in terms of basic singularity parameters of a matrix.

Consider a singular $n \times n$-matrix B of rank r and its eigenvalues μ_i with algebraic multiplicities $s_i = n - r'_i$ $(i = 1, \dots, q)$, $\mu_1 = 0$ (in the sequel, we omit the index $i = 1$ of a singular matrix parameters), for example, any *eigenmatrix* $B_i = B - \mu_i I$. From (27), the Hamilton-Cayley Theorem, with use of prime factorization and with replacement of the scalar coefficients by the Viète sums, as in (5), we have

$$K_1(B,n) = \sum_{t=0}^{n} (-B)^{n-t} k(B,t) = (-B)^{s'} \sum_{t=0}^{r'} (-B)^{r'-t} k(B,t) =$$

$$= (-B)^{s'} K_1(B, r') = (-B)^{s'} \prod_{i=2}^{q} (\mu_i I - B)^{s'_i} = Z \tag{32}$$

This is the *annulling characteristic polynomial* in B.

From the other hand, each characteristic coefficient of order r' is nonzero, that is why

$$K_1(B, r') = \prod_{i=2}^{q} (\mu_i I - B)^{s'_i} \neq Z, \quad k(B, r') = \prod_{i=2}^{q} \mu_i^{s'_i} \neq 0. \tag{33}$$

The recurrent Souriau formula (25) in the interval $r' < t \leq r''$ gives us the *nilpotent matrix coefficients*

$$K_1(B, t) = (-B)^{t-r'} K_1(B, r') = -K_2(B, t) \neq Z. \tag{34}$$

Further, if $t = r'' + 1$, then

$$K_1(B, r''+1) = (-B)^{r''-r'+1} K_1(B, r') = (-B)^{r''-r'+1} \prod_{i=2}^{q} (\mu_i I - B)^{s'_i} =$$

$$= Z = (-B)^{r''-r'+1} \prod_{i=2}^{q} (\mu_i I - B)^{s_i^0} = (-B)^{s^0} \prod_{i=2}^{q} (-B_i)^{s_i^0}, \tag{35}$$

where each s_i^0 is the exponent of the eigenmatrix B_i in a minimal annulling polynomial, it is called the *annulling multiplicity* of μ_i. From (35) and condition (see (34))

$$(-B)^{r''-r'} \prod_{i=2}^{q} (\mu_i I - B)^{s_i^0} \neq Z.$$

We obtain the **main result** – *formulae for the annulling multiplicities* of $\mu_1 = 0$ and consequently of all μ_i of the eigenmatrices B_i in the minimal annulling polynomial:

$$\boxed{s^0 = r'' - r' + 1, \qquad s_i^0 = r_i'' - r_i' + 1.} \tag{36}$$

The annulling multiplicities satisfy the classic inequalities $1 \leq s_i^0 \leq s_i'$ [2, p. 24] due to $r_i' \leq r_i''$ and (32). Replace s_i^0 in the classic inequalities by their values (36), obtain the weak inequality $r_i'' \leq n-1$. Therefore the classic inequalities may be strengthened, the upper bound is more precise:

$$1 \leq s_i^0 \leq r_i - r_i' + 1 \leq s_i'. \tag{37}$$

Now we can see that expressing the unknown 2-nd rock in terms of the given s_i^0 from (36) can not lead to restriction $r'' \leq r$. That is why the 1-st and the 2-nd rocks are the primary parameters of a singular matrix, while the annulling multiplicity is the secondary notion.

The upper bound in (37) is attained when $r_i'' = r_i$, in that number if $r_i'' = n - 1 = r_i \geq r_i'$.

Find condition for attaining the lower bound in (37), i. e. for equality $r_i' = r_i''$. Take advantage of the classic Sylvester Inequality [16, p. 394]:

$$\min(r_1, r_2) \geq rank\ (C_1 C_2) \geq r_1 + r_2 - n.$$

If $k \geq 2$ matrices are multiplied (or a power of a matrix is analysed), their singularities are more suitable than the ranks. Then the following *two inequalities in general forms* are expressed in terms of its factors singularities briefly and do not depend on n:

$$\max\langle sing\ C_i\rangle \leq sing\ \prod_{i=1}^{k} C_i \leq n, \quad sing\ \prod_{i=1}^{k} C_i \leq \sum_{i=1}^{k} sing\ C_i, \tag{38}$$

$$sing\ C \leq sing\ C^h \leq n, \quad sing\ C^h \leq h \cdot sing\ C, \tag{39}$$

where h is an arbitrary positive integer.

The upper bounds in right inequalities of (38) are attained if the following two conditions do hold together:
(i) $\langle ker\ C_i\rangle \oplus \langle im\ C_i\rangle \equiv \langle \mathcal{A}^n\rangle$,
(ii) $\langle ker\ C_i\rangle \subset \langle\ im\ \prod_{j=i+1}^{k} C_j\rangle,\ i = 1, \ldots, k-1$.
They seem sufficiently clear and are useful in further considerations. In particular, if C_i are the eigenmatrices, then their powers pairly commute and conditions above are transformed into

$$\langle ker\ B_i^{h_i}\rangle \cap \langle ker\ B_j^{h_j}\rangle = \langle \mathbf{0}\rangle,\ i \neq j.$$

Then, due to (38) and conditions $(i), (ii)$, for all $h_i \geq s_i^0$ there holds

$$n = sing\left(\prod_{1\leq i\leq q,\ h_i\geq s_i^0} B_i^{h_i}\right) = sing\ Z = \sum_{1\leq i\leq q,\ h_i\geq s_i^0} sing\ B_i^{h_i}.$$

From the other hand, $rank\ B_i^{h_i} \geq r'_i$ (and this is equivalent to $sing\ B_i^{h_i} \leq s'_i$) as the algebraic multiplicity and the 1-st rock are invariant under powering a matrix. Consequently, due to $\sum_{i=1}^{q} s'_i = n$, we obtain

$$\left.\begin{array}{l} sing\ B_i^{h_i} < s'_i \text{ iff } h_i < s_i^0, \\ sing\ B_i^{h_i} = s'_i \text{ iff } h_i \geq s_i^0. \end{array}\right\} \tag{40}$$

The value $s = n - r$ is the geometric multiplicity. In particular, $sing\ B^{s^0} = s'$, $sing\ B_i^{s_i^0} = s'_i$. This fact and (39) lead to the following special inequalities:

$$\left.\begin{array}{l} s_i^0 s_i \geq s'_i \ \ (s_i^0 \leq s'_i \text{ and } s_i \leq s'_i), \\ s^0 s \geq s' \ \ (s^0 \leq s' \text{ and } s \leq s'). \end{array}\right\} \tag{41}$$

The set $\langle sing\ B_i^{h_i}\rangle$ as well as the set $\langle rank\ B_i^{h_i}\rangle$ determines [2, p. 143] the set of the Jordan subcells in the ultrainvariant $s'_i \times s'_i$-cell, and the critical exponent of the matrix in (40) determines the maximal size of the Jordan $s_i^0 \times s_i^0$-subcell.

If $s_i^0 = 1$ (it is equivalent to $r'_i = r''_i$), then, due to (41), $s_i = s'_i$. Conversely, if $s_i = s'_i$, then $r'_i = r''_i = r_i$. Thus, for lower boundaries of s_i^0 there holds:

$$\left.\begin{array}{l} s_i^0 = 1 \ \Leftrightarrow\ r'_i = r''_i \ \Leftrightarrow\ r'_i = r_i, \\ s^0 = 1 \ \Leftrightarrow\ r' = r'' \ \Leftrightarrow\ r' = r. \end{array}\right\} \tag{42}$$

For example, the following fact is well known:

$$s_i^0 = 1,\ i = 1, \ldots, q,\ \Leftrightarrow\ s_i = s_i',\ i = 1, \ldots, q\ \Leftrightarrow\ B \in \langle P \rangle.$$

The Jordan form J_B is used for inferring them [2, p. 143], however it immediately follows from (42), if to let $i = 1, \ldots, q$.

On the other hand, for upper boundaries of s_i^0 there holds:

$$s_i^0 = s_i'\ \Leftrightarrow\ s_i = 1\ \Leftrightarrow\ r_i'' = n - 1 = r_i. \tag{43}$$

They are determined by (41).

So, the theory of minimal annuling polynomial is exposed more completely, and this polynomial is expressed in explicit form due to results obtained in the previous section.

1.6 Null-prime and null-defective singular matrices

A singular matrix is called *null-prime* if its 1-st rock is equal to its rank. We shall use notation Bp for null-prime matrices if necessary.

Of the fact above follows: if B is null-prime, then B' is null-prime. Obviously, for the eigenspace corresponding to its eigenvalue zero holds $\langle ker\ Bp \rangle \equiv \langle ker\ (Bp)^h \rangle$. In this subspace, the matrix Bp behaves as a prime one. Indicate more widely the properties and definitions of Bp.

The following assertions are equivalent:

(i) *a square matrix B of rank r is null-prime,*

(ii) $r' = r''$,

(iii) $r' = r$,

(iv) $rank(B^2) = r$,

(v) $\langle ker\ B \rangle \cap \langle im\ B \rangle \equiv \langle \mathbf{0} \rangle$,

(vi) $\langle ker\ B \rangle \cup \langle im\ B \rangle \equiv \langle ker\ B \rangle \oplus \langle im\ B \rangle \equiv \langle \mathcal{A}^n \rangle$.

Due to (vi), any null-prime matrix possesses the characteristic affine projectors in the linear spaces.

A square matrix B is called *null-defective* if $r' < r$ (its 1-st rock $r' = rank\ B^{s^0}$ also is the minimal value of $rank\ B^h$). According to (35), for a null-defective matrix B there exists the characteristic nilpotent matrix

$$O_1 = \{K_1(B^{s^0}, r_B')/k(B^{s^0}, r_B')\}B,\ O_1^{s^0} = Z,\ [(I \pm O_1)^{s^0} - I]^{s^0} = Z, \tag{44}$$

where all the matrices commute with each other as polynomials in B (see in details in sect. 2.2).

The nilpotent matrix O_B in (21) is, in its turn, the sum of all the eigenmatrices $O_1, \ldots, O_q$. The parameters of the nilpotent matrix for a null-defective matrix B are the following:

$$r' = 0, \qquad r'' = s^0 - 1,$$

where s^0 is the nilpotency degree, and

$$s^0 - 1 = r'' \leq rank\ O_1 \leq n[r''/(r''+1)] = n[(s^0-1)/s^0] \leq n-1, \tag{45}$$

$$0 \leq rank\ O_1 \leq r, \qquad n - s^0(n-r) \leq rank\ O_1. \tag{46}$$

Inequalities (45), (46) follow from (39). More precise bounds for the parameters

$$(n-1) - (s_i' - s_i^0) \leq r_i \leq n-1, \tag{47}$$

$$s_i \leq s_i' - (s_i^0 - 1), \quad s_i^0 \leq s_i' - (s_i - 1) \tag{48}$$

follow from (37).

In matrix Jordan form (see [10, part 2]), the value $s_i^0 - 1 = r_i'' - r_i'$ is the maximal quantity of nonseparated units in the adjacent diagonal of the i-th ultrainvariant $s_i' \times s_i'$-cell. The total number of units in the cell is $s_i' - s_i = r_i - r_i'$. This gives the sense to estimations (47) and (48), and the notions of the 1-st and 2-nd rock.

Inequality (41) may be interpreted in terms of the Jordan form too, namely, by the following arguments. The adjacent diagonal of the matrix Jordan form contains, as well known, only units and zeros; moreover, k nonseparated units in it correspond to the Jordan subcell of size $k+1$. Among them there exists a subcell (may be not unique) of the maximal size s_i^0. Consider this $s_i^0 \times s_i^0$-subcell and add to the end of its array of units one zero element (outside the subcell). When s_i^0 is fixed, the total number of units in the adjacent diagonal takes the maximal value if its partition into segments is almost uniform: all the segments (but may be one) are of length s_i^0, and the last segment may be shorter, its length is equal to the nonzero remainder of division s_i' by s_i^0. Each segment ends with a zero, all other its elements are units. Therefore,

$$\min s_i = \lfloor s_i'/s_i^0 \rfloor$$

and the equality in (41) holds iff s_i'/s_i^0 is an integer. Inequalities (41) are equivalent to each of the following:

$$(n - r_i)(r_i'' - r_i') \geq r_i - r_i', \tag{49}$$

$$r_i' + [(s_i' - s_i)/s_i] \leq r_i'' \leq (n-1) - (s_i' - s_i^0)/s_i^0]. \tag{50}$$

Hence estimations (41), (49), (50) for r'' and s_i^0 are effective only under condition $r_i'' < r$. In this case,

$$s_i < s_i', \quad s_i^0 < s_i', \quad s_i' > 3, \quad s_i > 2, \quad s_i^0 > 1, \quad n > 3.$$

The parameter $r_i - r''_i$ is called *the i-th different* of a matrix. A defective matrix is called *null-different* if $r'' < r$. The maximal value of the different (particular and total) is $(\sqrt{n}-1)^2$, it is less than $n-3$. This follows from (49). The different is maximal if the integer n is a square. In this case,

$$r = n - \sqrt{n}, \quad r'' = \sqrt{n} - 1, \quad r' = 0, \quad q = 1.$$

Due to (49), the matrix B is *null-indifferent* in the following special cases:

$$\left.\begin{array}{ll} (i) & r_i = 1 \ (\Leftrightarrow r''_i = 1); \\ (ii) & r_i = 2 \ (\Leftrightarrow r''_i = 2); \\ (iii) & n \leq 3; \\ (iv) & s'_i \leq 3. \end{array}\right\} \tag{51}$$

Therefore, the different is zero if the dimension of the whole space or the dimension of the ultrainvariant space does not exceed 3. This may be useful for constructing the minimal annulling polynomial in terms of the ranks. Note the sense of condition (ii): units in the adjacent diagonals of the Jordan cells can not be separated by zeros.

A singular square matrix B is null-indifferent iff

$$rank\ B^h = rank\ B^{h-1} - 1, \qquad h = 2, 3, \ldots, s^0$$

$(rank\ B^{s^0} = r'$ *is minimal*).

Null-prime and null-defective matrices as well as prime and defective ones according to their definitions are pure affine notions. But they relate only to the eigenvalue zero of singular matrices, in particular, of the eigenmatrices $B_i = B - \mu_i I$. For the definition, it is not meaning, the matrix is real-valued or complex-valued one.

These notions are important especially in theory of eigenprojectors connected with given singular matrix B, and in its numerous applications. One of them is spectral decomposition of a matrix B up to its invariant and ultrainvariant subspaces for each eigenvalue μ_i, with reducing original matrix into the basic canonical form or only into the null-cell form (see in sect. 2.3).

Further, we shall often deal with matrices-multiplications of the types $B = A_1 A'_2$ and $B' = A_2 A'_1$, where A_1 and A_2 are $n \times m$-matrices set certain geometric objects in a n-dimensional affine or metric space. In the case, angular geometric relations between these objects in the space are determine the matrix-multiplication B as a null-normal one or a null-defective one.

It is clear that in the minimal polynomial of a prime matrix P, all the eigenmatrices $P_i = P - \mu_i I$ are null-normal ones, and all they have powers 1 in it. However in the minimal polynomial of a defective matrix B, some of its eigenmatrices $B_i = B - \mu_i I$ are null-defective ones, and they have powers $s_i^0 > 1$ in it. Then $B_i^{s_i^0}$ became by null-normal matrix with this minimal power.

1.7 The reduced form of characteristic coefficients

We conclude the chapter with evaluating all the characteristic coefficients of a given matrix B in so called *reduced form*, where the fraction numerator and denominator in (1) are polynomials in $\epsilon = -\mu$ of the least degree. This reduced form is obtained through dividing by the greatest common divisor the numerator and the denominator. The similar method for computing the minimal annulling polynomial of a matrix is well known – see, for example, in [2, p. 123].

So, dividing the numerator and the denominator of fraction (1) by their greatest common divisor leads to reducing the Hamilton–Cayley zero polynomial as well as all the characteristic coefficients, their connection formulae, and the Souriau algorithm. Reducing in (24) yields the reduced analogues of the scalar and matrix characteristic polynomials $k_B(\epsilon)$ and $K_B(\epsilon)$ from (1):

$$q_B(\epsilon)I = (B + \epsilon I)Q_B(\epsilon). \tag{52}$$

These reduced polynomials have also the reduced scalar and matrix characteristic coefficients $q(B,t)$ and $Q_1(B,t)$, where t is the order of these coefficients:

$$q_B(\epsilon) = \sum_{t=0}^{n^0} q(B,t)\epsilon^{n^0-t},$$

$$Q_B(\epsilon) = \sum_{t=0}^{n^0-1} Q_1(B,t)\epsilon^{n^0-t-1}.$$

As well as (24), formula (52) is valid also for the matrix parameter E, and in special case $E = -B$ it gives the matrix minimal annulling polynomial of $E = -B$ (scalar one depends on $\epsilon = -\mu$), i. e., the reduced Hamilton–Cayley Theorem and the reduced secular equation:

$$q_B(-B) = Q_1(B,n^0) =$$

$$= \sum_{t=0}^{n^0} q(B,t)(-B)^{n^0-t} = \prod_{i=1}^{q}(\mu_i I - B)^{s_i^0} = Z, \tag{53}$$

$$q_B(-\mu) = \sum_{t=0}^{n^0} q(B,t)(-\mu)^{n^0-t} = \prod_{i=1}^{q}(\mu_i - \mu)^{s_i^0} = 0. \tag{54}$$

Thus n^0 is the order of the minimal annulling polynomial (53). Reducing results in only those parts of (53), (54) that do not contain μ_i and s_i^0. The values μ_i and s_i^0 are determined by solving the secular equation in (54).

When these values are known, the reduced Viète theorem

$$q(B,t) = \sum_{(C^t_{n^0} \text{ terms })} \prod_{(t \text{ values})} \mu_i \qquad (q \le n^0 = \sum_{i=1}^{q} s_i^0 \le n) \tag{55}$$

follows from (53). This leads to reducing (25)–(29). In the reduced Souriau algorithm, the initial values are as usually, but further computations use the reduced trace etc. (see the algorithm in sect. 1.4):

$$Q_1(B,t) = I, \quad Q_2(B,0) = Z, \quad Q_2(B,1) = B,$$

and $q(B,1) = \sum_{i=1}^{q} s_i^0 \mu_i$ is the matrix B reduced trace. The reduced determinant is

$$q(B,n^0) = \prod_{i=1}^{q} \mu_i^{s_i^0}.$$

The inverse nonsingular matrix is

$$B^{-1} = Q_1(B,n^0-1)/q(B,n^0).$$

Note that quantity of the eigenvalues decreases up to n^0.

The highest coefficients of the eigenmatrices $B_i = B - \mu_i I$ as functions of μ_i have the following reduced form:

$$Q_1(B_i,r_i^0) = \prod_{j=1}^{q} (\mu_j I - B)^{s_j^0}, \quad q(B_i,r_i^0) = \prod_{j=1}^{q} (\mu_j - \mu_i)^{s_j^0}, \ j \ne i, \tag{56}$$

where $r_i^0 = n^0 - s_i^0$ is the reduced 1-st rock. The second rock is equal to $n^0 - 1$ after reducing. Particular reducing (of the fixed eigenvalue μ_i quantity) is equal to $s_i' - s_i^0 = (n-1) - r_i''$, the total reducing (for all μ_i) is $n - n^0$.

The sum of the basic particular parameters satisfies inequalities

$$nq - 1 = \sum_{i=1}^{q} r_i' \le \sum_{i=1}^{q} r_i'' \le \sum_{i=1}^{q} r_i \le nq - q.$$

If the matrix is prime ($B \in \langle P \rangle$), then

$$n^0 = q, \quad s_i^0 = 1, \quad q(P^h,1) = \sum_{i=1}^{q} \mu_i^h, \quad q(P^h,n^0) = q^n(P,n^0) = \left(\prod_{i=1}^{q} \mu_i \right)^n,$$

and the coefficients for its eigenmatrices are

$$Q_1(P_i,n^0-1) = \prod_{j=1}^{q} (\mu_j I - P), \quad q(P_i,n^0-1) = \prod_{j=1}^{q} (\mu_j - \mu_i), j \ne i. \tag{57}$$

Note, the general spectral representation of a matrix (see in sect. 2.2) may apply the minimal annulling polynomial and, perhaps, other types of annulling polynomials, for example, these:

$$\prod_{j=1}^{q}(\mu_j I - B)^{\max s_j^0} = \prod_{j=1}^{q}(-B_j)^{\max s_j^0} = Z, \tag{58}$$

$$\prod_{j=1}^{q}(\mu_j^I - B)^{\max s'_j} = \prod_{j=1}^{q}(-B_j)^{\max s'_j} = Z. \tag{59}$$

Here the matrix $(-B_j)$ powers are null-prime matrices too.

These reduced forms of exact matrices scalar and matrix characteristic coefficients are important, of course, from the theoretical point of view. They demonstrate in some extent similarity between number theory and matrix algebra. In the both case, we deals with cancellation of greatest common divisor, but here it is as scalar and as matrix polynomials.

Contrary, from the practical point of view, the valuable results of this chapter are the general inequality of means and the fundamental inequality for basic singularity parameters of singular matrices with their dependence on structure of the scalar and matrix characteristic coefficients. Thus the chapter completely describes structure of all the characteristic coefficients of a square matrix, including the coefficients of the highest order of a singular matrix (note that all the eigenmatrices of an arbitrary square matrix $B_i = B - \mu_i I$ are always singular ones). Relationships between the singularity parameters will be used in the following theoretical considerations.

Other important results of the chapter are new opportunities for inferring explicit formulae expressing eigenprojectors and modal matrices in terms of the scalar and matrix characteristic coefficients. This advantage is used widely in development of tensor trigonometry in further divisions of the book.

Chapter 2

Affine (oblique) and orthogonal eigenprojectors

2.1 Affine (oblique) eigenprojectors and quasi-inverse matrix

Let $\langle \mathcal{A}^n \rangle$ be an affine n-dimensional space. Suppose Bp is a null-prime matrix of rank r, then $k(Bp, r) \neq 0$. Formula (26) is transforming into

$$K_1(Bp, r)/k(Bp, r) + K_2(Bp, r)/k(Bp, r) = \overrightarrow{Bp} + \overleftarrow{Bp} = I. \tag{60}$$

Further $\overrightarrow{Bp}$ and $\overleftarrow{Bp}$ stand for the so-called *affine eigenprojectors* of Bp. These projectors are also idempotent matrices (in general case, they are non-symmetric). In the Euclidean space they are also the *oblique eigenprojectors* in the metric sense. We claim that in the affine space $\overleftarrow{Bp}$ is a projector into the image $\langle im\ Bp \rangle$ parallel to the kernel $\langle ker\ Bp \rangle$, and $\overrightarrow{Bp}$ is a projector into $\langle ker\ Bp \rangle$ parallel to $\langle im\ Bp \rangle$. Indeed,

$$K_2(Bp, r) = BpK_1(Bp, r-1) = K_1(Bp, r-1)Bp;$$

$$\overrightarrow{Bp} + \overleftarrow{Bp} = I, \quad \overrightarrow{Bp} \cdot \overleftarrow{Bp} = \overleftarrow{Bp} \cdot \overrightarrow{Bp} = Z;$$

$$(\overrightarrow{Bp})^2 = \overrightarrow{Bp}(I - \overleftarrow{Bp}) = \overrightarrow{Bp}, \quad (\overleftarrow{Bp})^2 = \overleftarrow{Bp}(I - \overrightarrow{Bp}) = \overleftarrow{Bp};$$

$$\langle ker\ Bp \rangle \oplus \langle im\ Bp \rangle = \langle \mathcal{A}^n \rangle, \quad \mathbf{x} = \overrightarrow{Bp}\mathbf{x} + \overleftarrow{Bp}\mathbf{x} = \overrightarrow{\mathbf{x}} + \overleftarrow{\mathbf{x}}.$$

Any element $\mathbf{x}$ is uniquely decomposed into the sum of its projections in $\langle \mathcal{A}^n \rangle$ as above. Therefore,

$$\overrightarrow{Bp} = K_1(Bp, r)/k(Bp, r), \tag{61}$$

$$\overleftarrow{Bp} = K_2(Bp, r)/k(Bp, r) =$$

$$= BpK_1(Bp, r-1)/k(Bp, r) = K_1(Bp, r-1)Bp/k(Bp, r). \tag{62}$$

The matrix Bp and both its eigenprojectors commute with each another as polynomials in Bp (compare with formula (27)). In particular, for a scalar we get:

$$\overrightarrow{a} = 0, \quad \overleftarrow{a} = 1$$

and in some other trivial cases,

$$\overrightarrow{Z} = I, \quad \overrightarrow{I} = Z;$$

$$\left.\begin{array}{l} \langle im\ K_1(Bp, r) \rangle \equiv \langle ker\ K_2(Bp, r) \rangle \equiv \langle ker\ Bp \rangle, \\ \langle ker\ K_1(Bp, r) \rangle \equiv \langle im\ K_2(Bp, r) \rangle \equiv \langle im\ Bp \rangle; \end{array}\right\} \tag{63}$$

$$rank\ K_1(Bp, r) = sing\ Bp, \quad rank\ K_2(Bp, r) = rank\ Bp; \tag{64}$$

$$\overrightarrow{(Bp')} = (\overrightarrow{Bp})',\ \overleftarrow{(Bp')} = (\overleftarrow{Bp})',\ \overrightarrow{\overrightarrow{Bp}} = \overleftarrow{\overleftarrow{Bp}} = \overleftarrow{Bp},\ \overleftarrow{\overrightarrow{Bp}} = \overrightarrow{\overleftarrow{Bp}} = \overrightarrow{Bp}; \tag{65}$$

$$k(\overrightarrow{Bp}, t) = C^t_{n-r}, \quad k(\overleftarrow{Bp}, t) = C^t_r. \tag{66}$$

Then, for singular matrices B and Bp ($r = r'$), we have

$$k(B^h, r') = k^h(B, r') \neq 0,\ k(Bp^h, r) = k^h(Bp, r) \neq 0; \tag{67}$$

$$K_j((Bp)^h, r) = K^h_j(Bp, r) = k^{h-1}(Bp, r)K_j(Bp, r),\ j = 1, 2. \tag{68}$$

In an affine space, the *affine quasi-inverse matrix* for a matrix Bp is the following:

$$Bp^- = \overleftarrow{Bp}[K_1(Bp, r-1)/k(Bp, r)] = [K_1(Bp, r-1)/k(Bp, r)]\overleftarrow{Bp}$$

$$= Bp[K_1(Bp, r-1)/k(Bp, r)]^2 = [K_1(Bp, r-1)/k(Bp, r)]^2 Bp. \tag{69}$$

It commutes with Bp and in the subspace $\langle im\ Bp\rangle$ it behaves as an usual inverse matrix, in $\langle ker\ Bp\rangle$ it plays the role of the zero matrix. It is uniquely determined by equations

$$Bp^- Bp = BpBp^- = \overleftarrow{Bp}, \quad Bp^- = \overleftarrow{Bp}Bp^- = Bp^-\overleftarrow{Bp}. \tag{70}$$

The following formulae hold:

$$rank\ Bp^- = rank\ Bp;$$

$$\langle im\ Bp^-\rangle \equiv \langle im\ Bp\rangle, \quad \langle ker\ Bp^-\rangle \equiv \langle ker\ Bp\rangle;$$

$$BpBp^-Bp = Bp; \quad Bp^-BpBp^- = Bp^-;$$

$$(Bp^-)^- = Bp; \quad (Bp^h)^- = (Bp^-)^h; \quad (Bp')^- = (Bp^-)'.$$

Moreover,

$$B^- = B^{-1} \Leftrightarrow det\ B \neq 0.$$

Due to (1), (61), (62), and (69), the affine eigenprojectors and the quasi-inverse matrix are represented as limits

$$\overrightarrow{Bp} = \lim_{\epsilon\to 0}\,[\epsilon(Bp + \epsilon I)^{-1}] = \lim_{N\to\infty}(NBp + I)^{-1}, \tag{71}$$

$$\overleftarrow{Bp} = \lim_{\epsilon\to 0}\,[Bp(Bp + \epsilon I)^{-1}] = \lim_{N\to\infty}[NBp(NBp + I)^{-1}], \tag{72}$$

$$Bp^- = \lim_{\epsilon\to 0}\,[Bp(Bp + \epsilon I)^{-2}] = \lim_{N\to\infty}[(N^2Bp(NBp + I)^{-2}] \tag{73}$$

$$(\overrightarrow{Bp} + \overleftarrow{Bp} = I, \quad Bp^-Bp = BpBp^- = \overleftarrow{Bp}, \quad N = 1/\epsilon).$$

These limit formulae have most common *affine form*. They are gotten here by the algebraic manner using a resolvent of Bp (see also in sect. 3.4).

2.2 Spectral representation of an $n\times n$-matrix and its basic canonical form

In all ultrainvariant subspaces (their sums are direct), the affine eigenprojectors (61) of a prime matrix P may be represented, due to (57), by two manners as follows:

$$\overrightarrow{P_i} = \frac{K_1(P_i, r_i)}{k(P_i, r_i)} = \frac{Q_1(P_i, r^0)}{q(P_i, r^0)} = \prod_{1\le j\le q,\ j\ne i} \frac{\mu_j I - P}{\mu_j - \mu_i}, \tag{74}$$

where $r^0 = n^0 - 1 = q - 1$ (see the last manner also, for example, in [2, p. 156]). The affine projectors of a defective matrix B are represented due to (61), (33), (56), and (58)–(60), by two different manners as follows:

$$\overrightarrow{Bp_{(i)}} = \frac{K_1(B_i, r'_i)}{k(B_i, r'_i)} = \frac{Q_1(B_i, r_i^0)}{q(B_i, r_i^0)} =$$

$$= \prod_{1\le j\le q,\ j\ne i} \frac{(\mu_j I - B)^{s_j^0}}{(\mu_j - \mu_i)^{s_j^0}} = \prod_{1\le j\le q,\ j\ne i} \frac{(\mu_j I - B)^h}{(\mu_j - \mu_i)^h} = \overrightarrow{(B_i^h)}, \tag{75}$$

where $Bp_{(i)} = B_i^{s_i^0}$, $h \ge \max s_i^0$ (see the last manner, for example, in [12, p. 128–143]).

Note, that eigenmatrices $P_i = P - \mu_i I$ and the power matrices

$$B^h, h \ge s^0, \quad B_i^{h_i},\ h_i \ge s_i^0,$$

are trivial special cases of null-prime singular matrices Bp.

Spectral representation of a matrix B up to its ultrainvariant subspaces determines decomposition of B into the unique sum of two its characteristic matrices – prime one and nilpotent one (see before (21) and (44)):

$$B = B\sum_{i=1}^{q} \overrightarrow{Bp_{(i)}} = \sum_{i=1}^{q} \mu_i \overrightarrow{Bp_{(i)}} + \sum_{i=1}^{q} B_i \overrightarrow{Bp_{(i)}}$$

$$= \sum_{i=1}^{q} P_i + \sum_{i=1}^{q} O_i = P_B + O_B. \tag{76}$$

Note, $O_B^h = Z$ if $h \ge \max s_i^0$. This may be interpreted by the Jordan form.

In order to construct the canonical q-block-diagonal form of the matrix [2, p. 130], the modal matrix of transformation may be evaluated with use of the following coefficients (proportional to eigenprojectors) correspondingly, due to (33) and (56):
$K_1(B_i, r'_i) = \prod_{1\le j\le q,\ j\ne i} (\mu_j I - B)^{s'_j}$, $Q_1(B_i, r_i^0) = \prod_{1\le j\le q,\ j\ne i} (\mu_j I - B)^{s_j^0}$.

Then

$$\left.\begin{aligned} \langle im\ K_1(B_i, r'_i)\rangle \equiv \langle im\ Q_1(B_i, r_i^0)\rangle \equiv \langle ker\ B_i^{s_i^0}\rangle, \\ \langle ker\ K_1(B_i, r'_i)\rangle \equiv \langle ker\ Q_1(B_i, r_i^0)\rangle \equiv \langle im\ B_i^{s_i^0}\rangle. \end{aligned}\right\} \tag{77}$$

For a prime matrix, the coefficients are simplified according to $r'_i = r_i$, or due to (57).

All the coefficients are null-prime matrices. However, these matrices have nonzero scalar coefficients of the highest order, that is why they contain a basis minor. This minor is the intersection of the basis $s'_i \times n$-submatrix of the rows and the basis $n \times s'_i$-submatrix of the columns. Therefore the total covariant and contravariant modal matrices consist of all the column submatrices and, respectively, of all the row ones $(i = 1, \ldots, q)$:

$$V_{col}^{-1} B V_{col} = C_\mu(B), \qquad \tilde{E}_1 = V_{col}\tilde{E}, \tag{78}$$

$$V_{lig} B V_{lig}^{-1} = C_\mu(B), \qquad \tilde{E}_2 = V_{lig}^{-1}\tilde{E}, \tag{79}$$

$$(V'_{lig})^{-1} B' V'_{lig} = C'_\mu(B), \quad \tilde{E}_3 = V'_{lig}\tilde{E}, \tag{80}$$

$$(V^*_{lig})^{-1} B^* V^*_{lig} = C^*_\mu(B), \quad \tilde{E}_4 = V^*_{lig}\tilde{E}, \tag{81}$$

where C_μ is the q-block-diagonal form of B with respect to its eigenvalues $\mu_1, \ldots, \mu_q$, $\tilde{E}$ and $\tilde{E}_k$, $k = 1, \ldots, 4$, are the original basis and one of the canonical form. Each ultrainvariant space contains non invariant subspaces

$$\left.\begin{array}{c} \langle ker\ B_i^{s_i^0}\rangle \supset \langle im\ O_i^1\rangle \supset \cdots \supset \langle im\ O_i^{s_i^0-1}\rangle, \\ \langle ker\ B_i^{s_i^0}\rangle \supset \langle ker\ O_i^{s_i^0-1}\rangle \supset \cdots \supset \langle ker\ O_i^1\rangle, \end{array}\right\} \tag{82}$$

$$\left.\begin{array}{c} \langle im\ O_i^t\rangle \equiv \langle im\ K_1(B_i, r'_i)B_i^t\rangle \equiv \langle im\ Q_1(B_i, r_i^0)B_i^t\rangle, \\ \langle ker\ O_i^t\rangle \equiv \langle im\ B_i^t\rangle, \quad t = 1, \ldots, s_i^0 - 1. \end{array}\right\} \tag{83}$$

Take a certain ultrainvariant cell of projection (76) and subtract its prime diagonal part. The result is its nilpotent cells. It may be further transformed into subcells (82) till the final elementary subcells. After this the common process may be continued till the Jordan nilpotent form.

Formulae (78), (79) determine the various modal matrices for the prime matrix $P_B = \sum_{i=1}^q P_i$ in (76). The general formula of the covariant modal matrix is

$$\langle V_{col}\rangle \equiv V_{col}\langle C_q\rangle, \qquad V_{lig}^{-1} \in \langle V_{col}\rangle. \tag{84}$$

Here C_q is an arbitrary nonsingular cell matrix consisting of nonsingular blocks $c_1, \ldots, c_q$. The quantity of nilpotent Jordan $t \times t$-subcells in the i-th cell of the basic canonical form for the matrix B are

$$(rank\ O_i^t - rank\ O_i^{t+1}) - (rank\ O_i^{t+1} - rank\ O_i^{t+2}),$$

see, for example, [10, part 2, p. 95]. General spectral representation of the matrix B analytical functions may be computed with use of the Lagrange and Hermite interpolating polynomials with so called the component matrices [2, p. 155-159]:

$$B_{(ik)} = \frac{B_i^{k-1}}{(k-1)!}\overrightarrow{Bp_{(i)}}, \quad \langle im\ B_{(ik)}\rangle \equiv \langle im\ O_i^{k-1}\rangle, \quad k = 1, \ldots, s_i^0. \tag{85}$$

Substitute here $\overrightarrow{Bp_{(i)}}$ for (75), the result is the form depending only on the original matrix B.

2.3 Transforming a null-prime matrix in the null-cell canonical form

Let Bp be a null-prime $n \times n$-matrix and $rank\ Bp = r$. Further, define the canonical *null-cell (two-block-diagonal) form* of the matrix Bp (in certain new bases) as the modal transformed following $n \times n$-matrix Bc:

$$Bp\ \to\ Bc = \begin{bmatrix} Z_1 & Z \\ Z & B_1 \end{bmatrix}.$$

Here B_1 is a nonsingular $r \times r$-matrix ($det\ B_1 \neq 0$) and Z_1 is the zero $s \times s$-matrix, $s = n - r$ is the geometric and algebraic multiplicity of the eigenvalue 0 for Bp. Find the modal transformation of Bp into Bc. The high coefficients $K_1(Bp, r)$ and $K_2(Bp, r)$, where $r = rank\ Bp$, are proportional to the eigenprojectors (61) and (62), what are necessary here for the searched modal transformation. (But for their evaluating the eigenvalues of Bp are not necessary as for the full spectrum (76)). These coefficients are null-prime matrices, and thus they contain basis diagonal minors determining two basis $n \times s$- and $n \times r$-submatrices of columns. These submatrices generate the modal matrix of the base transformation:

$$V_{col}^{-1} Bp V_{col} = Bc, \quad \tilde{E}_1 = V_{col}\tilde{E}, \quad \langle V_{col}\rangle \equiv V_{col}\langle C_2\rangle. \tag{86}$$

Here C_2 is a two-cell analog of C_q from (84). *The transformation is found.*

Suppose we are given with two null-prime matrices Bp_1 and Bp_2 of the same order such that

$$\langle im\ Bp_1\rangle \equiv \langle im\ Bp_2\rangle,\ \langle im\ Bp_1'\rangle \equiv \langle im\ Bp_2'\rangle\ \ (\overrightarrow{Bp_1} = \overrightarrow{Bp_2},\ \overleftarrow{Bp_1} = \overleftarrow{Bp_2}).$$

Then, due to (86), we obtain

$$K_j(Bp_1 Bp_2, r) = K_j(Bp_2 Bp_1, r) = K_j(Bp_1, r) K_j(Bp_2, r),\ j = 1, 2,$$

$$k(Bp_1 Bp_2, r) = k(Bp_2 Bp_1, r) = k(Bp_1, r) k(Bp_2, r). \tag{87}$$

The formula generalize the well-known one for determinants of matrices multiplications

$$det\ (B_1 B_2) = det\ (B_2 B_1) = det\ B_1 \cdot det\ B_2.$$

One else simplest form for a null-prime matrix consists of zero $n \times (n-r)$-matrix and $n \times r$-matrix of the basis columns:

$$Bp = \left[\ Z_2 \mid A_2\ \right].$$

It may be also useful.

2.4 Null-normal singular matrices

There is an one-to-one correspondence between the pair of eigenprojectors $(\overrightarrow{Bp}, \overleftarrow{Bp})$ and the pair $(\langle im\ Bp\rangle, \langle ker\ Bp\rangle)$ of linear subspaces in an affine space $\langle \mathcal{A}^n\rangle$ with a certain base. Suppose this space is real. Consider the set of real so-called *null-normal* matrices $\langle Bm\rangle$ satisfying condition

$$\overrightarrow{Bm} = \overrightarrow{Bm'} = (\overrightarrow{Bm})' \Leftrightarrow \overleftarrow{Bm} = \overleftarrow{Bm'} = (\overleftarrow{Bm})'. \tag{88}$$

Geometrically, this means that

$$\langle ker\ Bm\rangle \equiv \langle ker\ Bm'\rangle \Leftrightarrow \langle im\ Bm\rangle \equiv \langle im\ Bm'\rangle. \tag{89}$$

The sum of $\langle im\ Bm\rangle$ and $\langle ker\ Bm\rangle$ in $\langle \mathcal{A}^n\rangle$ is direct as $k(Bm, r) \neq 0$. In the Euclidean space $\langle \mathcal{E}^n\rangle$ with an orthonormal base, we have

$$\left.\begin{array}{l}\langle ker\ Bm'\rangle \equiv \langle im\ Bm\rangle^{\perp} \equiv \langle ker\ Bm\rangle, \\ \langle im\ Bm'\rangle \equiv \langle ker\ Bm\rangle^{\perp} \equiv \langle im\ Bm\rangle;\end{array}\right\} \Leftrightarrow \tag{90}$$

$$\Leftrightarrow \langle im\ Bm\rangle \boxplus \langle ker\ Bm\rangle \equiv \langle \mathcal{E}^n\rangle.$$

This is the special geometric sense of matrices Bm: *In a real space $\langle \mathcal{E}^n\rangle$ with a fixed orthonormal base the characteristic eigenprojectors of a null-prime matrix Bp are symmetric iff its subspaces $\langle im\ Bp\rangle$ and $\langle ker\ Bp\rangle$ form the spherically orthogonal direct sum*, what is specially denoted above in (90) (i. e., iff they are orthocomplements of each to another in $\langle \mathcal{E}^n\rangle$.)

In the eigenspace corresponding to the eigenvalue $\mu = 0$, the matrix Bm is similar to a normal matrix. That is why it is called *null-normal*. In the Euclidean space its eigenprojectors are orthogonal. Special cases of null-normal matrices are normal, symmetric, skew-symmetric, and nonsingular ones. The following equivalences do hold:

$$\begin{array}{rclcrcl}\overrightarrow{Bm} & = & \overrightarrow{Bm'} = K_1(Bm, r)/k(Bm, r) & \Leftrightarrow & K_1(Bm, r) & = & K_1'(Bm, r) \quad (91)\\ & \Updownarrow & & & & \Updownarrow & \\ \overleftarrow{Bm} & = & \overleftarrow{Bm'} = K_2(Bm, r)/k(Bm, r) & \Leftrightarrow & K_2(Bm, r) & = & K_2'(Bm, r). \quad (92)\end{array}$$

In $\langle \mathcal{E}^n\rangle$, $\overrightarrow{Bm}$ and $\overleftarrow{Bm}$ project into $\langle ker\ Bm\rangle$ and respectively $\langle im\ Bm\rangle$ by the orthogonal way, and $\overrightarrow{Bm} \perp \overleftarrow{Bm}$.

The following conditions are equivalent (see sect. 2.1):
(i) *all the eigenmatrices B_i are real and null-prime;*
(ii) *all these matrices have the real affine projectors $\overrightarrow{B_i}$ and $\overleftarrow{B_i}$;*
(iii) *the matrix B is real and prime, and all its eigenvalues are real numbers.*

A real normal matrix $B = M$ may be transformed into diagonal real one by a real orthogonal modal matrix iff the matrix M is symmetric ($M = S$).

For any symmetric matrix S, the kernel and the image of each its eigenmatrix S_i are the orthogonal complements of each other, and kernels form the direct orthogonal sum. Therefore, *all the eigenmatrices of a real matrix B are real and null-normal iff B is real and symmetric.* In particular, null-normal matrices B and B' of rank $n-1$ have the common eigenvector $\langle ker\ B\rangle \equiv \langle ker\ B'\rangle$ iff $B^V = (B^V)'$.

Take a null-normal matrix Bm and apply the Gram–Schmidt orthogonalization algorithm to columns of the two blocks of the matrix $V_{col} = V'_{lig}$ in (86) separately. The result is the orthogonal modal matrix for constructing the null-cell canonical form (86), i. e. congruent modal transformation:

$$Bc = R'_{col} Bm R_{col} \tag{93}$$

($\langle R_{col}\rangle \equiv R_{col}\langle R_2\rangle$, but $\langle V_{col}\rangle \equiv R_{col}\langle C_2\rangle$, see (86)). Structure of R_2 here is similar to C_2 in (86). If the original base is, for example, Cartesian, then the new orthonormal base is expressed in terms of the columns of the modal matrix $\{R_{col}\} = \{R'_{lig}\}$. And orientation of the base is changed under multiplying R_{col} by the alternating unity matrix on the right for its restoring. The modal orthogonal matrix R_{col} for constructing the diagonal form of a symmetric matrix S is computed by the way similar to (78). If all the eigenvalues of S are distinct, then n its unity length eigenvectors determined by $\langle ker\ S_i\rangle$ form the desired matrix R_{col}. If some of them are degenerative (under condition $s_i > 1$), then the Gram–Schmidt orthogonalization is applied.

The following *examples of null-normal matrices* are used in the sequel. These matrices are generated by the special $n \times m$-matrix A ($n \neq m$):

$$Bm_1 = A_1 A'_2, \quad Bm'_1 = A_2 A'_1 \tag{94}$$

$$(\langle im\ A_1\rangle \equiv \langle im\ A_2\rangle, \quad rank\ A_1 = rank\ A_2 = m < n),$$

$$Bm_2 = A'_1 A_2, \quad Bm'_2 = A'_2 A_1 \tag{95}$$

$$(\langle ker\ A_1\rangle \equiv \langle ker\ A_2\rangle, \quad rank\ A_1 = rank\ A_2 = n < m).$$

Note some other properties of all null-normal matrices.

$$\left.\begin{array}{c} \overrightarrow{Bm'Bm} = \overrightarrow{BmBm'} = \overrightarrow{Bm}, \quad \overleftarrow{Bm'Bm} = \overleftarrow{BmBm'} = \overleftarrow{Bm}, \\ \langle ker\ Bm'Bm\rangle \equiv \langle ker\ BmBm'\rangle \equiv \langle ker\ Bm\rangle, \\ \langle im\ Bm'Bm\rangle \equiv \langle im\ BmBm'\rangle \equiv \langle im\ Bm\rangle. \end{array}\right\} \tag{96}$$

Apply (87) to null-normal matrices Bm and Bm', obtain

$$\left.\begin{array}{c} K_1(BmBm', r) = K_1(Bm'Bm, r) = K_1^2(Bm, r), \\ K_2(BmBm', r) = K_2(Bm'Bm, r) = K_2^2(Bm, r), \\ k(BmBm', r) = k(Bm'Bm, r) \quad = k^2(Bm, r). \end{array}\right\} \tag{97}$$

Formula (97) generalizes the well-known formula for determinants

$$det(BB') = det(B'B) = det^2 B.$$

(Singular matrices M and S are also the special cases of null-normal ones.)

2.5 Spherically orthogonal eigenprojectors and quasi-inverse matrices

In the previous section, we introduced the orthogonal eigenprojectors in addition to oblique ones. They were defined for null-normal matrices due to *spherical orthogonality* (90) of eigensubspaces in $\langle\mathcal{E}^n\rangle$. This property takes place only in $\langle\mathcal{E}^n\rangle$ and corresponds to to right tensor spherical angles (see in Ch. 5) between subspaces or lineors.

Let A be an real valued $n\times m$-matrix of rank r. Then $AA', A'A \in \langle Bm\rangle$ and their rank is equal to r. According to (91) and (92) we get

$$\overrightarrow{AA'} = K_1(AA',r)/k(AA',r), \quad \overrightarrow{A'A} = K_1(A'A,r)/k(A'A,r), \tag{98}$$

$$\left.\begin{array}{c} \overleftarrow{AA'} = K_2(AA',r)/k(AA',r) = AA^+, \\ \overleftarrow{A'A} = K_2(A'A,r)/k(A'A,r) = A^+A, \\ \{k(AA',t)\} = \{k(A'A,t)\}, \end{array}\right\} \tag{99}$$

$\overrightarrow{AA'}$ is the orthogonal projector onto $\langle ker\ A'\rangle$, $\overrightarrow{\mathbf{aa}'} = I - \mathbf{aa}'/\mathbf{a}'\mathbf{a}$,
$\overleftarrow{AA'}$ is the orthogonal projector onto $\langle im\ A\rangle \equiv \langle ker\ A'\rangle^{\perp}$, $\overleftarrow{\mathbf{aa}'} = \mathbf{aa}'/\mathbf{a}'\mathbf{a}$,
A^+ is the classical quasi-inverse Moor–Penrose $m\times n$-matrix [23–25], $rank\ A^+ = r$.
For these two complementary subspaces of $\langle\mathcal{A}^n\rangle$ we obviously get

$$\overleftarrow{AA'} + \overrightarrow{AA'} = I, \qquad (\ \overleftarrow{BB'} + \overrightarrow{BB'} = I\ ), \tag{100}$$

what are equivalent to $\langle im\ A\rangle \oplus \langle ker\ A'\rangle \equiv \langle\mathcal{A}^n\rangle$, $(\langle im\ B\rangle \oplus \langle ker\ B'\rangle \equiv \langle\mathcal{A}^n\rangle)$!

In $\langle\mathcal{E}^n\rangle$ the subspaces are *orthogonally complementary.* According to (99) the matrix A^+ satisfies the two Penrose equations, which determinate it independently:

$$A^+AA^+ = A^+, \qquad AA^+A = A.$$

From the latter and (62) we have

$$A^+ = A'\cdot\frac{K_1(AA',r-1)}{k(AA',r)} = \frac{K_1(A'A,r-1)}{k(A'A,r)}\cdot A'. \tag{101}$$

This is Decell's formula inferred in [26] from the Souriau algorithm. The equalities can be checked by representing the matrix coefficients by polynomials (27). So, in particular, $\{\mathbf{a}\}^+ = \mathbf{a}'/(\mathbf{a}'\mathbf{a})$. The matrix A^+ behaves as the inverse matrix in $\langle im\ A\rangle$ and as the zero one in $\langle ker\ A'\rangle$ with respect to multiplication from the left:

$$A^+C = A^+[(\overleftarrow{AA'} + \overrightarrow{AA'})C] = A^+(\overleftarrow{AA'}C). \tag{102}$$

With respect to multiplication from the right, it plays the role of the inverse matrix in $\langle im\ A'\rangle$ and the zero matrix in $\langle ker\ A\rangle$:

$$CA^+ = [C(\overrightarrow{A'A} + \overleftarrow{A'A})]A^+ = (C\overleftarrow{A'A})A^+. \tag{103}$$

In particular, a matrix B commutes with B^+ exactly in $\langle im\ B\rangle \cap \langle im\ B'\rangle$. That is why the following equivalences hold for the matrix B^- from (69) (see sect. 2.1):

$$B^- = B^+ \Leftrightarrow B \in \langle Bm\rangle \Leftrightarrow B^+B = BB^+. \tag{104}$$

In the Euclidean space with a certain orthonormal base, a quasi-inverse orthogonal matrix has the following geometric sense: its Frobenius norm (the matrix norm of the 1-st order, see sect. 9.1) is minimal among all quasi-inverse matrices determined by the Penrose equation $AXA = A$, i. e., this matrix is the normal solution of this equation from the left and from the right [23, 25] (see also below). Moreover, this matrix gives the normal solutions (with the minimal Frobenius norm) of the left, right, and mixed general linear equations

$$A_1(n \times m)X = A(n \times t) \ \Rightarrow \overset{\bullet}{X}\ (m \times t) = A_1^+A, \tag{105}$$

$$YA_2(n \times m) = A(t \times m) \ \Rightarrow \overset{\bullet}{Y}\ (t \times n) = AA_2^+, \tag{106}$$

$$\left.\begin{array}{c} A_1(n_1 \times m_1)XA_2(n_2 \times m_2) = A(n_1 \times m_2) \\ \Rightarrow \overset{\bullet}{X}\ (m_1 \times n_2) = A_1^+AA_2^+. \end{array}\right\} \tag{107}$$

Equations *residuals* for full solutions have the minimal Frobenius norm too:

$$\left.\begin{array}{r} \overset{\bullet}{\Delta}_1 = -\overrightarrow{A_1A_1'}A, \\ \overset{\bullet}{\Delta}_1 = Z \ \Leftrightarrow \ A \in \langle \overleftarrow{A_1A_1'}\mathcal{E}^{n\times t}\rangle \equiv \langle KER_R\overrightarrow{A_1A_1'}\rangle, \end{array}\right\} \tag{108}$$

$$\left.\begin{array}{r} \overset{\bullet}{\Delta}_2 = -A\overrightarrow{A_2'A_2}, \\ \overset{\bullet}{\Delta}_2 = Z \ \Leftrightarrow \ A \in \langle \mathcal{E}^{t\times m}\overleftarrow{A_2'A_2}\rangle \equiv \langle KER_L\overrightarrow{A_2'A_2}\rangle, \end{array}\right\} \tag{109}$$

$$\left.\begin{array}{r} \overset{\bullet}{\Delta} = -\overrightarrow{A_1A_1'}A - A\overrightarrow{A_2'A_2} + \overrightarrow{A_1A_1'}A\overrightarrow{A_2'A_2}, \qquad \overset{\bullet}{\Delta} = Z \ \Leftrightarrow \\ A \in \langle \overleftarrow{A_1A_1'}\mathcal{E}^{n_1\times m_2}\overleftarrow{A_2'A_2}\rangle \equiv \langle KER_R\overrightarrow{A_1A_1'} \cap KER_L\overrightarrow{A_2'A_2}\rangle. \end{array}\right\} \tag{110}$$

Intersection of the set of all left quasi-inverse matrices and the set of all right ones determined by (99) consists of the unique element A^+ [23, 27]:

$$\langle A_R^-\rangle \equiv A^+ \oplus \langle \overrightarrow{A'A}\mathcal{E}^{m\times n}\overleftarrow{AA'}\rangle \tag{111}$$

(all these matrices produce orthoprojectors in (108), in particular, A^+),

$$\langle A_L^-\rangle \equiv A^+ \oplus \langle \overleftarrow{A'A}\mathcal{E}^{m\times n}\overrightarrow{AA'}\rangle \tag{112}$$

(all these matrices produce orthoprojectors (109), in particular, A^+),

$$A^+ = \langle A_R^-\rangle \cap \langle A_L^-\rangle. \tag{113}$$

From (108)–(110) we obtain

$$\left.\begin{aligned} rank\ A_1 = n\ &\Rightarrow \overset{\bullet}{\Delta}_1 = Z, \\ rank\ A_2 = m\ &\Rightarrow \overset{\bullet}{\Delta}_2 = Z, \\ (rank\ A_1 = n,\ rank\ A_2 = m)\ &\Rightarrow \overset{\bullet}{\Delta} = Z. \end{aligned}\right\} \tag{114}$$

Consider in details the exact normal solution of the classical linear equation $A\mathbf{x} = \mathbf{a}$ in the general form with the use of formula (101):

$$||A\mathbf{x} - \mathbf{a}|| \to \min, \quad \overset{\bullet}{\mathbf{x}} = A^{+}\mathbf{a} = [\overset{\bullet}{A}(r)/k(AA', r)]\ \mathbf{a}, \tag{115}$$

$$\overset{\bullet}{\mathbf{d}} = -\overrightarrow{AA'}\ \mathbf{a}. \tag{116}$$

We have

$$\overset{\bullet}{\mathbf{d}} = \mathbf{0}\ \Leftrightarrow \mathbf{a} \in \langle ker \overrightarrow{AA'} \rangle \equiv \langle ker\ K_1(AA', r) \rangle. \tag{117}$$

Here we get exact formulae (115) and (116) for the *normal solution and minimal residual* of the classical linear equation $A\mathbf{x} = \mathbf{a}$. The residual is antiprojection (116). Consequently, its Euclidean norm satisfies

$$||\ \overset{\bullet}{\mathbf{d}}\ ||^2 = -\ \overset{\bullet}{\mathbf{d}}{}'\ \cdot \mathbf{a} = -\mathbf{a}' \cdot \overset{\bullet}{\mathbf{d}}, \tag{118}$$

$$||\ \overset{\bullet}{\mathbf{d}}\ || = \sin\varphi\ ||\mathbf{a}||, \quad (\varphi \in [0; \pi/2]) \tag{119}$$

where φ is the scalar angle between the vector $\mathbf{a}$ and the subspace $\langle im\ A \rangle$.

We conclude the section with inferring from formula (101) the explicit expression for a (p, q)-element of the $m \times n$-matrix $\overset{\bullet}{A}(r)$ in (115). The most general Hermitean-like form of this element in the case of a complex initial linear equation is

$$(p, q) = \sum_{(C^r_{m-1}\ \text{terms})} \sum_{(C^r_{n-1}\ \text{terms})} \overline{\left\{ \frac{(q,p) \in}{minor(r)} A \right\}} Ad_{q',p'} \left\{ \frac{(q,p) \in}{minor(r)} A \right\},$$

where $p = 1, \ldots, m,\ q = 1, \ldots, n,\ p'$ and q' are new indexes of a_{qp} in minors of A. Therefore, (115) generalizes here the Cramer formulae. *In special case* $r = n = m$, (115) represents the matrix solution of a nonsingular linear equation $A\mathbf{x} = \mathbf{a}$, because $\overset{\bullet}{A}(n) = \overline{det\ A} \cdot A^V$, $k(AA^*, n) = \overline{det\ A} \cdot det\ A$ and, consequently, the solution is $\mathbf{x} = (A^V/det\ A)\mathbf{a} = A^{-1}\mathbf{a}$ (the special classical case see, for example, in [2, p. 38]).

Chapter 3

Main scalar invariants of singular matrices

3.1 The minorant of a matrix and it applications

Let A_1 and A_2 be $n \times m$-matrices. Then $k(A_1A_2', t) = k(A_2A_1', t)$. The scalar coefficients of order t for $n \times n$-matrix A_1A_2' were shown to be the sums of all diagonal minors of order t. Represent each matrix of diagonal minors of A_1A_2' as the following multiplication of $t \times m$-matrices of rows:

$$\{D\text{-minor}(t)A_1A_2'\} = \{\text{lig}(t)A_1\}\{\text{lig}(t)A_2\}'.$$

By the Binet–Cauchy formula [2, p. 39], this minor (i. e., determinant of the left matrix) is the sum of all pair multiplications of all minors from the right submatrices of order t with the same set of columns. For the $m \times m$-matrix $A_1'A_2$, *in all these assertions, rows are changed for columns and columns are changed for rows.* Consider the two sets of $C_n^tC_m^t$ pair multiplications of order-t minors of A_1 and A_2. They form the two sums. The first sum is equal to the scalar coefficient $k(A_1A_2', t)$, the second sum is equal to the scalar coefficient $k(A_1'A_2, t)$. There exists a bijection between these two sets, just as above, thus for external and internal multiplications of these matrices we have

$$k(A_1A_2', t) = k(A_1'A_2., t) = k(A_2A_1', t) = k(A_2'A_1, t). \tag{120}$$

In the special case $A_1 = A_2 = A$, i. e., for these both *homomultiplications*, there holds

$$k(AA', t) = \sum_{(C_m^t \text{ terms})} \sum_{(C_n^t \text{ terms})} \text{minor}^2(t)A \ = k(A'A, t) \geq 0. \tag{121}$$

Introduce the highest positive characteristic of an $n \times m$-matrix, its *minorant*:

$$\mathcal{M}t(r)A = \sqrt{k(AA', r)} = \sqrt{k(A'A, r)} = \mathcal{M}t(r)A' > 0.$$

It is the square root of the sum of all squared basic minors A, this follows from (121).

Note *the special cases.*

1. If $n > m = r$, then $\mathcal{M}t^2(m)A = det\ A'A$ (the Gram determinant for columns A).
2. If $m = 1$, then $\mathcal{M}t(1)\mathbf{a} = ||\mathbf{a}||_E$ (the Euclidean module $\mathbf{a}$).
3. If $n = m = r$, then $\mathcal{M}t(n)A = |det\ A|$ (the determinant A).

Formulae for the matrix poly-step homomultiplication minorant follow from (67):

$$\mathcal{M}t(r)\{\underbrace{AA'A\ldots}_{h}\} = \mathcal{M}t(r)\{\underbrace{A'AA'\ldots}_{h}\} =$$

$$= \sqrt{k[(AA')^h, r]} = \sqrt{k^h(AA', r)} = \mathcal{M}t^h(r)A.$$

Consider equation (115) and the matrix $\{A|\mathbf{a}\}$. If $n = m = r$, due to (117), $\overset{\bullet}{\mathbf{d}} = \mathbf{0}$. When $n \geq m \geq r$, (116) and (119) give the general result:

$$\mathcal{M}t(r+1)\{A|\mathbf{a}\} = \sin\varphi \cdot ||\mathbf{a}|| \cdot \mathcal{M}t(r)A = ||\overset{\bullet}{\mathbf{d}}|| \cdot \mathcal{M}t(r)A. \qquad (122)$$

This leads to the Kronecker–Capelli Theorem, *expressed by formula* in terms of the squared minorant (121) of the matrix $\{A|\mathbf{a}\}$, i. e., from order $(r+1)$ squared minors:

$$\mathcal{M}t^2(r+1)\{A|\mathbf{a}\} = \sum_{(C_{m+1}^{r+1})} \sum_{(C_n^{r+1})} \text{minor}^2(r+1)\{A|\mathbf{a}\} = 0 \Leftrightarrow \overset{\bullet}{\mathbf{d}} = \mathbf{0} \Leftrightarrow \sin\varphi = 0.$$

If $n > m = r$, then the Gram determinant is also the analogous criterion:

$$\mathcal{M}t^2(r+1)\{A|\mathbf{a}\} = det[\{A|\mathbf{a}\}'\{A|\mathbf{a}\}] = ||\overset{\bullet}{\mathbf{d}}||^2 \cdot \mathcal{M}t^2(r)A.$$

Formula (122) in the trigonometric form (where $\varphi \in (0;\pi/2]$) is

$$0 \leq \sin\varphi = \mathcal{M}t(r+1)\{A|\mathbf{a}\}/(\mathcal{M}t(r)A \cdot \mathcal{M}t(1)\mathbf{a}) \leq 1. \qquad (123)$$

In particular, for the angle between two vectors ($\varphi_{12} \in (0;\pi/2]$) in $\langle\mathcal{E}^n\rangle$, we have:

$$0 \leq \sin\varphi_{12} = \mathcal{M}t(2)[\mathbf{a_1}|\mathbf{a_2}] \;/(\mathcal{M}t(1)\mathbf{a_1} \cdot \mathcal{M}t(1)\mathbf{a_2}) =$$

$$= \sqrt{det\{[\mathbf{a_1}|\mathbf{a_2}]' \cdot [\mathbf{a_1}|\mathbf{a_2}]\}}/(||\mathbf{a_1}|| \cdot ||\mathbf{a_2}||) = ||\mathbf{a_1} \times \mathbf{a_2}||/(||\mathbf{a_1}|| \cdot ||\mathbf{a_2}||) \leq 1. \qquad (124)$$

On the left we gives a *scalar multiplication of sine type for two vectors* and on the right we gives identical to it a module of their vector multiplication. In the first variant, *for two vectors on a plane* ($n = 2$), *may be eigen*, i. e., in $\langle\mathcal{E}^2\rangle$!), the determinant in formula (124) disintegrates in two equal determinants. As result, there holds the simplified formula for the angle between two vectors on a plane with the angle sign:

$$-1 \leq \sin\varphi_{12} = det[\mathbf{a_1}|\mathbf{a_2}]/(||\mathbf{a_1}|| \cdot ||\mathbf{a_2}||) \leq +1, \quad (\varphi_{12} \in [-\pi/2; +\pi/2]).$$

Relation between the minorant of an $n \times r$-matrix A and the square root of the Gram determinant of its r columns enables one to clarify the geometric sense of the minorant as the volume of the parallelepiped, constructed on the vector-columns of the matrix A [3, p. 216]. In particular, put $m = r$. We often deal with such matrices in part II. They represent special linear geometric objects *lineors* of greater dimension ($r > 1$), then vectors. Consider the columns of a matrix A. Denote the submatrix formed by first j columns as A_j. Then $A_{j+1} = \{A_j|\mathbf{a_{j+1}}\}$ for each j. Apply formulae (119) and (122) to A_{j+1}, also the geometric interpretation of the Gram determinant square root may be used. Subsequent application of this operation gives the formula

$$\mathcal{M}t(r)A = v_r = ||\mathbf{a}_1|| \cdot \sin\varphi_{1,2} \cdot ||\mathbf{a}_2|| \cdot \sin\varphi_{1,2,3} \cdots ||\mathbf{a}_r|| \leq ||\mathbf{a}_1|| \cdot ||\mathbf{a}_2|| \cdots ||\mathbf{a}_r||, \quad (125)$$

where v_r is the volume of the r-dimensional parallelepiped with sides $\mathbf{a}_1, \ldots, \mathbf{a}_r$, and $\varphi_{1,2}, \varphi_{1,2,3}, \cdots \in (0;\pi/2]$.

If $n = m = r$, then from (125) the sine Hadamard Inequality in its usual form [16, p. 35] is valid; and, if $r = 2$, it has particular form (124). Due to (74), the following does hold:

$$\mathcal{M}t(r)A = \sqrt{k(AA', r)} = \prod_{j=2}^{q} \sigma_j^{s_j} > 0, \quad \overrightarrow{AA'} = \prod_{j=2}^{q} (\sigma_j^2 I_{n\times n} - AA')/\sigma_j^2, \tag{126}$$

where $\sigma_j^2 > 0$ are the nonzero eigenvalues of AA' or $A'A$.

In general ($n \geq m \geq r \geq t$), the coefficients $k(AA', t) = k(A'A, t)$ may be expressed either geometrically as the sums of squared t-dimensional volumes (t-measures) or algebraically as the Viète sums of the eigenvalues of AA':

$$\left.\begin{aligned} k(AA', t) &= \sum_{(C_m^t \text{ terms})} v_{t(p)}^2 = s_t(\sigma_j^2) = v_t^2 > 0, \\ k(AA', 1) &= \sum_{(m \text{ terms})} \ell_{(p)}^2 = s_1(\sigma_j^2) = \ell^2 = ||A||_F^2 > 0, \end{aligned}\right\} (\mathcal{M}t^2(r)A = v_r^2). \tag{127}$$

Here, in Cartesian coordinates, $v_{t(p)}$ is the volume v_t of the orthoprojection of rank t. If $m = r$, then the ratio $v_{t(p)}/v_t = \cos\alpha_p$ is the p-th direction cosine.

Formulae (127) express the Pythagorean Theorem for the linear objects represented by $n \times r$-matrices. Further, they are called *lineors*. All the characteristics are always positive and invariant under orthogonal transformations of columns or rows of the matrix A and its Cartesian base. In particular, there holds

$$\mathcal{M}t(r)A = \mathcal{M}t(r)\{R_1 A R_2\} = \mathcal{M}t(r)\sqrt{AA'} = \mathcal{M}t(r)\sqrt{A'A}. \tag{128}$$

Therefore, a minorant may be used as geometric characteristic for these lineors of different dimensions and ranks. In Ch. 9 this opportunity will be realized for introducing general norms of similar linear objects.

The arithmetic roots in (128) may be singular; in general, they are related with the matrix A by the *quasi-polar decompositions of A (i. e. QR-factorization)*:

$$A = S_1^{\oplus} \cdot Rq = \sqrt{AA'} \cdot \{(\sqrt{AA'})^+ \cdot A\}, \tag{129}$$

$$A = Rq \cdot S_2^{\oplus} = \{A \cdot (\sqrt{A'A})^+\} \cdot \sqrt{A'A}. \tag{130}$$

$$S_1^{\oplus} = Rq \cdot S_2^{\oplus} \cdot Rq' \;\Leftrightarrow\; AA' = Rq \cdot A'A \cdot Rq',$$

$$Rq = A \cdot (\sqrt{A'A})^+ = (\sqrt{AA'})^+ \cdot A \;\Rightarrow$$

$$Rq \cdot Rq' = \overleftarrow{AA'}, \quad Rq'Rq = \overleftarrow{A'A}, \quad Rq' = Rq^+.$$

The transformation $A \to Rq$ gives the same result as the Gram–Schmidt unity orthogonalization of m linearly independent vectors:

$$A = \{\mathbf{a_1}, \ldots, \mathbf{a_m}\} \to \{\mathbf{e_1}, \ldots, \mathbf{e_m}\} = Rq.$$

This algebraic transformation is the uniquely determined variant of the Gram–Schmidt orthogonalization (provided that the sequence of vectors is fixed).

In Euclidean space, the Gram–Schmidt orthogonalization may be also expressed geometrically clearly with use of orthoprojectors:

$$\mathbf{v_1} = \mathbf{a_1}, \quad \mathbf{v_i} = \mathbf{a_i} - \sum_{\mathbf{k=1}}^{\mathbf{i-1}} [\mathbf{e_k} \cdot \mathbf{e'_k}] \cdot \mathbf{a_i} = \{I - \sum_{\mathbf{k=1}}^{\mathbf{i-1}} [\mathbf{e_k} \cdot \mathbf{e'_k}]\} \cdot \mathbf{a_i}, \tag{131}$$

where $\mathbf{e_k} \cdot \mathbf{e'_k} = \overleftarrow{\mathbf{e_k} \cdot \mathbf{e'_k}}$ – see sect. 2.1. The results of this procedure are the following $\mathbf{e_i} = \mathbf{v_i}/||\mathbf{v_i}||$, $i = 1, \ldots, m$, and additionally we have the matrix Rq for A.

For the *special* kind of $n \times m$-matrices, with $n > m = r$, prove the split formula for the minorant of their external multiplications:

$$\mathcal{M}t(r)A_1A'_2 = \mathcal{M}t(r)A_1 \cdot \mathcal{M}t(r)A_2 = \sqrt{det\ (A'_1A_1) \cdot det\ (A'_2A_2)}. \tag{132}$$

With the definition of a minorant, the quasi-polar decompositions such as (129), (130), and also formula (128), we subsequently obtain

$$\mathcal{M}t^2(r)\{A_1A'_2\} = k[(A_1A'_2A_2A'_1), r] = k[(Rq_1 \cdot S_1^{\oplus} \cdot S_2^{\oplus} \cdot S_2^{\oplus} \cdot S_1^{\oplus} \cdot Rq'_1), r] =$$

$$= k[(S_1^{\oplus} \cdot S_2^{\oplus} \cdot S_2^{\oplus} \cdot S_1^{\oplus}), r] = det\ (A'_1A_1) \cdot det\ (A'_2A_2) = \mathcal{M}t^2(r)A_1 \cdot \mathcal{M}t^2(r)A_2.$$

Further, for *external* and *internal multiplications* of $n \times m$-matrices of the special kind we use notations:

$$B = A_1A'_2, \quad B' = A_2A'_1; \qquad C = A'_1A_2, \quad C' = A'_2A_1.$$

For B, if $\langle im\ A'_2\rangle \cap \langle ker\ A_1\rangle = \mathbf{0}$, $\langle im\ A'_1\rangle \cap \langle ker\ A_2\rangle = \mathbf{0}$, there holds:
$\langle im\ B\rangle \equiv \langle im\ A_1\rangle \ \Leftrightarrow\ \langle ker\ B'\rangle \equiv \langle ker\ A'_1\rangle$ – see also (100),
$\langle im\ B'\rangle \equiv \langle im\ A_2\rangle \ \Leftrightarrow\ \langle ker\ B\rangle \equiv \langle ker\ A'_2\rangle$ – see also (100).

Due to additional condition $m = rank\ A_1 = rank\ A_2 = r$, the following does hold:

$$\left.\begin{array}{l} \overleftarrow{BB'} = \overleftarrow{A_1A'_2A_2A'_1} = \overleftarrow{A_1A'_1} = \overleftarrow{Rq_1Rq'_1}, \\ \overleftarrow{B'B} = \overleftarrow{A_2A'_1A_1A'_2} = \overleftarrow{A_2A'_2} = \overleftarrow{Rq_2Rq'_2}, \\ \overrightarrow{BB'} = \overrightarrow{A_1A'_2A_2A'_1} = \overrightarrow{A_1A'_1} = \overrightarrow{Rq_1Rq'_1}, \\ \overrightarrow{B'B} = \overrightarrow{A_2A'_1A_1A'_2} = \overrightarrow{A_2A'_2} = \overrightarrow{Rq_2Rq'_2}. \end{array}\right\} \tag{133}$$

Besides, $det\ C = det(A'_1A_2) \neq 0$. (See this in details in Part II, sect. 5.4.)

Then formulae

$$\left.\begin{array}{l} K_j[(A_1A'_2A_2A'_1), r] = det\ (A'_2A_2) \cdot K_j(A_1A'_1, r), \\ K_j[(A_2A'_1A_1A'_2), r] = det\ (A'_1A_1) \cdot K_j(A_2A'_2, r), \\ j = 1, 2, \end{array}\right\} \tag{134}$$

follow from (61), (62), (132), (133).

3.2 Sine characteristics of matrices

Let $E = \{\mathbf{e}_i\}_{n\times n}$ be some $n \times n$-matrix, given as a linear unity geometric object in the 1-st quadrant of Cartesian base $\{I\}$ in a space $\langle\mathcal{E}^n\rangle$, where $||\mathbf{e}_i|| = 1$ for all i. Namely the matrix $E = \{\mathbf{e}_i\}_{n\times n}$ determines an n-edges polyhedral tensor angle in the Euclidean space; $det\ E = \mathcal{M}t(n)E \leq 1$ is, due to the trigonometric value in Hadamard Inequality (125), its *sine characteristic.* This polyhedral angle corresponds one-to-one the unique mutual tensor angle, given by the matrix $\hat{E} = \{\hat{\mathbf{e}}_i\}_{n\times n} = \{\overrightarrow{E_iE_i'}\mathbf{e}_i \sec\beta_i\}$, where E_i is obtained from E by change of the column $\mathbf{e}_i$ on zero one, and for this tensor angle $\hat{E}$ unity its calibration by $\sec\beta_i$ is used. The orthoprojector of type $\overrightarrow{E_iE_i'}$ projects into the kernel $\langle ker\ E_i'\rangle$ orthogonally to the image $\langle im\ E_i\rangle$ (see sect. 2.5). There holds: $\cos\beta_i = \mathbf{e}_i'\hat{\mathbf{e}}_i = \hat{\mathbf{e}}_i'\mathbf{e}_i$ $(0 < \cos\beta_i \leq 1)$, $\mathbf{e}_i'\hat{\mathbf{e}}_j = 0$ or $E'\hat{E} = D_{\cos\beta} = \hat{E}'E$, $\to \cos^2\beta_i = \mathbf{e}_i'\overrightarrow{E_iE_i'}\mathbf{e}_i$, and the all values of $\cos\beta_i$ are finding. Further,

$$det\ E \cdot det\ \hat{E}\ = det\ D_{\cos\beta} = \prod_{i=1}^{n} \cos\beta_i,\ |det\ E| \leq 1,\ |det\ \hat{E}| \leq 1;$$

$$E'E = D_{\cos\beta} \cdot (\hat{E}'\hat{E})^{-1} \cdot D_{\cos\beta}, \quad \hat{E}'\hat{E} = D_{\cos\beta} \cdot (E'E)^{-1} \cdot D_{\cos\beta},$$
$$G = \sqrt{D_{\sec\beta}} \cdot E'E \cdot \sqrt{D_{\sec\beta}} = \hat{G}^{-1} = [\sqrt{D_{\sec\beta}} \cdot \hat{E}'\hat{E} \cdot \sqrt{D_{\sec\beta}}\ ]^{-1}.$$

Here G and $\hat{G}$ are metric tensors in the stretched of these angles mutual affine bases, given in $\{I\}$ by modal matrices $\{E\sqrt{D_{\sec\beta}}\}$ and $\{\hat{E}\sqrt{D_{\sec\beta}}\}$.

However, in the book, we deal with tensor angles of the binary type, i. e., angles formed by pairs of linear subspaces (straight lines if $r = 1$) or linear objects A_1, A_2 (vectors if $r = 1$) in spaces with quadratic metrics.

At first, consider the *sine characteristic* of binary angles. Now we suppose that $r_1 = rank\ A_1$ and $r_2 = rank\ A_2$, but $r_1 + r_2 \leq n$. The block matrix $\{A_1|A_2\}$ is called the *external summation* of A_1 and A_2. Introduce for the rectangular matrices (or lineors) A_1 and A_2 the scalar characteristic *sine ratio* (see more in sect. 8.4):

$$|\{A_1|A_2\}|_{\sin} = \mathcal{M}t(r_1 + r_2)\{A_1|A_2\}/(\mathcal{M}t(r_1)A_1 \cdot \mathcal{M}t(r_2)A_2) = \tag{135}$$
$$= \sqrt{det\ \left[\begin{array}{c|c} A_1'A_1 & A_1'A_2 \\ \hline A_2'A_1 & A_2'A_2 \end{array}\right]} / \sqrt{det\ (A_1'A_1) \cdot det\ (A_2'A_2)} = detG_{1,2}/\mathcal{M}t(r)A_1A_2' \geq 0.$$

It generalizes (123) and ratio (124) for the sine of the angle between two vectors. The matrix in numerator generalizes *the internal multiplication of two vectors of sine type* used in (124). This ratio is the *sine positively definite norm* for a pair of A_1 and A_2.

The Kronecker–Capelli Theorem may be generalized to matrix linear equations such as (105)–(107). The generalization is expressed also in terms of the minorant:

$$\mathcal{M}t^2(r_1 + r_2 + 1)\left[\begin{array}{c|c} A_1 & A \\ \hline Z & A_2 \end{array}\right] = 0\ \Leftrightarrow\ \dot{\Delta} = Z. \tag{136}$$

3.3 Cosine characteristics of matrices

Denote the highest scalar characteristic of a square singular matrix, its *dianal* :

$$\mathcal{D}l(r)B = k(B, r) = \mathcal{D}l(r)B' \quad (det\ B = 0),$$

So, $\mathcal{D}l(r)\{AA'\} = \mathcal{D}l(r)\{A'A\} = k(AA', r) = \mathcal{M}t^2(r)A$ – see sect. 3.1. And from (122) we have: $\mathcal{M}t^2(r+1)\{A|\mathbf{a}\} = \mathcal{D}l(r+1)\{[A|\mathbf{a}][A|\mathbf{a}]'\} = 0 \Leftrightarrow \overset{\bullet}{\mathbf{d}} = \mathbf{0} \Leftrightarrow \sin\varphi = 0$!

Then the new scalar characteristic for a singular square matrix B, its sign-indefinite *cosine ratio* (see more in Ch. 8), is expressed in terms of the minorant and the dianal:

$$\{B\}_{\cos} = \mathcal{D}l(r)B/\sqrt{\mathcal{D}l(r)BB'} = \mathcal{D}l(r)B/\mathcal{M}t(r)B = \prod_{i=2}^{q_1} \mu_i^{s'_{1,i}} \Big/ \prod_{j=2}^{q_2} \sigma_j^{s_{2,j}}. \quad (137)$$

We may preliminary introduce the *cosine norm* for B as follows (see more in sect. 8.1):

$$1 \geq |\{B\}|_{\cos} = |\mathcal{D}l(r)B|/\mathcal{M}t(r)B = \prod_{i=2}^{q_1} |\mu_i|^{s'_{1,i}} \Big/ \prod_{j=2}^{q_2} \sigma_j^{s_{2,j}} \geq 0. \quad (138)$$

The cosine ratio of null-defective B is 0 ($r' < r$), and it is $+1$ or -1 for null-normal B. Formula (137) to the right contains the eigenvalues μ_i with their algebraic multiplicities $s'_{1,i}$ for the matrix B and its singular numbers $\sigma_j > 0$ (for the square root of the matrix BB' or $B'B$) with their algebraic (geometric) multiplicities $s_{2,j}$ in $\mathcal{M}t(r)B$ as in (126).

Let A_1 and A_2 be $n \times m$-matrices with their *external and internal multiplications of cosine type* $B = A_1A'_2$ and $B' = A_2A'_1$, $C = A'_1A_2$ and $C' = A'_2A_1$. Then the cosine ratio for a pair of matrices (or lineors) A_1 and A_2 may be expressed as

$$\{A_1 \cdot A'_2\}_{\cos} = \{A_2 \cdot A'_1\}_{\cos} = \mathcal{D}l(r)\{A_1 \cdot A'_2\}/\mathcal{M}t(r)\{A_1 \cdot A'_2\}. \quad (139)$$

If A_1 and A_2 are equirank $n \times r$-matrices, then, due to (120) and (132),

$$\{A_1 \cdot A'_2\}_{\cos} = \mathcal{D}l(r)\{A_1 \cdot A'_2\}/(\mathcal{M}t(r)A_1 \cdot \mathcal{M}t(r)A_2) =$$

$$= det\ \{A'_1A_2\}/[\sqrt{det\ \{A'_1A_1\}} \cdot \sqrt{det\ \{A'_2A_2\}}]. \quad (140)$$

In particular, for the angle between two vectors in the Euclidean space $\langle \mathcal{E}^n \rangle$ we have

$$-1 \leq \cos\varphi_{12} = \mathbf{a}'_1\mathbf{a}_2/||\mathbf{a}_1|| \cdot ||\mathbf{a}_2|| = \mathbf{a}'_2\mathbf{a}_1/||\mathbf{a}_2|| \cdot ||\mathbf{a}_1|| \leq +1, \quad (\varphi_{12} \in (0; \pi]). \quad (141)$$

In part II, the trigonometric sense of the sine and cosine ratios will be explained, the trigonometric spectrum of a singular matrix B is necessary for it. We note here especially, that both left and right sides in formulae (135) or (140) may be considered as some identical algebraic expressions of trigonometric (sine or cosine) nature for coordinates of geometric objects (lineors) represented by $n \times r$-matrices A_1 and A_2. The angle sign is defined only for two vectors on a plane, *may be eigen*, i. e., in $\langle \mathcal{E}^2 \rangle$.

For two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ (i. e. if $r = 1$), the expressions in (135), (140) at $n \geq 2$ separately or as the sum of their squared forms give a number of algebraic inequalities or identities of trigonometric (sine and cosine) nature. Their examples are well-known as sine Hadamard Inequality, for example in form (125) at $r = 2$; cosine Cauchy Inequality, for example in form (141). The scalar multiplications of two vectors of sine type in (124) and cosine type in (141) give these Summary identity for their coordinates (here in Euclidean space), which equivalent to Lagrange Identity also for two vectors:

$$[\mathcal{M}t(2)[\mathbf{a_1}|\mathbf{a_2}]/(\mathcal{M}t(1)\mathbf{a_1} \cdot \mathcal{M}t(1)\mathbf{a_2})]^2 + [tr\ \mathbf{a_1}\mathbf{a_2'}/(\mathcal{M}t(1)\mathbf{a_1} \cdot \mathcal{M}t(1)\mathbf{a_2})]^2 =$$

$$= [det([\mathbf{a_1}|\mathbf{a_2}]'[\mathbf{a_1}|\mathbf{a_2}])]/[\mathbf{a_1'}\mathbf{a_1} \cdot \mathbf{a_2'}\mathbf{a_2}] + [(\mathbf{a_1'}\mathbf{a_2})^2]/[\mathbf{a_1'}\mathbf{a_1} \cdot \mathbf{a_2'}\mathbf{a_2}] = \qquad (142)$$

$$= \sin^2\varphi_{1,2} + \cos^2\varphi_{1,2} = 1 = (\mathbf{a_1} \times \mathbf{a_2})^2/||\mathbf{a_1}||^2 \cdot ||\mathbf{a_2}||^2 + (\mathbf{a_1} \cdot \mathbf{a_2})^2/||\mathbf{a_1}||^2 \cdot ||\mathbf{a_2}||^2,$$

(where the last variant is a classical sine-cosine Identity of Lagrange for two vectors). Formula (142) enables one to normalize the angles between vectors in Euclidean spaces. In part II of the book, similar constructions for more general linear objects as *lineors*, represented by $n \times m$-matrices A_1 and A_2, will be analyzed.

3.4 Limit methods for evaluating projectors and quasi-inverse matrices

According to (1) and (101), the following limit formulae do hold:

$$A^+ = \lim_{\epsilon \to 0}[A'(AA' + \epsilon I)^{-1}] = \lim_{\epsilon \to 0}[(A'A + \epsilon I)^{-1}A'] = \qquad (143)$$

$$= \lim_{N \to \infty}[NA'(NAA' + I)^{-1}] = \lim_{N \to \infty}[(NA'A + I)^{-1}NA'], \qquad (144)$$

$(\overrightarrow{A'A}A' = Z = A'\overrightarrow{AA'} \quad \Rightarrow \quad K_1(A'A, r)A' = Z = A'K_1(AA', r) \ ).$

As well as general formulae (71)–(73), the special limit formulae (143), (144) are inferred by purely algebraic way, with use of the resolvent (1).

A. N. Tikhonov [29] was the first who expressed the normal solution of the linear equation $A\mathbf{x} = \mathbf{a}$ as a limit. He used his regularization method in the special case of a conditional extremum problem: find the value of the argument with the minimal Euclidean norm on a given set corresponding to the minimal residual of equation

$$U(\mathbf{x}, \epsilon) = \epsilon F_1(\mathbf{x}) + F_2(\mathbf{x}) = \min, \quad dU/d\mathbf{x} = \mathbf{0} \ \ (\epsilon \to 0). \qquad (145)$$

($F_1(\mathbf{x}) = \mathbf{x}'\mathbf{x}, F_2(\mathbf{x}) = \mathbf{d}'(\mathbf{x}) \cdot \mathbf{d}(\mathbf{x})$, where the residual is $\mathbf{d}(\mathbf{x}) = A\mathbf{x} - \mathbf{a}$).

Note, that similar results, but in limit form (144), might be obtained long before the publication of A. N. Tikhonov by Courant's penalty function method [11]:

$$W(\mathbf{x}, N) = F_1(\mathbf{x}) + N \cdot F_2(\mathbf{x}) = \min, \quad dW/d\mathbf{x} = \mathbf{0} \ \ (N \to \infty). \qquad (146)$$

In this task, both the methods are in one-to-one correspondence consisting in multiplying or dividing by a scalar limit parameter.

Courant's penalty function method finds the conditional extremum of $F_1(\mathbf{x})$ with the gradient $1 \times n$-vector function in the constraint equation $\mathbf{h}'(\mathbf{x}) = dF_2/d\mathbf{x} = \mathbf{0}$. Integration converts the usual vector form into the equivalent scalar form:

$$h(\mathbf{x}) = \int_{\mathbf{x_s}}^{\mathbf{x}} \mathbf{h}'(\mathbf{x})d\mathbf{x} = 0 = const. \tag{147}$$

Then in (146) we obtain the Lagrange function $W(\mathbf{x}, N)$ and the *unique* scalar Lagrange multiplier $N \to \infty$, as

$$(dh/d\mathbf{x})N = \mathbf{h}'(\mathbf{x})N = \mathbf{0} \cdot N = -dF_1/d\mathbf{x} \neq \mathbf{0}$$

follows from the differential equation (146), and consequently $N \to \infty$.

In particular, these limit methods are applicable for finding conditional extremum of $F_1(\mathbf{x})$ on the stationary set of $F_2(\mathbf{x})$. Chains in equations (145) and (146) may be continued by polynomials in ϵ or N. The sufficient condition for applicability these two limit methods in the *differential form* (with the small or large parameter) is, due to (147), integrability of the $1 \times n$-vector function $\mathbf{h}'(\mathbf{x})$ from the constraint equation and consequently symmetry of its Jacobi matrix: $(d\mathbf{h}/d\mathbf{x})' = d\mathbf{h}/d\mathbf{x}$. If the normal solution of equation $A\mathbf{x} = \mathbf{a}$ is searched for, this symmetric Jacobi matrix is $A'A$.

Due to General optimization limit method *differential equation* $\epsilon dF_1/d\mathbf{x} + \mathbf{h}'(\mathbf{x}) = \mathbf{0}$, $\epsilon \to 0$ *or* $dF_1/d\mathbf{x} + N\mathbf{h}'(\mathbf{x}) = \mathbf{0}$, $N \to \infty$, *determines a complete solution according to conditional stationarity of* $F_1(\mathbf{x})$ *under constraint* $\mathbf{h}'(\mathbf{x}) = \mathbf{0}$ *iff the Jacobi matrix of the constraint vector function* $\mathbf{h}'(\mathbf{x})$ *is null-normal*, i. e., $\langle ker\ d\mathbf{h}/d\mathbf{x}\rangle \equiv \langle ker\ (d\mathbf{h}/d\mathbf{x})'\rangle$. (At the stationarity point of $F_1(\mathbf{x})$ for the $1 \times n$-vector of the conditional gradient, obviously, there holds: $dF_1/d\mathbf{x} \cdot \overrightarrow{d\mathbf{h}/d\mathbf{x}} \in \langle ker\ d\mathbf{h}/d\mathbf{x}\rangle$.)

The conditional stationarity nature of $F_1(\mathbf{x})$ *(i. e., either a conditional minimum or a conditional maximum, or a conditional saddle without extremum) is determined by the limit conditional Hesse matrix of* $F_1(\mathbf{x})$ *up to scalar parameter* ϵ *or* N.

See detailed exposition of this General optimization limit method and its applications in other our monograph [18, p. 97–112]. In particular, this method give, by simple way, the exact solutions for a conditional extremum of the second-order scalar function $Q(\mathbf{x})$ under the linear constraint equation $Bm \cdot \mathbf{x} = \mathbf{a}$, including $Bm = S$.

Moreover, the concrete constant singular Jacoby null-prime matrix Bp for the linear constraint equation $Bp \cdot \mathbf{x} = \mathbf{a}$ may be transformed into the null-normal matrix Bm by a suitable modal transformation of the initial base (further, this limit method may by applied). As example, for a null-prime matrix Bp, its affine quasi-inverse matrix Bp^-, see (69), may be computed by the same functional limit way with preliminary use of linear base transformation for converting Bp into Bm. Then one calculates Bm^- by the limit method according its value in (104), i. e., in fact, as the Moor-Penrose quasi-inverse matrix. Having finished these operations, one returns to the initial base by the reverse modal transformation, and get the matrix Bp^-.

Chapter 4

Two alternative complexification variants

4.1 Comparing two variants

Nature of complex numbers gives rise to main two and quite different approaches for implementing operations over initially given complex numerical or algebraic elements. Besides, the complex elements may have due to these operations the corresponding form of their representations.

By the *adequate* approach, operations over complex-number elements are formally the same as over real-number ones. This allows one to use results previously obtained for real-number analogous objects. However, there are some exceptions: inequalities (unless parameters are only real), module notions. The special case is *pseudoization*, when real and imaginary parts of complex elements form direct sums of the same type.

The *symbiotic* approach supposes the use of standard operations applied to real numbers as well as the additional operation of complex conjugation independent on usual ones. In particular, it takes place in the *Hermitean* approach for vectors and matrices with complex entries: their transposition is always accompanied by complex conjugation. The Hermitean variant of complexification allows one to use in the self-conjugate form notions of the real positive module or norm as well as similar self-conjugate form for a lot of inequality relations.

These variants of complexification point out two independent directions for further development of theories and their applications in complex spaces.

So, relations $\langle im\ B\rangle \equiv \langle im\ B'\rangle$ and $\langle im\ B\rangle \equiv \langle imB^*\rangle$ determine adequately and Hermitean null-normal matrices. But adequately and Hermitean orthogonal eigenprojectors and quasi-inverse matrices are defined by differ ways using (98)–(101). Adequate complex characteristics no always exist in such determined form in what Hermitean ones exist. As example, $\mathcal{M}t^2(r)A = k(AA', r) = k(A'A, r)$ for a complex matrix, where $r = rangA$, may have any complex values including zero.

But for pseudoized vectors and matrices their squared minorant may have only real values — positive, negative and zero. From the other hand, in the Hermitean variant there holds $k(AA^*, t) = k(A^*A, t) > 0, \quad t \leq r$.

In any case, all eigenprojectors of a null-prime matrix Bp exist and are spectrally nonnegative semi-definite matrices, because their eigenvalues are equal to $+1$ and 0. Moreover, for matrices Bp affine eigenprojectors and quasi-inverse matrices do not depend on the complexification variant. If a matrix B is complex and nonsingular, then $\langle im\ B\rangle \equiv \langle imB'\rangle \equiv \langle imB^*\rangle \equiv \langle \mathcal{A}^n\rangle$, that is why the complex inverse matrix B^{-1} for such quadratic matrix B is uniquely determined.

Forms of representing any complex number "a" with the imaginary unit "i" are well-known. They are simplest arithmetic form, trigonometric Moivre's and polar forms, exponential Euler's form, pseudoized vectorial form, stereographic Riemann's form. For further aims, use a *normal* 2×2-matrix form without the imaginary unit:

$$\left.\begin{array}{l} W(a) \equiv F(\rho, \varphi), \ (\varphi \in [-\pi; +\pi]) : \\ \\ \left[\begin{array}{c|c} p & -q \\ \hline +q & p \end{array}\right] = \rho \begin{bmatrix} \cos\varphi & -\sin\varphi \\ +\sin\varphi & \cos\varphi \end{bmatrix} = S + K \\ \\ (a = p + iq). \\ \\ W'(a) \equiv F'(\rho, \varphi) : \\ \\ \left[\begin{array}{c|c} p & +q \\ \hline -q & p \end{array}\right] = \rho \begin{bmatrix} \cos\varphi & +\sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} = S - K \\ \\ (\bar{a} = p - iq), \end{array}\right\} \quad (148)$$

Then, $W(a) \cdot W'(a) = W'(a) \cdot W(a) = \rho^2 \cdot I_{2\times 2}$, $S = S'$, $K = -K'$, $SK = KS$.

Note especially, this real form $W(a)$ is also single-valued. (In particular, it may consider for representations of paired solutions of a real valued algebraic equation also in real valued paired conjugate forms!)

There holds $W(a_1){\cdot}W(a_2) = W(a_1{\cdot}a_2) \equiv F(\rho_1, \varphi_1){\cdot}F(\rho_2, \varphi_2) = F[\rho_1{\cdot}\rho_2, (\varphi_1{+}\varphi_2)]$. The form $W(a)$ executes summation and multiplication so as the arithmetic form a.

Besides, the real form $W(a)$ of complex number a as well as the scalar complex form of one is commutative in their summations and multiplications, and satisfy all formulae and identities for complex numbers. They compile the pairs of mutually transposed matrices in (148) similarly to the pairs of conjugate complex numbers. Formally $W(a)$ represents a complex number a in the real affine space of the binary type $\langle \mathcal{A}^{2\times 2} \rangle$, i. e., in the *matrix space*. The trigonometric form in (148) represents the complex number a in the real Euclidean space of the binary type $\langle \mathcal{E}^{2\times 2} \rangle$.

From this point of view, a *real-valued normal* $n \times n$-matrix M represents in a certain Cartesian base $2k \leq \lfloor n \rfloor$ complex conjugate numbers and $n - 2k$ real-valued ones, i. e., $M = RWR'$. A *real-valued prime* matrix $P = VWV^{-1}$ represents in a certain affine base the numbers, where W is a *canonical normal monobinary* form of the matrix P – see, for example, in [3, p. 106]. The decomposition, as a direct sum, contains only real 1×1- and 2×2-cells.

Generally, the matrix W, up to permutations of its cells, is the simplest *real solution* of secular equation $c(\mu) = 0$. Applying the Cayley-Hamilton Theorem to the prime matrix P gives $V^{-1}\{c(P)\}V = c(W) = Z$. These similar W-forms will be widely use in Part II of the book for clear inferring tensor trigonometric formulae!

In its turn, real matrix form (148) may be complexified too, either in the adequate or Hermitian variant. In the first case, there holds

$$\left.\begin{array}{l} W(z_1): \\ \left[\begin{array}{c|c} u & -v \\ \hline v & u \end{array}\right] = \rho \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} = S + K \\ (z_1 = u + iv), \\ W'(z_1) = W(z_2): \\ \left[\begin{array}{c|c} u & v \\ \hline -v & u \end{array}\right] = \rho \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} = S - K \\ (z_2 = u - iv). \end{array}\right\} \quad (149)$$

$[W(z) \cdot W'(z) = W'(z) \cdot W(z) = \rho^2 \cdot I_{2\times 2},\ S = S',\ K = -K',\ SK = KS.]$

Complex adequately normal W-form (149) is implemented in a certain adequately Cartesian base of the *special* Euclidean space $\langle \mathcal{E}^{2\times 2} \rangle$ over $\mathbb{C}$. A complex adequately normal $n \times n$-matrix $M = R\,WR'$ may represent double quantity of non-conjugate complex numbers (i. e., as z_1 and z_2) in the similar bases. All the elements of its W-form are complex numbers, including the module ρ and the angle ψ. The complex normal matrix M may be simplified with some adequately orthogonal transformation R (also complex) and represented in complex canonical W-form (149).

In the second case,, in the Hermitean variant, there holds

$$\begin{array}{ll} W(z): & W^*(z) = W'(\bar{z}): \\ \left[\begin{array}{c|c} u & -\bar{v} \\ \hline v & \bar{u} \end{array}\right] = H + Q & \left[\begin{array}{c|c} \bar{u} & \bar{v} \\ \hline -v & u \end{array}\right] = H - Q, \\ (z = u + iv), & (\bar{z} = \bar{u} - i\bar{v}), \end{array} \quad (150)$$

$[W(z) \cdot W^*(z) = W^*(z) \cdot W(z),\ H = H^*,\ Q = -Q^*,\ HQ = QH].$

Complex Hermitean normal W-form (150) is implemented in a certain Cartesian base of the unitary space $\langle \mathcal{U}^{2\times 2} \rangle$. Its two eigenvalues are the complex conjugate numbers so as in (148). Hence, this complex normal form is simplified with some Hermitean orthogonal transformation U till converting into real W-form of type (148). The full set $\langle UWU^* \rangle$ is the *specified set of complex normal matrices*, that may by reduced by some modal transformations till canonical forms (150) and (148).

These normal matrices are interesting in Hermitean tensor trigonometry. Their conjugate eigenvalues are $d_t = \rho_t \exp(\pm i\beta_t)$, $\rho_t \in (-\infty + \infty)$, $\beta_t \in [-\pi/2; +\pi/2]$; for *Hermitean orthogonal matrices:* $d_t = \exp(\pm i\beta_t)$. Moreover, a pair of conjugate elements in their diagonal forms correspond to a trigonometric 2×2-cell of some Hermitean rotation for the geometric transformation of elements in a basic unitary space. (But general complex $n \times n$-normal matrices are simplified with some unitary transformations till their diagonal forms with n entries of the type $d_t = \rho_t \exp(i\beta_t)$!)

These questions are discussed in details in Part II, Ch. 10.

4.2 Examples of adequate complexification

Typical examples of *adequate complexification* are the following:

• formulae for roots of algebraic equations with complex coefficients,

• algebraic identities including ones of trigonometric nature,

• trigonometric formulae for complex angles and their functions,

• analytical (holomorphic) functions, their expansions into power series,

• formulae for derivatives, differentials and integrals for functions of scalar and vectorial complex arguments.

(Everywhere real-number elements are substituted by complex ones.)

In a space over $\mathbb{C}$ with an adequate metric, the measures of length and angles are necessary complex. However, in a pseudo-Euclidean space, these measures may be real, zero or imaginary. Give below the following main examples for the pseudo-Euclidean space of index $q = 1$ (see more in Part II, Chs. 6, 11, 12, and in the large Appendix):

• Minkowskian Geometry and pseudo-Euclidean tensor trigonometry in *elementary forms* as the additional important part of this Geometry,

• pseudo-spherical geometries for spheres of imaginary and real radius (i. e., of two types), embedded into pseudo-Euclidean space. (These two geometries with tensor hyperbolic and spherical functions in elementary forms are isometric to Lobachevsky–Bolyai and Beltrami geometries).

Consider examples of applications of the *adequate complexification in theory of analytical functions of scalar and vectorial complex variable* and in theory of matrices.

Let $\mathbf{x}, \mathbf{y} \in \langle \mathcal{E}^n \rangle$, and $\mathbf{z} = \mathbf{x} + \mathbf{iy}$ be an $(n \times 1)$-vector argument in a n-dimensional complex Euclidean space, $F(\mathbf{z}) = F_1(\mathbf{x}, \mathbf{y}) + iF_2(\mathbf{x}, \mathbf{y})$ be a certain scalar complex analytical function of $\mathbf{z}$. Differentiation and integration with respect to an $(n \times 1)$-vector-argument in the Euclidean space are expressed in Cartesian coordinates. Total derivatives, differentials, and integrals have adequate analogues from which partial characteristics and their relations are clear and obviously inferred:

$$dF = \mathbf{h}(\mathbf{z})d\mathbf{z} \ \Leftrightarrow \ dF = dF_1 + idF_2 = (\mathbf{h_1}(\mathbf{x}, \mathbf{y}) + i\mathbf{h_2}(\mathbf{x}, \mathbf{y}))(d\mathbf{x} + id\mathbf{y}) =$$

$$= [\mathbf{h_1}(\mathbf{x}, \mathbf{y})d\mathbf{x} - \mathbf{h_2}(\mathbf{x}, \mathbf{y})d\mathbf{y}] + i[\mathbf{h_1}(\mathbf{x}, \mathbf{y})d\mathbf{y} + \mathbf{h_2}(\mathbf{x}, \mathbf{y})d\mathbf{x}].$$

Here the $1 \times n$-vector partial derivatives (gradients) form pairs:

$$\left.\begin{aligned} \mathbf{h_1}(\mathbf{x},\mathbf{y}) &= \frac{\partial F_1}{\partial \mathbf{x}} = \frac{\partial F_2}{\partial \mathbf{y}}, \\ \mathbf{h_2}(\mathbf{x},\mathbf{y}) &= -\frac{\partial F_1}{\partial \mathbf{y}} = \frac{\partial F_2}{\partial \mathbf{x}}. \end{aligned}\right\} \qquad (a)$$

This is the vector-form of classical d'Alembert–Euler Equations for the scalar functions F_1, F_2 totally differentiable with respect to arguments $\mathbf{x}$, $\mathbf{y}$ (or for totality of two differential expressions above in square brackets).

Apply the same scheme of reasoning to the $1 \times n$-vector function

$\frac{dF}{d\mathbf{z}} = \mathbf{h}(\mathbf{z}) = \mathbf{h_1}(\mathbf{x},\mathbf{y}) + i\mathbf{h_2}(\mathbf{x},\mathbf{y})$:

$$\left.\begin{aligned} \frac{\partial \mathbf{h_1}}{\partial \mathbf{x}} &= \frac{\partial \mathbf{h_2}}{\partial \mathbf{y}} = \frac{\partial^2 F_1}{\partial \mathbf{x}^2} = -\frac{\partial^2 F_1}{\partial \mathbf{y}^2} = \frac{\partial^2 F_2}{\partial \mathbf{x}\partial \mathbf{y}} = \frac{\partial^2 F_2}{\partial \mathbf{y}\partial \mathbf{x}} = \left(\frac{\partial \mathbf{h_1}}{\partial \mathbf{x}}\right)', \\ \frac{\partial \mathbf{h_1}}{\partial \mathbf{y}} &= -\frac{\partial \mathbf{h_2}}{\partial \mathbf{x}} = \frac{\partial^2 F_2}{\partial \mathbf{y}^2} - \frac{\partial^2 F_2}{\partial \mathbf{x}^2} = \frac{\partial^2 F_1}{\partial \mathbf{y}\partial \mathbf{x}} = \frac{\partial^2 F_1}{\partial \mathbf{x}\partial \mathbf{y}} = \left(\frac{\partial \mathbf{h_1}}{\partial \mathbf{y}}\right)'. \end{aligned}\right\} \qquad (b)$$

The first equalities in chains (b) are the matrix-form d'Alembert–Euler equations for the vector functions $\mathbf{h_1}$ and $\mathbf{h_2}$ totally differentiable in terms of $\mathbf{x}$, $\mathbf{y}$. Together they express, as well as symmetry of Jacobi matrices due to symmetry of Hesse matrices, necessary and sufficient conditions for totality of the second differential F also in terms of $\mathbf{x}$, $\mathbf{y}$. The matrix-forms Laplace Equations for the harmonic functions F_1, F_2 of the real variables $\mathbf{x}$, $\mathbf{y}$ follow from the additional matrix equations in (b).

In a pseudo-Euclidean space $\langle \mathcal{E}^{n+q} \rangle$ (in the binary complex form), due to its special structure, the characteristics described above are changed:

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ i\mathbf{y} \end{bmatrix}; \quad dF = \mathbf{h}(\mathbf{z})d\mathbf{z} \ \Leftrightarrow \ dF = dF_1 + idF_2 =$$

$$= \left([\ \mathbf{h_1} \mid \mathbf{t_1}\ ] + i[\ \mathbf{h_2} \mid \mathbf{t_2}\ ] \right) \begin{bmatrix} d\mathbf{x} \\ \hline id\mathbf{y} \end{bmatrix} =$$

$$= [\mathbf{h_1}(\mathbf{x},\mathbf{y})d\mathbf{x} - \mathbf{t_2}(\mathbf{x},\mathbf{y})d\mathbf{y}] + i[\mathbf{t_1}(\mathbf{x},\mathbf{y})d\mathbf{y} + \mathbf{h_2}(\mathbf{x},\mathbf{y})d\mathbf{x}].$$

Here

$$\left.\begin{aligned} \mathbf{h_1}(\mathbf{x},\mathbf{y}) &= \frac{\partial F_1}{\partial \mathbf{x}}, \quad \mathbf{h_2}(\mathbf{x},\mathbf{y}) = \frac{\partial F_2}{\partial \mathbf{x}}, \\ \mathbf{t_1}(\mathbf{x},\mathbf{y}) &= \frac{\partial F_2}{\partial \mathbf{y}}, \quad \mathbf{t_2}(\mathbf{x},\mathbf{y}) = -\frac{\partial F_1}{\partial \mathbf{y}}; \end{aligned}\right\} \qquad (a')$$

$$\left.\begin{aligned} \frac{\partial \mathbf{h_1}}{\partial \mathbf{x}} &= \frac{\partial^2 F_1}{\partial \mathbf{x}^2} = \left(\frac{\partial \mathbf{h_1}}{\partial \mathbf{x}}\right)', \quad \frac{\partial \mathbf{t_1}}{\partial \mathbf{y}} = \frac{\partial^2 F_2}{\partial \mathbf{y}^2} = \left(\frac{\partial \mathbf{t_1}}{\partial \mathbf{y}}\right)', \\ \frac{\partial \mathbf{h_2}}{\partial \mathbf{x}} &= \frac{\partial^2 F_2}{\partial \mathbf{x}^2} = \left(\frac{\partial \mathbf{h_2}}{\partial \mathbf{x}}\right)', \quad \frac{\partial \mathbf{t_2}}{\partial \mathbf{y}} = -\frac{\partial^2 F_1}{\partial \mathbf{y}^2} = \left(\frac{\partial \mathbf{t_2}}{\partial \mathbf{y}}\right)', \\ \frac{\partial \mathbf{h_1}}{\partial \mathbf{y}} &= \frac{\partial^2 F_1}{\partial \mathbf{x}\partial \mathbf{y}} = \left(\frac{\partial^2 F_1}{\partial \mathbf{y}\partial \mathbf{x}}\right)' = -\left(\frac{\partial \mathbf{t_2}}{\partial \mathbf{x}}\right)', \\ \frac{\partial \mathbf{t_1}}{\partial \mathbf{x}} &= \frac{\partial^2 F_2}{\partial \mathbf{y}\partial \mathbf{x}} = \left(\frac{\partial^2 F_2}{\partial \mathbf{x}\partial \mathbf{y}}\right)' = \left(\frac{\partial \mathbf{h_2}}{\partial \mathbf{y}}\right)'. \end{aligned}\right\} \qquad (b')$$

In this case, $F_1(\mathbf{x},\mathbf{y})$, $F_2(\mathbf{x},\mathbf{y})$ are not harmonic in the Sense of Laplace.

The real analogues exist for *purely real* parameters used previously. In particular, for matrices they are the rank, the 1-st and 2-nd rock. Parallelism of linear objects is an affine property, that is why it does not depend on the complexification variant. However, optimal procedures for parallelism checking in a real space and complex one may differ.

Suppose that $n \times m$-matrices A_1 and A_2 determine linear subspaces (or linear objects) in the affine space $\langle \mathcal{A}^n \rangle$. The procedure for parallelism recognizing uses here characteristic symmetric projectors. If ranks of A_1 and A_2 are equal, then process (94) may be run in the simplest variant. In more general case, consider an $n \times n$-matrix with the same image, i. e., $\langle im\ AC \rangle \equiv \langle im\ A \rangle$, where C is an $m \times n$-matrix such that:

1) $\langle im\ C \rangle \cap \langle ker\ A \rangle = \mathbf{0} \ \Leftrightarrow\ rank\ AC = rank\ A$,

2) $k(AC, r) \neq 0$.

In a space over $\mathbb{R}$ one may put $C = A'$, in a space over $\mathbb{C}$ put $C = A^*$. In general, the following holds:

1. $\langle im\ A_2 \rangle \subseteq \langle im\ A_1 \rangle \ \Leftrightarrow\ \overleftarrow{A_1C_1} \cdot A_2 = A_2 \ \Leftrightarrow\ \overrightarrow{A_1C_1} \cdot A_2 = Z,$

   $\langle im\ A_1 \rangle \subseteq \langle im\ A_2 \rangle \ \Leftrightarrow\ \overleftarrow{A_2C_2} \cdot A_1 = A_1 \ \Leftrightarrow\ \overrightarrow{A_2C_2} \cdot A_1 = Z.$

2. $\langle im\ A_2 \rangle \equiv \langle im\ A_1 \rangle \ \Leftrightarrow\ \overrightarrow{A_1C_1} \cdot A_2 = Z = \overrightarrow{A_2C_2} \cdot A_1.$

On the other hand, orthogonality of linear objects is the notion depending on a metric in a given space.

In a real Euclidean space or in a complex Euclidean space with the adequate metric variant, orthogonality is recognized by the condition:

$$\langle im\ A_1 \rangle \perp \langle im\ A_2 \rangle \ \Leftrightarrow\ A_1'A_2 = Z = A_2'A_1.$$

But in a complex Euclidean space with the Hermitian metric variant, it is recognized by the condition:

$$\langle im\ A_1 \rangle \perp \langle im\ A_2 \rangle \ \Leftrightarrow\ A_1^*A_2 = Z = A_2^*A_1.$$

Here the both (left and right) conditions equations are equivalent.

4.3 Examples of Hermitean and symbiotic complexification

Hermitean complexification may be used almost in any case when it is necessary to decide problems in a complex space with vectorial objects. Therefore we indicate only some examples, most close to our theme:

• positive norms for lengths, surfaces, volumes etc. of a different geometric objects in the Hermitean space;

• positive norms for the angle and its functions in an Hermitean plane;

• previous results expressed in the self-conjugate form, in particular, formulae and inequalities (98)–(103), (115)–(130), (132)–(144), especially,

• • minorant positivity for the linear objects in an Hermitean space,

• • formulae (122) and (136) expressing the Kronecker–Capelli Theorem,

• • Hadamard and Cauchy Inequalities, they are important for the trigonometry in an Hermitean plane as the basis for definition of Hermitean spherical trigonometric functions of angles between vectors using the sine and cosine normalizing inequalities);

• limit functional methods (sect. 3.4);

• *maximum-modulus principle*, it holds for scalar and vectorial complex functions of complex variables.

Most general is the *symbiotic approach*. Its application to the classical theory of analytical functions and basic operations of calculus (orthogonal differentiation and integration) gives the following *symbiotic analogues*:

• expansions into power series in conjugate variables $\mathbf{z}$ and $\overline{\mathbf{z}}$ for special analytical nonholomorphic functions of $\mathbf{z}$ and $\overline{\mathbf{z}}$, i. e., these functions are not analytical in the Sense of Riemann,

• special rules of symbiotic (conjugate) differentiation and integration,

• special conditions for differentiability and analyticity for functions of conjugate variables $\mathbf{z}$ and $\overline{\mathbf{z}}$,

• special conditions for integrability of a certain differential expression (i. e., of the differential totality),

• symbiotic methods for finding extrema of scalar real functions of conjugate variables (the preliminary necessary condition to such scalar function is its symmetry with respect to the conjugate arguments). This is further development of formal derivatives idea (see, for example, [15]) in analysis of nonholomorphic complex-variable functions. We illustrate the extremal problems by the following two examples, close to our theme,

• • extrema of the scalar real functions (from sect. 1.2) expressing the differences or ratios of corresponding means formed of all the algebraic equation roots, if the roots are positive and complex conjugate (see methods of solving similar tasks in other our monograph [18, p. 124–135]),

• • minimizing squared Hermitean module of complex equation residual (116), i. e., scalar real function $F = ||A\mathbf{x} - \mathbf{a}||^2_H$ with inferring complex limit formulae (143, 144).

* * *

In conclusion of this introductory part I with the base for subsequent development of the tensor trigonometry in part II, the author considers it necessary to note the following. A lot of new provisions, characteristics and formulae of this part were established by the author else at the beginning of 1981. However, they were not accepted then to publications by reason of lack of understanding their scientific value; and this content was publicized many later, in 2004, in the author monograph [17]. In particular, this has place for the structure of matrix characteristic coefficients, for the new parameters of matrices singularity with fundamental relations and inequalities connecting them, for the explicit form of a minimal annulling polynomial, for the explicit formulae of all eigenprojectors and quasi-inverse matrices in terms of elements of an initial matrix, for the definition and applications of null-prime and null-normal matrices, for the new algebraic notions as a minorant and a dianal of a matrix with their useful properties in the theory of linear algebraic equations and so one.

But some content from this series began to appear later in publications from the same circle of mathematicians which did not accept all indicated above. For this reason, I did not consider it necessary to make references to these publications with their non-original results. The same applies to publications with borrowings from [17] after 2004. All of plagiarists were surpassed by the Ukrainian publishing house "Освіта України" issued my "Tensor Trigonometry -2004" after 10 years, in 2015, without changes and under other "author" name, with reviews of two professors–doctors of sciences!!!

Part II

Tensor Trigonometry: fundamental contents

This main part of the book begins by large Chapter 5 in which the Tensor Trigonometry in its concrete defining forms is exposed at first in Euclidean space and then in *quasi-Euclidean* one. These space has the same quadratic metric, but its transformations have to correspond not only to metric tensor $\{I^+\}$, but else to a set *reflector tensor*. This tensor is a symmetric matrix with eigenvalues -1 and $+1$, it is $\{I^\pm\}$ in the simplest form. It divides a space into direct sum of two parts corresponding to these eigenvalues!

In the first half of Chapter 5 (sect. 5.1–5.6) the projective version of Tensor Trigonometry is constructed. It is developing with using of eigenprojectors for the rectangular or square matrices. Projective spherical trigonometric functions and reflectors for the tensor angle between lineors A_1 and A_2 or their images $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ (i. e., planars of rank r_1 and r_2) are defined and considered. In other alternative interpretation, the tensor angle is defined by the same manner between two images of the singular null-prime $n \times n$-matrix $\langle im\ B\rangle$ and $\langle im\ B'\rangle$ (i. e., planars of rank r). Then canonical structures of projective tensor trigonometric functions and reflectors are installed.

In the second half of Chapter 5 (sect. 5.7–5.12) the motive (rotational and deformational) version of Tensor Trigonometry is constructed also in Euclidean and quasi-Euclidean spaces. The tensor trigonometric functions in this version represent rotations and deformations (i. e., sine-cosine and tangent-secant transformations). They are gotten also in the *elementary* forms, i. e., only with one main spherical *eigen angle* of motions and corresponding to the reflector tensor with index $q = 1$. The reflector tensor determines the mono-binary canonical structure in some Cartesian base for all the main notions of the *quasi-Euclidean trigonometry* besides its Euclidean metric.

In Chapter 6 the pseudo-Euclidean Tensor Trigonometry in pseudo-Euclidean space with the identical reflector and metric tensors $I^\pm$ and with corresponding to it sign-indefinite quadratic metric is constructed with the use of *abstract and concrete spherical-hyperbolic analogies*. The concrete analogy in covariant form as a very important one-to-one correspondence is defined and widely used for the sequential development of Tensor Trigonometry and its applications in non-Euclidean geometries. In addition, we exposed separately the complete solution of the pseudo-Euclidean time-like and space-like right triangles in a pseudoplane and in a pseudo-Euclidean space with relations between complementary angles. The abstract and concrete connections of the spherical and hyperbolic principal angles and their functions were given in the special Quart circle. For all geometries with the hyperbolic principal motions, the especial angle (number) ω is introduced as the hyperbolic analog of $\pi/4$ and which corresponds to a hyperbola focus for more descriptive analysis with scalar and tensor trigonometric approach.

In Chapter 7 the trigonometric nature of matrices commutativity and anticommutativity is considered as the separate important application.

In Chapter 8 complete trigonometric spectrums of a null-prime matrix is established, which serve as a basis for inferring the general cosine and sine normalizing inequalities. They give opportunity for correct defining trigonometric norms for the trigonometric functions and define bound with them sine and cosine relations for matrices.

In Chapter 9 the correct quadratic norms of matrices and lineors as some geometric objects are defined with the use of the general inequality for average values from Chapter 1.

In Chapters 10, 11 and 12 the Tensor Trigonometry is developed in different complex arithmetic metric spaces with alternative variants of complexification. Large attention is spared from the Tensor Trigonometry point of view to studying the motions in pseudo-Euclidean spaces and separately in the Minkowskian pseudo-Euclidean space with index $q = 1$, and in embedded into them two concomitant hyperbolic subspaces. Different trigonometric models of hyperbolic geometries in the large are considered. In conclusion, the Special mathematical principle of relativity is formulated for its use in the subsequent large Appendix.

Chapter 5

Euclidean and quasi-Euclidean tensor trigonometry

5.1 Objects of tensor trigonometry and their space relations

According to the Cantor–Dedekind Continuum Axiom [16, p. 99], affine and arithmetic spaces of the same dimensional are isomorphic, therefore their same metric forms are isomorphic too. Due to this, results, obtained by algebraic ways, may be geometrically interpreted; and vice versa. Primary elements of the n-dimensional affine space are, according to the axiomatic determination by Hermann Weyl [81, p. 26–33], points and free vectors. Their coordinates in a certain base are represented by n-tuples of numbers. Points and vectors form geometric objects. There are centralized and noncentralized geometric objects. Each centralized geometric object has its application point in the center of a given coordinates system. There is the following correspondence between the equivalent algebraic and geometric forms of linear objects in these two spaces $\langle \mathcal{A}^n \rangle$:

a vector $\mathbf{a}$	–	a straight line segment,
an image $\langle im\ \mathbf{a} \rangle$	–	a straight line,
a kernel $\langle ker\ \mathbf{a}' \rangle$	–	a hyperplane,
$n \times r$-lineor A of rank r	–	an r-simplex,
an image $\langle im\ A \rangle$	–	a planar of rank r,
a kernel $\langle ker\ A' \rangle$	–	a planar of rank $n - r$.

Note, that due to (100) there holds $\langle im\ A \rangle \oplus \langle ker\ A' \rangle \equiv \langle \mathcal{A}^n \rangle$ (direct orthogonal sum).

All these simplest linear objects of developing tensor trigonometry have a valency 1. A valency for nonanalitic functions of objects may be other. For example, the internal and external multiplications of two vectors have the valency respectively 0 and 2:

$$\mathbf{a_1'a_2} = c = \mathbf{a_2'a_1}, \tag{151}$$

$$\mathbf{a_1a_2'} = B = \{\mathbf{a_2a_1'}\}'. \tag{152}$$

Relations of parallelism and orthogonality (with eigenprojectors see also in Ch. 2) in affine and Euclidean spaces for the planars ($rank\ A_1 = r_1,\ rank\ A_2 = r_2$) are the following:

$$\left.\begin{array}{l} \langle im\ A_1 \rangle \equiv \langle im\ A_2 \rangle \ \Leftrightarrow\ \overleftarrow{A_1A_1'} = \overleftarrow{A_2A_2'} \ \Leftrightarrow \\ \Leftrightarrow\ \overrightarrow{A_1A_1'} = \overrightarrow{A_2A_2'} \ \Leftrightarrow\ \langle ker\ A_1' \rangle \equiv \langle ker\ A_2' \rangle, \end{array}\right\} \quad (r_1 = r_2); \tag{153}$$

$$\left.\begin{array}{l} \langle im\ A_2 \rangle \subseteq \langle im\ A_1 \rangle \ \Leftrightarrow\ \overleftarrow{A_1A_1'} \cdot \overleftarrow{A_2A_2'} = \overleftarrow{A_1A_1'} = \\ = \overleftarrow{A_2A_2'} \cdot \overleftarrow{A_1A_1'} \ \Leftrightarrow\ \overleftarrow{A_1A_1'} \cdot A_2 = A_2 \ \Leftrightarrow \\ \Leftrightarrow \overrightarrow{A_1A_1'} \cdot A_2 = Z = A_2' \cdot \overrightarrow{A_1A_1'} \Leftrightarrow \langle ker\ A_1' \rangle \subseteq \langle ker\ A_2' \rangle, \end{array}\right\} \quad (r_2 \leq r_1); \tag{154}$$

$$\left.\begin{array}{l} \langle im\ A_2\rangle \subseteq \langle ker\ A_1'\rangle \ \Leftrightarrow\ A_1'A_2 = Z_1,\ A_2'A_1 = Z_2 \ \Leftrightarrow \\ \Leftrightarrow\ \langle im\ A_1\rangle \subseteq \langle ker\ A_2'\rangle \ \Rightarrow\ \langle im\ A_1\rangle \cap \langle im\ A_2\rangle = \mathbf{0}, \end{array}\right\} \Rightarrow (r_1 + r_2 \leq n), \quad (155)$$

$$\left.\begin{array}{l} \langle ker\ A_1'\rangle \subseteq \langle im\ A_2\rangle \ \Leftrightarrow\ \overleftarrow{A_2A_2'} \cdot \overrightarrow{A_1A_1'} = \overrightarrow{A_1A_1'} \ \Leftrightarrow \\ \Leftrightarrow\ \overrightarrow{A_2A_2'} \cdot \overrightarrow{A_1A_1'} = Z = \overrightarrow{A_1A_1'} \cdot \overrightarrow{A_2A_2'} \Leftrightarrow \\ \Leftrightarrow\ \langle ker\ A_2'\rangle \subseteq \langle im\ A_1\rangle \ \Rightarrow\ \langle ker\ A_1'\rangle \cap \langle ker\ A_2'\rangle = \mathbf{0}, \end{array}\right\} \Rightarrow (r_1 + r_2 \geq n), \quad (156)$$

If the linear subspaces are defined by null-prime $n \times n$-matrices Bp (Part I, sect. 1.6), then their affine eigenprojectors may be used also, for example,

$$\langle im\ Bp_1\rangle \equiv \langle im\ Bp_2\rangle, \langle ker\ Bp_1\rangle \equiv \langle ker\ Bp_2\rangle \ \Leftrightarrow\ \overleftrightarrow{Bp_1} = \overleftrightarrow{Bp_2}; \qquad (157)$$

$$\left.\begin{array}{l} \langle im\ Bp_2\rangle \subseteq \langle im\ Bp_1\rangle \ \Leftrightarrow\ \overleftarrow{Bp_1} \cdot Bp_2 = Bp_2 \ \Leftrightarrow \\ \Leftrightarrow \overrightarrow{Bp_1} \cdot Bp_2 = Z = Bp_2' \cdot \overrightarrow{Bp_1'} \Leftrightarrow \langle ker\ Bp_1'\rangle \subseteq \langle ker\ Bp_2'\rangle. \end{array}\right\} \qquad (158)$$

Affine relations (153)–(156) between planars determined by lineors A_1 and A_2 of rank r_1 and r_2 may be naturally widen as follows. In the first extreme case, we have:

$$\left.\begin{array}{l} \langle im\ A_1\rangle \cap \langle im\ A_2\rangle = \mathbf{0} \ \Leftrightarrow\ rank\ (\overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'}) = \\ = r_1 + r_2 = rank\ (\overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'}) \leq n. \end{array}\right\} \qquad (159)$$

The image of the matrix $(\overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'})$ in any Cartesian base $\tilde{E}$ of an Euclidean space $\langle \mathcal{E}^n\rangle$ is the direct orthogonal sum $\langle im\ A_1\rangle \oplus \langle im\ A_2\rangle$ of dimension $(r_1 + r_2)$, and its kernel is the orthocomplement in the same $\langle \mathcal{E}^n\rangle$ to the image of dimension $n - (r_1 + r_2)$. In the second extreme case, we have:

$$\left.\begin{array}{l} \langle ker\ A_1'\rangle \cap \langle ker\ A_2'\rangle = \mathbf{0} \ \Leftrightarrow\ rank\ (\overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'}) = \\ = rank\ (\overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'}) = (n - r_1) + (n - r_2) \leq n. \end{array}\right\} \qquad (160)$$

Here the same matrix image, but in other interpretation $(\overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'})$, is the direct sum $\langle ker\ A_1'\rangle \oplus \langle ker\ A_2'\rangle$ of dimension $[(n - r_1) + (n - r_2)] = 2n - (r_1 + r_2)$, and its kernel is the orthocomplement in $\langle \mathcal{E}^n\rangle$ of dimension $(r_1 + r_2) - n$. Note, that (155) and (156) are only the special extreme cases of (159) and (160). Formulae (159) and (160) are compatible iff $n = r_1 + r_2$, i. e., in this especial case, there holds

$$\langle im\ A_1\rangle \oplus \langle im\ A_2\rangle \equiv \langle \mathcal{A}^n\rangle \equiv \langle ker\ A_1'\rangle \oplus \langle ker\ A_2'\rangle.$$

Under this condition, the matrix $(\overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'}) = (\overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'})$ is nonsingular. Similarly, in other cases, we have:

$$\langle im\ A_1\rangle \cap \langle im\ A_2\rangle \neq \mathbf{0} \ \Leftrightarrow\ rank\ (\overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'}) < r_1 + r_2, \qquad (161)$$

$$\langle ker\ A_1'\rangle \cap \langle ker\ A_2'\rangle \neq \mathbf{0} \ \Leftrightarrow\ rank\ (\overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'}) < 2n - (r_1 + r_2). \qquad (162)$$

This wonderful matrix (in brackets) give us the way for defining further the spherical trigonometric functions of tensor angles in terms of eigenprojectors.

5.2 The projective sines, cosines, and spherically orthogonal reflectors

The following matrix characteristic

$$\sin\tilde{\Phi}_{12} = \overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'} = \overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'} = \sin'\tilde{\Phi}_{12} = -\sin\tilde{\Phi}_{21} \tag{163}$$

is called the *projective tensor sine* of the angle between the planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ (or between the lineors A_1 and A_2. The projective nature of the angle is pointed out by the tilde upper character. We have:

$$\tilde{\Phi}_{12} = (\tilde{\Phi}_{12})' = -\tilde{\Phi}_{21}. \tag{164}$$

These properties (164) of a projective angle will be inferred further after converting with its sine into the canonical monobinary and diagonal forms.

According to (163), the angle between $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ is additively opposite to the angle between $\langle ker\ A_1'\rangle$ and $\langle ker\ A_2'\rangle$. These two angles together form the whole *binary structure* of $\tilde{\Phi}_{12}$. For example, the tensor sine of the angle between two non-oriented vectors or straight lines is

$$\sin\tilde{\Phi}_{12} = \overleftarrow{\mathbf{a_2a_2'}} - \overleftarrow{\mathbf{a_1a_1'}} = \frac{\mathbf{a_2a_2'}}{\mathbf{a_2'a_2}} - \frac{\mathbf{a_1a_1'}}{\mathbf{a_1'a_1}}. \tag{165}$$

In addition, its algebraic structure on an *Euclidean plane* $\langle\mathcal{E}^2\rangle$ is

$$\sin\tilde{\Phi}_{12} = \sin\varphi_{12}\sqrt{I_{2\times 2}}, \quad \sqrt{I_{2\times 2}} = R\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot R',$$

where φ_{12} is the counter-clockwise angle in the right Cartesian base, $|\varphi_{12}| \le \pi$ for vectors or $|\varphi_{12}| \le \pi/2$ for straight lines, R is some orthogonal modal matrix.

Condition $\sin\tilde{\Phi}_{12} = \tilde{\Phi}_{12} = Z$ means parallelism (153) of the planars. In common these planars may be noncentralized as $<\mathbf{a}_1 + \langle im\ A_1\rangle>$ and $<\mathbf{a}_2 + \langle im\ A_2\rangle>$.

Relations similar to (154) have trigonometric analogues too:

$$\langle im\ A_1\rangle \subseteq \langle im\ A_2\rangle \ \Leftrightarrow\ \sin^2\tilde{\Phi}_{12} = +\sin\tilde{\Phi}_{12}, \tag{166}$$

$$\langle im\ A_2\rangle \subseteq \langle im\ A_1\rangle \ \Leftrightarrow\ \sin^2\tilde{\Phi}_{12} = -\sin\tilde{\Phi}_{12}. \tag{167}$$

Indeed,

$$\sin^2\tilde{\Phi}_{12} = \overleftarrow{A_1A_1'}\cdot\overrightarrow{A_2A_2'} + \overleftarrow{A_2A_2'}\cdot\overrightarrow{A_1A_1'} = \overrightarrow{A_1A_1'}\cdot\overleftarrow{A_2A_2'} + \overrightarrow{A_2A_2'}\cdot\overleftarrow{A_1A_1'}. \tag{168}$$

For example, in the case of formula (167), it may be inferred as:

$$\langle im\ A_2\rangle \subseteq \langle im\ A_1\rangle \ \Leftrightarrow\ \langle ker\ A_1'\rangle \subseteq \langle ker\ A_2'\rangle \ \Leftrightarrow$$

$$\Leftrightarrow\ \overleftarrow{A_1A_1'}\cdot\overleftarrow{A_2A_2'} = \overleftarrow{A_2A_2'}\ ,\ \overleftarrow{A_2A_2'}\cdot\overrightarrow{A_1A_1'} = Z \ \Leftrightarrow\ \sin^2\tilde{\Phi}_{12} = -\sin\tilde{\Phi}_{12}.$$

In special case (166), the tensor sine is a symmetric projector (its eigenvalues are only 0 and +1); in special case (167) it is an antiprojector (the eigenvalues are 0 and −1).

Separate the class of *equirank* $n \times r$-lineors and planars. The planars may be determined also by any singular null-prime $n \times n$-matrices Bp (we shall denote these matrices briefly as B unless another sense is noted). The tensor angle between $\langle im\ B' \rangle$ and $\langle im\ B \rangle$ is additively opposite to the tensor angle between $\langle ker\ B \rangle$ and $\langle ker\ B' \rangle$. These two angles form entirely the whole binary structure of the projective tensor angle $\tilde{\Phi}_B$. Similarly to (163) and (164), there holds

$$\sin \tilde{\Phi}_B = \overleftarrow{B'B} - \overleftarrow{BB'} = \overrightarrow{BB'} - \overrightarrow{B'B} = \sin' \tilde{\Phi}_B = -\sin \tilde{\Phi}_{B'}; \tag{169}$$

$$\tilde{\Phi}_B = (\tilde{\Phi}_B)' = -\tilde{\Phi}_{B'}. \tag{170}$$

Condition $\sin \tilde{\Phi}_B = Z$ is equivalent to $\tilde{\Phi}_B = Z$ and $B \in \langle Bm \rangle$, it is the trigonometric interpretation of null-normal matrices (Part I, sect. 2.4).

By similar way, the trigonometric relations between the image and the kernel of two matrices A_1 and A_2 or B and B' are characterized by the *projective tensor cosine* of tensor angle $\tilde{\Phi}_{12}$ or $\tilde{\Phi}_B$:

$$\left.\begin{aligned} \cos \tilde{\Phi}_{12} &= \overleftarrow{A_2A_2'} - \overrightarrow{A_1A_1'} = \overleftarrow{A_1A_1'} - \overrightarrow{A_2A_2'} = \\ &= \overleftarrow{A_1A_1'} + \overleftarrow{A_2A_2'} - I = I - \overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'} = \\ &= \cos' \tilde{\Phi}_{12} = \cos \tilde{\Phi}_{21} = \cos\,(-\tilde{\Phi}_{12}), \end{aligned}\right\} \tag{171}$$

$$\left.\begin{aligned} &\cos \tilde{\Phi}_B = \overleftarrow{B'B} - \overrightarrow{BB'} = \overleftarrow{BB'} - \overrightarrow{B'B} = \overleftarrow{BB'} + \overleftarrow{B'B} - I = \\ &= I - \overrightarrow{BB'} - \overrightarrow{B'B} = \cos' \tilde{\Phi}_B = \cos \tilde{\Phi}_{B'} = \cos\,(-\tilde{\Phi}_{B'}). \end{aligned}\right\} \tag{172}$$

For two non-oriented vectors or straight lines on the Euclidean plane there holds:

$$\cos \tilde{\Phi}_{12} = \cos \varphi_{12} \sqrt{I_{2\times 2}}, \quad \sqrt{I_{2\times 2}} = R \cdot \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} \cdot R', \quad (\cos \varphi_{12} \geq 0).$$

The trigonometric analogues of conditions (155) and (156) follow from formula

$$\cos^2 \tilde{\Phi}_{12} = \overleftarrow{A_1A_1'} \cdot \overleftarrow{A_2A_2'} + \overrightarrow{A_2A_2'} \cdot \overrightarrow{A_1A_1'} = \overrightarrow{A_1A_1'} \cdot \overrightarrow{A_2A_2'} + \overleftarrow{A_2A_2'} \cdot \overleftarrow{A_1A_1'}. \tag{173}$$

Similarly to (168), equalities for the singular cosine (as projector and antiprojector)

$$\cos^2 \tilde{\Phi}_{12} = +\cos \tilde{\Phi}_{12} \leftrightarrow (156), \ \cos^2 \tilde{\Phi}_{12} = -\cos \tilde{\Phi}_{12} \leftrightarrow (155) \tag{174}$$

are equivalent to formulae (156) and (155), this follows from (173).

In an affine space $\langle \mathcal{A}^n \rangle$, the tensor angles have no quantitative sense unless they are zero or open. In an Euclidean space $\langle \mathcal{E}^n \rangle$ the angle have metric characteristics and determine the quantitative angular relations between lineors and between planars.

As one important example, for the tensor angle and its complement (with respect to *concordant* tensor right angle) $\tilde{\Xi} = (\tilde{\pi}/2 - \tilde{\Phi})$, there hold the usual relations:

$$\cos\tilde{\Phi} = \sin\tilde{\Xi}, \quad \sin\tilde{\Phi} = \cos\tilde{\Xi}. \tag{175}$$

$\sin\tilde{\Phi}_{Bm} = Z\ (\tilde{\Phi} = \tilde{Z}) \to \cos\tilde{\Phi}_{Bm} = \overleftarrow{Bm} - \overrightarrow{Bm} = I - 2\overrightarrow{Bm} = 2\overleftarrow{Bm} - I, \cos^2\tilde{\Phi}_{Bm} = Z.$ Right tensor angles are formed by the planars $\langle im\ A\rangle$ and $\langle ker\ A'\rangle$, $\langle im\ B\rangle$ and $\langle ker\ B'\rangle$ (see Part I, (100)). Hence, this relates to multiplicative null-normal matrices:

$$\overleftarrow{A_1A_1'} - \overrightarrow{A_1A_1'} = Ref\{A_1A_1'\} = Ref\{A_1A_1'\}^{-1} = \cos\tilde{\Phi}_{12} - \sin\tilde{\Phi}_{12} = \cos\tilde{Z}_1, \tag{176}$$

$$\overleftarrow{A_2A_2'} - \overrightarrow{A_2A_2'} = Ref\{A_2A_2'\} = Ref\{A_2A_2'\}^{-1} = \cos\tilde{\Phi}_{12} + \sin\tilde{\Phi}_{12} = \cos\tilde{Z}_2, \tag{177}$$

$$\overleftarrow{BB'} - \overrightarrow{BB'} = Ref\{BB'\} = Ref\{BB'\}^{-1} = \cos\tilde{\Phi}_B - \sin\tilde{\Phi}_B = \cos\tilde{Z}_B, \tag{178}$$

$$\overleftarrow{B'B} - \overrightarrow{B'B} = Ref\{B'B\} = Ref\{B'B\}^{-1} = \cos\tilde{\Phi}_B + \sin\tilde{\Phi}_B = \cos\tilde{Z}_{B'}. \tag{179}$$

Here $\cos^2\tilde{Z} = I$. They are the zero tensor angles cosines corresponding to planars $\langle im\ A_1\rangle, \langle im\ A_2\rangle, \langle im\ B\rangle, \langle im\ B'\rangle$ and the sines of the indicated tensor right angles. The symmetric square roots (176)–(179) such as $\sqrt{I} = (\sqrt{I})'$ are called *orthogonal spherical eigenreflectors.* The variant $Ref\{Bm\} = \pm(\cos\tilde{\Phi} \mp \sin\tilde{\Phi})$ is used also for them ($\tilde{\Phi}$ is variable projective tensor angle). The symmetric tensor reflectors carry out the *orthogonal reflection*, namely: $+Ref\{BB'\}$ with respect to the *mirror* $\langle ker\ B'\rangle$ (*to the orthocomplement of* $\langle im\ B\rangle$); $-Ref\{BB'\}$ with respect to the *mirror* $\langle im\ B\rangle$. This is inferred with the use of (178) and (100). Note special extreme cases:

$$\sin\tilde{\Phi} = \tilde{Z} \ \Leftrightarrow\ \cos\tilde{\Phi} \subset \langle\sqrt{I_{n\times n}}\ \rangle,\ \cos\tilde{\Phi} = Z \ \Leftrightarrow\ \tilde{\Phi} = \tilde{\pi}/2 \ \Leftrightarrow\ \sin\tilde{\Phi} \subset \langle\sqrt{I_{n\times n}}\ \rangle,$$

$$\cos\tilde{\Phi} = +I \ \Leftrightarrow\ rank\ A = rank\ B = n, \quad \cos\tilde{\Phi} = -I \ \Leftrightarrow\ rank\ A = rank\ B = 0;$$

$$\sin\tilde{\Phi}_{12} = +I \ \Leftrightarrow\ r_1 = 0,\ r_2 = n, \ \ \sin\tilde{\Phi}_{12} = -I \ \Leftrightarrow\ r_1 = n,\ r_2 = 0; \quad (\sin\tilde{\Phi}_B \neq \pm I).$$

The following identities equivalent to $I\cdot I = I = I \cdot I$ are clearly valid:

$$(\overleftarrow{A_1A_1'} + \overrightarrow{A_1A_1'})(\overleftarrow{A_2A_2'} + \overrightarrow{A_2A_2'} = I = (\overleftarrow{A_2A_2'} + \overrightarrow{A_2A_2'})(\overleftarrow{A_1A_1'} + \overrightarrow{A_1A_1'}), \tag{180}$$

$$(\overleftarrow{BB'} + \overrightarrow{BB'})(\overleftarrow{B'B} + \overrightarrow{B'B} = I = (\overleftarrow{B'B} + \overrightarrow{B'B})(\overleftarrow{BB'} + \overrightarrow{BB'}). \tag{181}$$

They give trigonometric formulae for a sine-cosine pair in the projective version:

$$\sin^2\tilde{\Phi} + \cos^2\tilde{\Phi} = I = \cos^2\tilde{\Xi} + \sin^2\tilde{\Xi} \ \text{ (Ptolemy Invariant)}, \tag{182}$$

$$\sin\tilde{\Phi}\cdot\cos\tilde{\Phi} = -\cos\tilde{\Phi}\cdot\sin\tilde{\Phi}, \tag{183}$$

$$\sin^2\tilde{\Phi}\cdot\cos^2\tilde{\Phi} = \cos^2\tilde{\Phi}\cdot\sin^2\tilde{\Phi}, \tag{184}$$

$$\sin^{2k}\tilde{\Phi}\cdot\cos^t\tilde{\Phi} = \cos^t\tilde{\Phi}\cdot\sin^{2k}\tilde{\Phi}, \quad \sin^t\tilde{\Phi}\cdot\cos^{2k}\tilde{\Phi} = \cos^{2k}\tilde{\Phi}\cdot\sin^t\tilde{\Phi}. \tag{185}$$

Note, that *the projective sine-cosine tensor pair is anticommutative.*

Trigonometric formulae may be inferred more easily and clearly with applicaton of the *Table of multiplication for miscellaneous eigenprojectors*:

$$\boxed{\begin{array}{ll}
\overleftarrow{B}\cdot\overleftarrow{BB'}=\overleftarrow{BB'}=\overleftarrow{BB'}\cdot\overleftarrow{B'}, & \overrightarrow{B}\cdot\overrightarrow{B'B}=\overrightarrow{B'B}=\overrightarrow{B'B}\cdot\overrightarrow{B'},\\
\overleftarrow{B'}\cdot\overleftarrow{B'B}=\overleftarrow{B'B}=\overleftarrow{B'B}\cdot\overleftarrow{B}, & \overrightarrow{B'}\cdot\overrightarrow{BB'}=\overrightarrow{BB'}=\overrightarrow{BB'}\cdot\overrightarrow{B},\\
\overleftarrow{B}\cdot\overleftarrow{B'B}=\overleftarrow{B}=\overleftarrow{BB'}\cdot\overleftarrow{B}, & \overrightarrow{B}\cdot\overrightarrow{BB'}=\overrightarrow{B}=\overrightarrow{B'B}\cdot\overrightarrow{B},\\
\overleftarrow{B'}\cdot\overleftarrow{BB'}=\overleftarrow{B'}=\overleftarrow{B'B}\cdot\overleftarrow{B'}, & \overrightarrow{B'}\cdot\overrightarrow{B'B}=\overrightarrow{B'}=\overrightarrow{BB'}\cdot\overrightarrow{B'}.
\end{array}}$$

This Table may be inferred according to the main properties of eigenprojectors, in that number with use of transposition operations.

Projective nature of introduced trigonometric functions is illustrated by the cosine formulae, associated with solving a right triangle:

$$\overleftarrow{BB'}=+\overleftarrow{B}\cdot\cos\tilde{\Phi}=+\cos\tilde{\Phi}\cdot\overleftarrow{B'}, \tag{186}$$

$$\overleftarrow{B'B}=+\overleftarrow{B'}\cdot\cos\tilde{\Phi}=+\cos\tilde{\Phi}\cdot\overleftarrow{B}, \tag{187}$$

$$\overrightarrow{B'B}=-\overrightarrow{B}\cdot\cos\tilde{\Phi}=-\cos\tilde{\Phi}\cdot\overrightarrow{B'}, \tag{188}$$

$$\overrightarrow{BB'}=-\overrightarrow{B'}\cdot\cos\tilde{\Phi}=-\cos\tilde{\Phi}\cdot\overrightarrow{B}, \tag{189}$$

In the Euclidean space $\overleftarrow{B}$ and $\overrightarrow{B}$ are the oblique eigenprojectors (see Part I, sect. 2.1). Here they play a role of the hypotenuse in such tensor triangles.

But the sine formulae give the surprising four nilpotent legs:

$$\overleftarrow{B}-\overleftarrow{BB'}=+(\sqrt{Z})_1=+\overleftarrow{B}\cdot\sin\tilde{\Phi}=+\overleftarrow{B}\cdot\overrightarrow{BB'}=-\overleftarrow{BB'}\cdot\overrightarrow{B}, \tag{190}$$

$$\overrightarrow{B}-\overrightarrow{B'B}=+(\sqrt{Z})_2=+\overrightarrow{B}\cdot\sin\tilde{\Phi}=-\overrightarrow{B'B}\cdot\overleftarrow{B}=+\overrightarrow{B}\cdot\overleftarrow{B'B}, \tag{191}$$

$$\overleftarrow{B'}-\overleftarrow{B'B}=-(\sqrt{Z})'_2=-\overleftarrow{B'}\cdot\sin\tilde{\Phi}=-\overleftarrow{B'B}\cdot\overrightarrow{B'}=+\overleftarrow{B'}\cdot\overrightarrow{B'B}, \tag{192}$$

$$\overrightarrow{B'}-\overrightarrow{BB'}=-(\sqrt{Z})'_1=-\overrightarrow{B'}\cdot\sin\tilde{\Phi}=+\overrightarrow{B}\cdot\overleftarrow{BB'}=-\overrightarrow{BB'}\cdot\overleftarrow{B'}, \tag{193}$$

(When these formulae are transposed, then the sine sign changes.) The differences of oblique and orthogonal projectors of the same type are nilpotent matrices of order 2.

Quadrating and multiplying of simple formulae (186)–(189) give the cosine formulae for the multiplications of oblique as well as orthogonal projectors of the same type:

$$\overleftarrow{BB'}=(\overleftarrow{B}\cdot\cos\tilde{\Phi})^2=\overleftarrow{B}\cdot\overleftarrow{B'}\cdot\cos^2\tilde{\Phi}=\overleftarrow{B}\cdot\cos^2\tilde{\Phi}\cdot\overleftarrow{B'}=\cos^2\tilde{\Phi}\cdot\overleftarrow{B}\overleftarrow{B'}, \tag{194}$$

$$\overrightarrow{BB'}=(-\cos\tilde{\Phi}\cdot\overrightarrow{B})^2=\overrightarrow{B'}\cdot\overrightarrow{B}\cdot\cos^2\tilde{\Phi}=\overrightarrow{B'}\cdot\cos^2\tilde{\Phi}\cdot\overrightarrow{B}=\cos^2\tilde{\Phi}\cdot\overrightarrow{B'}\overrightarrow{B}, \tag{195}$$

$$\overleftarrow{BB'}\cdot\overleftarrow{B'B}=(\overleftarrow{B}\cdot\cos\tilde{\Phi})\cdot(\overleftarrow{B'}\cdot\cos\tilde{\Phi})=\cos^2\tilde{\Phi}\cdot\overleftarrow{B}=\overleftarrow{B}\cdot\cos^2\tilde{\Phi}, \tag{196}$$

$$\overrightarrow{B'B}\cdot\overrightarrow{BB'}=(-\cos\tilde{\Phi}\cdot\overrightarrow{B'})\cdot(-\cos\tilde{\Phi}\cdot\overrightarrow{B})=\cos^2\tilde{\Phi}\cdot\overrightarrow{B}=\overrightarrow{B}\cdot\cos^2\tilde{\Phi}. \tag{197}$$

Projective trigonometric nature of the tensor angles is illustrated with the *symbolic tensor octahedron* formed by eight eigenprojectors of null-prime B in 2-valent $\langle \mathcal{E}^{n\times n}\rangle$ (Figure 1). For null-normal B, this octahedron is reduced to the right triangle.

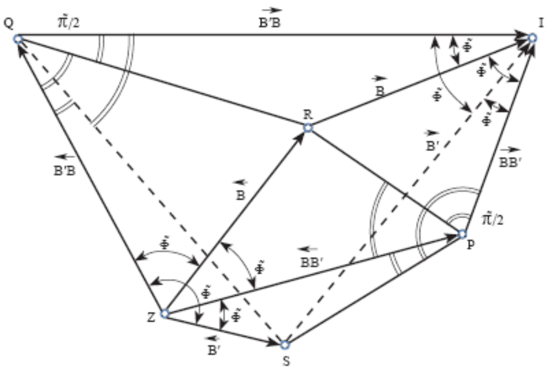

Figure 1. Symbolic tensor octahedron from 8 eigenprojectors for illustration of the projective tensor angles.

5.3 The projective secant, tangent, and affine (oblique) reflectors

The tensor secant (and further tangent) of a projective angle is defined in terms of oblique eigenprojectors. The matrix trigonometric function

$$\left.\begin{array}{l} \sec\tilde{\Phi}_B = \overleftarrow{B'} - \overrightarrow{B} = \overleftarrow{B} - \overrightarrow{B'} = \overleftarrow{B} + \overleftarrow{B'} - I = \\ = I - \overrightarrow{B} - \overrightarrow{B'} = \sec'\ \tilde{\Phi}_B = \sec\tilde{\Phi}_{B'} = \sec\,(-\tilde{\Phi}_B) = \\ = (\overleftarrow{B'})' - \overrightarrow{B} = \overleftarrow{B} - (\overrightarrow{B})' \end{array}\right\} \qquad (198)$$

is called *the projective tensor secant* of the tensor angle $\tilde{\Phi}_B$.

These formulae are easily inferred by the following way. Summation of (186) and (189), (187) and (188) gives

$$(\overleftarrow{B'} - \overrightarrow{B})\cdot\cos\tilde{\Phi} = \cos\tilde{\Phi}\cdot(\overleftarrow{B'} - \overrightarrow{B}) = I = \cos\tilde{\Phi}\cdot(\overleftarrow{B} - \overrightarrow{B'}) = (\overleftarrow{B} - \overrightarrow{B'})\cdot\cos\tilde{\Phi}.$$

These equalities determine the tensor secant.

According to (172), $\cos\tilde{\Phi}_B$ is nonsingular iff $\langle im\ B\rangle \cap \langle ker\ B\rangle = \mathbf{0}$, i. e., $B \in \langle Bp\rangle$ is a null-prime matrix (see Part I, sect. 1.6), therefore,

$$\sec\tilde{\Phi}_{Bp} = \cos^{-1}\tilde{\Phi}_{Bp}, \quad \sec\tilde{\Phi}_{Bp}\cdot\cos\tilde{\Phi}_{Bp} = I = \cos\tilde{\Phi}_{Bp}\cdot\sec\tilde{\Phi}_{Bp}; \qquad (199)$$

The matrix B may be null-defective, and there may exist no oblique eigenprojectors. Then the cosine of angle $\tilde{\Phi}_B$ is the zero matrix on the subspace $\langle im\ B\rangle \cap \langle ker\ B\rangle$ and

$$\sec\ \tilde{\Phi}_B = \cos^{+}\tilde{\Phi}_B, \quad \sec\tilde{\Phi}\cdot\cos\tilde{\Phi} = \overleftarrow{\cos\tilde{\Phi}} = \cos\tilde{\Phi}\cdot\sec\tilde{\Phi}. \qquad (200)$$

The formal definition of the tensor secant as *quasi-secant* takes advantage of the quasi-inverse Moor–Penrose matrix (see Part I, sect. 2.5) for the inversion of the singular tensor cosine. (Recall, that its matrix is symmetrical.) In this case, the multiplication of the tensor cosine and quasi-secant is the orthoprojector in formula (200). From the other hand, for a null-defective matrix B, the cosine of the angle between the subspaces $\langle im\ B^{s^0}\rangle$ and $\langle im\ (B')^{s^0}\rangle$ is a nonsingular matrix. Note, that for the null-normal matrix the tensor angle between $\langle im\ B\rangle$ and $\langle ker\ B\rangle$ is right. But for the main tensor angle and its functions, in the case, we have:

$$\sin\tilde{\Phi}_B = Z \ \Leftrightarrow\ \cos\tilde{\Phi}_B = \sqrt{I},\ \cos^2\tilde{\Phi}_B = I,\ \sec\tilde{\Phi}_B = \cos^{-1}\tilde{\Phi}_B.$$

For the tensor sine in the especial case, if $B \in \langle Bp\rangle$ and $r_B = n/2$, there holds

$$det\sin\tilde{\Phi}_B \neq 0 \ \Leftrightarrow\ \langle im\ B\rangle \cap \langle im\ B'\rangle = \mathbf{0},\ \langle ker\ B\rangle \cap \langle ker\ B'\rangle = \mathbf{0}. \tag{201}$$

If the same tensor angle is defined by lineors A_1 and A_2, then conditions (159) and (160) should hold simultaneously. In other cases, the tensor sine is a singular matrix, and the *quasi-cosecant* is defined in terms of the quasi-inverse Moor–Penrose matrix:

$$\operatorname{cosec}\tilde{\Phi}_B = \sin^{+}\tilde{\Phi}_B = \operatorname{cosec}'\ \tilde{\Phi}_B = -\operatorname{cosec}\tilde{\Phi}_{B'} = -\operatorname{cosec}(-\tilde{\Phi}_B) = \sec\tilde{\Xi}. \tag{202}$$

Further, subtracting (186) and (187) gives

$$\sin\tilde{\Phi}_B = -\cos\tilde{\Phi}_B\cdot(\overleftarrow{B'} - \overleftarrow{B}) = +(\overleftarrow{B'} - \overleftarrow{B})\cdot\cos\tilde{\Phi}_B.$$

These equalities determine the tensor function

$$\left.\begin{aligned} &i\tan\tilde{\Phi}_B = \overleftarrow{B'} - \overleftarrow{B} = \overrightarrow{B} - \overrightarrow{B'} = (\overleftarrow{B})\,' - \overleftarrow{B}) = \\ &= \overrightarrow{B} - (\overrightarrow{B})\,' = -(i\tan\tilde{\Phi}_B)' = -i\tan\Phi_{B'} = -i\tan(-\Phi_B), \end{aligned}\right\} \tag{203}$$

called the *projective realificated tensor tangent* of $\tilde{\Phi}_B$. In the realificated form it is a *real valued skewsymmetric matrix* with the eigenvalues $\mu_j = \pm i\tan\varphi_j$. Moreover (see also sect. 5.5 and 7), there hold the following anticommutative paired relations (!):

$$\left.\begin{aligned} &i\tan\tilde{\Phi} = +\sin\tilde{\Phi}\cdot\sec\tilde{\Phi} = -\sec\tilde{\Phi}\cdot\sin\tilde{\Phi} \leftrightarrow \\ &\leftrightarrow\ \sin\tilde{\Phi} = +i\tan\tilde{\Phi}\cdot\cos\tilde{\Phi} = -\cos\tilde{\Phi}\cdot i\tan\tilde{\Phi} \rightarrow \\ &\rightarrow +\sin\tilde{\Phi}\cdot i\tan\tilde{\Phi} = -i\tan\tilde{\Phi}\cdot\sin\tilde{\Phi}. \end{aligned}\right\} \tag{204}$$

For two vectors or two straight lines, due to (151) and (152), there holds

$$i\tan\tilde{\Phi}_B = \frac{B'}{trB'} - \frac{B}{trB} = \frac{B' - B}{trB} = \frac{\mathbf{a_2a_1'}}{\mathbf{a_1'a_2}} - \frac{\mathbf{a_1a_2'}}{\mathbf{a_2'a_1}} = \frac{\mathbf{a_2a_1'} - \mathbf{a_1a_2'}}{\mathbf{a_1'a_2}} = i\tan\tilde{\Phi}_{12}. \tag{205}$$

Its structure is $[i\tan\tilde{\Phi}_{12} = \tan\varphi_{12}\sqrt{I_{2\times2}}\ ]$, $\sqrt{I_{2\times2}} = R\cdot\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}\cdot R'$.

The *realificated quasi-cotangent* is defined, in the general case, as

$$i\cot\tilde{\Phi}_B = i\tan^{+}\tilde{\Phi}_B = -i\cot'\tilde{\Phi}_B = -i\cot\tilde{\Phi}_{B'} = -i\cot(-\tilde{\Phi}_B) = i\tan\tilde{\Xi}_B. \tag{206}$$

The following identities are affine (oblique) analogs of identities (181):

$$(\overleftarrow{B} + \overrightarrow{B}) \cdot (\overleftarrow{B'} + \overrightarrow{B'}) = I = (\overleftarrow{B'} + \overrightarrow{B'}) \cdot (\overleftarrow{B} + \overrightarrow{B}) \tag{207}$$

They are clearly valid for the null-prime matrices. Further, trigonometric formulae

$$\sec^2 \tilde{\Phi} - \tan^2 \tilde{\Phi} = I = \sec^2 \tilde{\Phi} + (i \tan \tilde{\Phi})^2 = \operatorname{cosec}^2 \tilde{\Xi} - \cot^2 \tilde{\Xi}, \tag{208}$$

$$= +i \tan \tilde{\Phi} \cdot \sec \tilde{\Phi} = -\sec \tilde{\Phi} \cdot i \tan \tilde{\Phi}, \tag{209}$$

$$= \tan^2 \tilde{\Phi} \cdot \sec^2 \tilde{\Phi} = \sec^2 \tilde{\Phi} \cdot \tan^2 \tilde{\Phi}. \tag{210}$$

complement formulae (182)–(184) for the tensor sine-cosine *anticommutative pair.* Note, $\tan \tilde{\Phi}$ is a *true projective tensor tangent* with the eigenvalues $\mu_j = \pm \tan \varphi_j$.

Thus we expressed by formulae all the trigonometric functions of tensor angles in the projective version of the tensor trigonometry. On the Euclidean plane the formulae are applicable too. Here they determine exactly the orientation of tensor angles, but their invariants determine also classic scalar trigonometry. A main angle is formed by two vectors, a mutual angle is formed by their orthogonal complements-vectors.

Rule 1. *Square and any even degrees of all the tensor trigonometric functions of the same angle (i. e., for the same pair of lineors or planars) commute with each other, with all the eigenprojectors and all the eigenreflectors.*

If B is null-prime matrix (but not null-normal one), then its *oblique spherical eigenreflectors* (as deforming with reflection) are defined similarly to formulae (176)–(179) in terms of the oblique eigenprojectors (see Part I, (60)):

$$\overleftarrow{B} - \overrightarrow{B} = I - 2\overrightarrow{B} = Ref\{B\} = Ref\{B\}^{-1} = \sec \tilde{\Phi}_B - i \tan \tilde{\Phi}_B = \sec \tilde{Z}_B \ , \tag{211}$$

$$\overleftarrow{B'} - \overrightarrow{B'} = I - 2\overrightarrow{B'} = Ref\{B'\} = Ref\{B'\}^{-1} = \sec \tilde{\Phi}_B + i \tan \tilde{\Phi}_B = Ref'\{B\}. \tag{212}$$

Here $\sec \tilde{Z}_B = \cos^{-1} \tilde{Z}_B \Leftrightarrow B \in \langle Bp \rangle$, $\tilde{Z}_B$ as in (178). From the algebraic point of view, they are nonsymmetric square roots such as $\sqrt{I}$. The functional variant $Ref\{Bp\}(\Phi) = \pm(\sec \tilde{\Phi} \pm i \tan \tilde{\Phi})$ is used too (as in sect. 5.2). These *nonsymmetric tensorial eigenreflectors* carry out the *oblique reflection*, namely:

$+Ref\{Bp\}$ with respect to the linear mirror $\langle ker\ B \rangle$ parallel to $\langle im\ B \rangle$,

$-Ref\{Bp\}$ with respect to the linear mirror $\langle im\ B \rangle$ parallel to $\langle ker\ B \rangle$.

They are inferred with use of (211) and (60). But iff Bp is a null-normal matrix Bm, then square roots (211) and (212) are symmetric, i. e., transformed into (178), (179).

Each symmetric and nonsymmetric prime square roots of I geometrically are orthogonal and oblique reflectors. Moreover, each pair of the same roots corresponds to a unique pair of mutual eigenprojectors and to a unique pair of mutual (i. e., sine-cosine and tangent-secant) projective tensor trigonometric functions (see more in sect. 5.6).

Reflectors are nonsingular matrices, because in their defining formulae we have that ranks of both matrices (left and right) are summated with their sum equaled to n. (These questions will be consider in details in the following sect. 5.6., 5.7, 5.10.)

5.4 Comparison of two ways for defining projective tensor angles

These ways for the angles $\tilde{\Phi}_{12}$ and $\tilde{\Phi}_B$, are the following:

• in terms of $n \times m$-matrices of lineors A_1 and A_2, as geometric objects;

• in terms of $n \times m$-matrices B and B' (as multiplication of the lineors). Both these ways have already been used before (see Part I, sect. 3.3).

Find general conditions under which *tensor angle* $\tilde{\Phi}$ *and its trigonometric functions do not depend on a choice of the way from these two ways of the tensor angle defining.*

In accordance with the initial definitions in Part I (sect. 3.1), put:

$$B = A_1 A_2', \quad B' = A_2 A_1'; \qquad (213),(214)$$

$$C = A_1' A_2, \quad C' = A_2' A_1. \qquad (215),(216)$$

Then the matrices A_1 and A_2 should have the same sizes. Moreover, from the identity of the two tensor angles, i. e., $\tilde{\Phi}_{12} = \tilde{\Phi}_B$, the equalities of their projective sine-cosine trigonometric functions follow as well as the equalities of the corresponding orthogonal eigenprojectors (bound with the angles by exact formulae) follow too; and vice versa:

$$\tilde{\Phi}_{12} = \tilde{\Phi}_B \Leftrightarrow (\sin \tilde{\Phi}_{12} = \sin \tilde{\Phi}_B,\ \cos \tilde{\Phi}_{12} = \cos \tilde{\Phi}_B) \Leftrightarrow$$

$$\Leftrightarrow (\overleftrightarrow{A_1 A_1'} = \overleftrightarrow{BB'},\ \overleftrightarrow{A_2 A_2'} = \overleftrightarrow{B'B}).$$

Note, however, the equalities of the corresponding affine (oblique) eigenprojectors $\overleftrightarrow{A_1 A_2'} = \overset{\rightleftarrows}{B}$ (bound with the angle by other formulae) follow from definitions (213)–(214). These additional equalities are valid due to only existence of concrete affine projectors (sect. 2.1). For their existence in the case, see below condition (230).

Equality of the orthoprojectors is equivalent to the following relations:

$$\langle im\ A_1 \rangle \equiv \langle im\ B \rangle \quad \Leftrightarrow \quad \langle ker\ A_1' \rangle \equiv \langle ker\ B' \rangle, \qquad (217),(218)$$

$$\langle im\ A_2 \rangle \equiv \langle im\ B' \rangle \quad \Leftrightarrow \quad \langle ker\ A_2' \rangle \equiv \langle ker\ B \rangle. \qquad (219),(220)$$

In their turn, the pairs of relations (217), (218) and (219), (220) are equivalent each to another due to the well-known fact, that the left and right sub-spaces in these pairs are complements each to another in $\langle \mathcal{A}^n \rangle$ and orthogonal ones in $\langle \mathcal{E}^n \rangle$ – see in Part I this well-known property (100).

At first, consider, when conditions (217) are valid. Obviously, that

$$\langle im\ B \rangle \equiv A_1 \langle im\ A_2' \rangle \ \Leftarrow\ B = A_1 A_2',$$

$$\langle im\ A_1 \rangle \equiv A_1 \langle \mathcal{A}^{r_2} \rangle \equiv A_1 (\langle im\ A_2' \rangle \oplus \langle ker\ A_2 \rangle).$$

Therefore (217) is equivalent to the pair of obvious conditions in (213):

$$\langle im\ A_2' \rangle \cap \langle ker\ A_1 \rangle = \mathbf{0}, \quad \langle ker\ A_2 \rangle \subset \langle ker\ A_1 \rangle. \qquad (221)$$

Similarly, (219) is equivalent to the pair of obvious conditions in (214):

$$\langle im\ A_1'\rangle \cap \langle ker\ A_2\rangle = \mathbf{0}, \quad \langle ker\ A_1\rangle \subset \langle ker\ A_2\rangle. \tag{222}$$

It is seen that independent conditions (217), (219) hold simultaneously iff

$$\left.\begin{array}{l} \langle ker\ A_1\rangle \equiv \langle ker\ A_2\rangle \ \Leftrightarrow\ \langle im\ A_1'\rangle \equiv \langle im\ A_2'\rangle \Leftrightarrow \\ \Leftrightarrow\ \overrightarrow{A_1'A_1} = \overrightarrow{A_2'A_2} \ \Leftrightarrow\ \overleftarrow{A_1'A_1} = \overleftarrow{A_2'A_2} \end{array}\right\} \tag{223}$$

and where it is necessary $r_1 = r_2 \le m$.

Thus (223) is the necessary and sufficient condition answering the problem from beginning of the section. Obviously, (223) also implies the very simple and useful sufficient condition $r_1 = r_2 = r = m$. This condition, in its turn, has simple corollaries

$$\langle ker\ A_1\rangle \equiv \langle ker\ A_2\rangle = \mathbf{0}, \qquad \langle im\ A_1'\rangle \equiv \langle im\ A_2'\rangle \equiv \langle \mathcal{A}^r\rangle.$$

This special case is implied when one deals with external and internal multiplications such as (213)–(216) for these so called *equirank lineors* A_1 and A_2 under condition

$$r_1 = r_2 = r = m < n. \tag{224}$$

(This holds always for two vectors.) From (120) and (213)–(216) we have

$$k(B, r) = k(B', r) = det\ C = det\ C'. \tag{225}$$

If B is null-prime matrix, then $\langle im\ B\rangle \cap \langle ker\ B\rangle = \mathbf{0}$ and $k(B, r) = det\ C \neq 0$. In the case, if B is null-normal matrix, then $\langle im\ B\rangle \equiv \langle im\ B'\rangle$ and due to (97) (see Part I, sect. 2.4) we have $k(BB', r) = k(B'B, r) = k^2(B, r) = det^2 C > 0$. However if B is null-defective matrix, then $\langle im\ B\rangle \cap \langle ker\ B\rangle \neq \mathbf{0}$ and $k(B, r) = det\ C = 0$.

Under general condition (223) or particular condition (224), there holds

$$\overleftrightarrow{A_1A_1'} = \overleftrightarrow{BB'}, \qquad \overleftrightarrow{A_2A_2'} = \overleftrightarrow{B'B}. \tag{226}$$

In an affine space, the characteristic $det\ G = det[(A_1|A_2)'(A_1|A_2)]$ is the criterion for *at least* partial parallelism of these planars or partial *coplanarity* of these lineors – see this in sect. 9.4. In an Euclidean space, the characteristic $det\ C = det(A_1'A_2)$, under condition (224), is the criterion for *at least* their partial orthogonality.

$$det\ G = 0 \ \Leftrightarrow\ \langle im\ A_1\rangle \cap \langle im\ A_2\rangle \neq \mathbf{0}, \tag{227}$$

$$det\ G \neq 0 \ \Leftrightarrow\ \langle im\ A_1\rangle \cap \langle im\ A_2\rangle = \mathbf{0}, \tag{228}$$

$$det\ C = 0 \Leftrightarrow \langle im\ A_1\rangle \cap \langle ker\ A_2'\rangle \neq \mathbf{0} \Leftrightarrow \langle im\ A_2\rangle \cap \langle ker\ A_1'\rangle \neq \mathbf{0}, \tag{229}$$

$$det\ C \neq 0 \Leftrightarrow \langle im\ A_1\rangle \cap \langle ker\ A_2'\rangle = \mathbf{0} \Leftrightarrow \langle im\ A_2\rangle \cap \langle ker\ A_1'\rangle = \mathbf{0}. \tag{230}$$

In an Euclidean space there holds $\langle ker\ A'\rangle \equiv \langle im\ A\rangle^{\perp}$ – see, for example, in (100).

Total parallelism of planars (153) or *colplanarity* of equirank lineors − see this in sect. 8.4, under condition (224), means that the matrix $B = A_1 A_2'$ is null-normal, i. e., $B \in \langle Bm \rangle$. Due to (97) and (132), this is equivalent to the relations:

$$\left.\begin{array}{l} |det\ C| = \sqrt{k(Bm \cdot Bm', r)} = |k(Bm, r)| = \\ = \mathcal{M}t(r)(A_1 \cdot A_2') = \mathcal{M}t(r)A_1 \cdot \mathcal{M}t(r)A_2 = \\ = \sqrt{det(A_1' \cdot A_1)} \cdot \sqrt{det(A_2' \cdot A_2)} \end{array}\right\} \quad (231)$$

and is also equivalent to parallelism (153) in an affine space. Formulae (227)–(231) may be interpreted trigonometrically, it will be done later.

Total orthogonality of planars or lineors, under condition (224), here means that $B = A_1 A_2'$ is a nilpotent matrix of order 2: $B^2 = Z$, or $C = Z$. The latter is also equivalent to orthogonality (155), if $r_1 = r_2$, in an Euclidean space. Their partial orthogonality means that B is a null-defective matrix.

The tensor angle $\tilde{\Phi}_{12}$ and its trigonometric functions are, of course, more general than the angle $\tilde{\Phi}_B$ and its functions, as matrices A_1 and A_2 may have distinct sizes $n \times r_1$ and $n \times r_2$ admissible only for $\tilde{\Phi}_{12}$. Moreover, if the lineors are partially or totally orthogonal, then only the angles $\tilde{\Phi}_{12}$ exist. Therefore the type of a tensor angle more convenient in the problem solving should be chosen.

5.5 Canonical cell-forms of trigonometric functions and reflectors

Parallelism and orthogonality correspond to extreme values of tensor angles between linear objects. In order to completely analyze all relations between objects, it is necessary to represent the trigonometric functions in canonic forms, to find their eigenvalues and to define informative scalar invariant characteristics for the tensor angle.

Consider differences of orthoprojectors similar to (163) and (171). They express the projective sine and cosine by two manners. According to (182)–(184) the sine and cosine eigenvalues are real paired (±) numbers belonging to $(-1; +1)$:

$$\mu_{i\sin}^2 + \mu_{i\cos}^2 = 1. \quad (232)$$

The paired sine and cosine eigenvalues in an *Euclidean space* correspond to a metric characteristic of scalar angles on the trigonometric eigenplanes. Four eigen orthoprojectors in both variants of differences (163) and (171) are pairly orthogonal. Hence, the projectors correspond one-to-one to four pairly orthogonal subspaces: $\langle im\ A_1 \rangle$, $\langle ker\ A_1' \rangle$ and $\langle im\ A_2 \rangle$, $\langle ker\ A_2' \rangle$ – see (100) in Part I:

$$\left.\begin{array}{l} \langle im\ A_1 \rangle \perp \langle ker\ A_1' \rangle, \langle im\ A_1 \rangle \oplus \langle ker\ A_1' \rangle \equiv \langle \mathcal{E}^n \rangle \Leftrightarrow \\ \Leftrightarrow\ \langle im\ A_2 \rangle \perp \langle ker\ A_2' \rangle, \langle im\ A_2 \rangle \oplus \langle ker\ A_2' \rangle \equiv \langle \mathcal{E}^n \rangle. \end{array}\right\} \quad (233)$$

In the first variant of (163), i. e., as in (159), the sine is considered in the subspace $\langle im\ A_1 \cup im\ A_2 \rangle$; in the second variant of (163), i. e., as in (160), the sine is considered in the subspace $\langle ker\ A'_1 \cup ker\ A'_2 \rangle$. Similarly, in the first variant of (171), the cosine is considered in $\langle im\ A_2 \cup ker\ A'_1 \rangle$; in the second variant of (171), the cosine is considered in $\langle im\ A_1 \cup ker\ A'_2 \rangle$.

The illustration is given in Figure 2. It is supposed without loss of generality that the first variant as in (154), i. e., $r_1 \leq r_2,\ r_1 + r_2 \leq n$ (or $2r \leq n$), takes place. The original space $\langle \mathcal{E}^n \rangle$ is partitioned with respect to this variant of differences (163) and (171) into four basic subspaces (both for the sine function and for the cosine function).

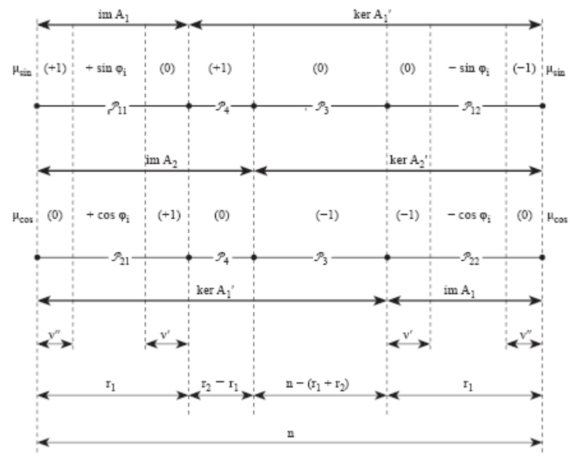

Figure 2. Distribution of projective sine and cosine eigenvalues in all the eigensubspaces of tensor angle between two lineors.

All indicated subspaces are pairly orthogonal provided that in the *trigonometric subspace* of the tensor angle of dimension 2τ there holds

$$\sin\varphi_i \neq \pm 1, \quad \sin\varphi_i \neq\ 0, \quad (\cos\varphi_i \neq\ 0, \quad \cos\varphi_i \neq \pm 1) \tag{234}$$

otherwise orthogonalization may be used.

This binary trigonometric subspace is defined as the following direct sums of these four particular orthogonal subspaces (in the sine and cosine variants):

$$\langle \mathcal{P}_{11} \oplus \mathcal{P}_{12} \rangle \equiv \langle \mathcal{P}_{21} \oplus \mathcal{P}_{22} \rangle. \tag{235}$$

(These four subspaces are formed by eigenvectors of the tensor sine and cosine.)

Its even dimension 2τ is called *the trigonometric rank of a tensor angle*, where $\tau = \min\{r_1, r_2, n - r_1, n - r_2\}$. Here we have $\tau = r_1$. The eigenvalues of the sine and cosine functions in (232) have the same absolute values in the two *mutual subspaces* (235), as the two sides of the binary angle in (163) are, due to (233), orthogonal; but their signs are opposite, as the projectors are ordered inversely in the two variants of differences (163) and (171) – see Figure 2.

If additional conditions (234) and $r_1 \leq r_2$ hold, the two intersections subspaces (the zero sine and the zero cosine) and their dimensions are expressed as follows:

$$\langle \mathcal{P}_3 \rangle \equiv \langle ker\ A_1' \cap ker\ A_2' \rangle, \quad dim \langle \mathcal{P}_3 \rangle = n - (r_1 + r_2), \quad (\sin\varphi = 0,\ \cos\varphi = -1);$$
$$\langle \mathcal{P}_4 \rangle \equiv \langle im\ A_2 \cap ker\ A_1' \rangle\ (\nu'' = 0), \quad dim \langle \mathcal{P}_4 \rangle = r_2 - r_1,\ (\cos\varphi = 0,\ \sin\varphi = +1).$$

The projective tensor cosine and sine are symmetric (anticommutative) matrices, so they may be converted separately into their D-forms with certain modal orthogonal matrices R_1 and R_2 respectively in the bases $\tilde{E}_1 = R_1 \cdot \tilde{E}$ and $\tilde{E}_2 = R_2 \cdot \tilde{E}$. In order to give the trigonometric sense to the eigenvalues (232), we use an Euclidean space $\langle \mathcal{E}^n \rangle$ with the original base $\tilde{E}$ and then find an *local unity Cartesian base for the canonical W-forms* of the tensor trigonometric functions for the angle $\tilde{\Phi}$. We establish it below.

Each i-th trigonometric 2×2-cell with an unique pair of the cosine and sine ($\pm$) eigenvalues in the trigonometric subspace of a tensor angle $\tilde{\Phi}_{12}$ corresponds to its i-th eigenplane. It is determined here in $\tilde{E}$ by a pair of the *cosine orthogonal unity eigenvectors u_i* and v_i. They are two Cartesian axes of the *tensor cosine D-form base* (not yet oriented) and correspond to its eigenvalues $\pm\cos\varphi_i$, where $\varphi_i \in [-\pi/2; +\pi/2]$ are the eigenvalues of the tensor angle between planars or non-oriented lineors. In order to construct the canonical forms of the tensor trigonometric functions, dispose the trigonometric cells along the matrix diagonal with increasing the values $|\cos\varphi_i|$. Then along the diagonal dispose the 1×1-cells corresponding to the intersection subspaces $\langle \mathcal{P}_3 \rangle$ and $\langle \mathcal{P}_4 \rangle$. Denote the original base $\tilde{E}$ axes as $x_1, \ldots, x_n$, and the trigonometric part of the new axes as $u_1, \ldots, u_\tau;\ v_1, \ldots, v_\tau$ such that $x_1 \leftrightarrow u_1,\ x_2 \leftrightarrow v_1, \ldots,$ $x_{2i-1} \leftrightarrow u_i,\ x_{2i} \leftrightarrow v_i, \ldots,\ x_{2\tau-1} \leftrightarrow u_\tau,\ x_{2\tau} \leftrightarrow v_\tau$. Direct the new axes in such way that each u_i and v_i form an acute angle. We found R_W for translating into new $\tilde{E}_1 = R_W\{\tilde{E}\} = \{I\}$.

In any trigonometric cell, $\sin^2\tilde{\Phi}_{12}$ and $\cos^2\tilde{\Phi}_{12}$ have the two positive (quadric) multiple eigenvalues from (232). As $\sin^2\tilde{\Phi}_{12}$ and $\cos^2\tilde{\Phi}_{12}$ commute in (184) and (185), then their and cosine D-forms are implemented together in the same *local base* $\tilde{E}_1$:

$$\sin^2\tilde{\Phi}_{12} = \begin{bmatrix} \ddots & & & & & & \\ & \begin{matrix} \sin^2\varphi_i & 0 \\ 0 & \sin^2\varphi_i \end{matrix} & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{bmatrix} \begin{matrix} \\ \\ \\ \langle \mathcal{P}_3 \rangle \\ \\ \langle \mathcal{P}_4 \rangle \\ \\ \end{matrix} \qquad \cos^2\tilde{\Phi}_{12} = \begin{bmatrix} \ddots & & & & & & \\ & \begin{matrix} \cos^2\varphi_i & 0 \\ 0 & \cos^2\varphi_i \end{matrix} & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{bmatrix}.$$

Due to symmetry of $\sin\tilde{\Phi}_{12}$ and $\cos\tilde{\Phi}_{12}$, and anticommutativity condition (183) one represents these functions in the following *canonical W-forms* – see ((148) in sect. 1.4) *in the local base $\tilde{E}_1 = \{I\}$ of the diagonal cosine* (provided that $r_1 \leq r_2,\ r_1 + r_2 \leq n$):

$\sin\tilde{\Phi}_{12}$ $\qquad\qquad$ $\cos\tilde{\Phi}_{12}$ $\qquad\qquad$ (236), (237)

$\ddots$							
	$0 \quad +\sin\varphi_i$ $+\sin\varphi_i \quad 0$						
		$\ddots$					
			0				$\langle\mathcal{P}_3\rangle$
				$\ddots$			
					$+1$		$\langle\mathcal{P}_4\rangle$
						$\ddots$	

$\ddots$						
	$+\cos\varphi_i \quad 0$ $0 \quad -\cos\varphi_i$					
		$\ddots$				
			-1			
				$\ddots$		
					0	
						$\ddots$

,

In (236) and (237), the signs of the projective sine and cosine in the trigonometric cells are chosen out four possible variants, according to their definitions in (163), (171); but the signs of them in $\langle\mathcal{P}_3\rangle$, and $\langle\mathcal{P}_4\rangle$ are chosen, according to the additional conditions.

For the angle $\tilde{\Phi}_B$ ($B \in \langle Bp\rangle$, there holds $\langle\mathcal{P}_4\rangle = \mathbf{0}$, $dim\langle\mathcal{P}_3\rangle = n-2r$. According to (199) we obtain the same base $\tilde{E}_1$ of the diagonal secant as for the cosine. From anti-symmetry of $i\tan\tilde{\Phi}_{12}$ and anticommutativity condition in (203)-(204) one represents these functions in $\tilde{E}_1$ in the following *canonical W-forms* (provided that $2r < n$):

$\sec\tilde{\Phi}_B$ $\qquad\qquad$ $i\tan\ \tilde{\Phi}_B$ $\qquad\qquad$ (238), (239)

$\ddots$					
	$+\sec\ \varphi_i \quad 0$ $0 \quad -\sec\ \varphi_i$				
		$\ddots$			
			-1		$\langle\mathcal{P}_3\rangle$
				$\ddots$	

$\ddots$				
	$0 \quad -\tan\varphi_i$ $+\tan\varphi_i \quad 0$			
		$\ddots$		
			0	
				$\ddots$

,

Formulae (236)–(239) are the canonical W-forms for all projective trigonometric functions in the directed base of the diagonal cosine $\tilde{E}_1$. This base is called *trigonometric* and used in W-forms representations. These forms also illustrate **Rule 1**.

Under conditions (234) also there holds:

$$\left.\begin{array}{l} \overrightarrow{S_i} = \overrightarrow{\{\cos^2\tilde{\Phi} - \cos^2\varphi_i \cdot I\}} = \overrightarrow{S_{i1}} + \overrightarrow{S_{i2}}; \\ \\ \overrightarrow{S_{i1}} = \overrightarrow{\{\cos\tilde{\Phi} - \cos\varphi_i \cdot I\}}, \quad \overrightarrow{S_{i2}} = \overrightarrow{\{\cos\tilde{\Phi} + \cos\varphi_i \cdot I\}}; \\ \\ \overrightarrow{S_3} = \overrightarrow{\sin\tilde{\Phi}}, \quad \overrightarrow{S_4} = \overrightarrow{\cos\tilde{\Phi}}; \quad (\overrightarrow{S_1} + \overrightarrow{S_2} + \overrightarrow{S_3} + \overrightarrow{S_4} = I) \end{array}\right\}. \tag{240}$$

These are the orthoprojectors onto the following characteristic subspaces: the i-th trigonometric cell, the axes $u_i \subset \langle\mathcal{P}_{21}\rangle$ and $v_i \subset \langle\mathcal{P}_{22}\rangle$, $\langle\mathcal{P}_3\rangle$, and $\langle\mathcal{P}_4\rangle$. Their basic columns (as well as the basic rows) determine the subspaces indicated.

If some angle φ_i is multiple, then the i-th trigonometric cells are united, and orthogonalization of their homogeneous axes are necessary for preserving the binary trigonometric structure. Moreover, if simplest eigenvalues (0 and ± 1) of the projective cosine or sine are equal to the same ones in $\langle \mathcal{P}_3 \rangle$ and $\langle \mathcal{P}_4 \rangle$, one may also use orthogonalization for dividing the mixed trigonometric partial subspaces. (See sect. 3.1.)

Below we consider the extreme cases of the angles and the cases with the other primary additional assumptions (see Figure 2).

Return to conditions (234). They facilitate partitioning an Euclidean space $\langle \mathcal{E}^n \rangle$ into trigonometric subspaces due to the unary and binary parts of W-forms. At first, consider the additional case, when the eigenvalues $\sin\varphi_i = 0$ of the multiplicity $2\nu\,'$ are in $\langle \mathcal{P}_{11} \rangle$ and $\langle \mathcal{P}_{12} \rangle$. Besides they corresponds to the sine value 0 belonging to $\langle \mathcal{P}_3 \rangle$. Also they corresponds to the pair eigenvalues of the multiplicity $\nu' \cos\varphi_i = +1$ in $\langle \mathcal{P}_{21} \rangle$ and $\cos\varphi_i = -1$ in $\langle \mathcal{P}_{22} \rangle$. The last value of the cosine corresponds to the cosine value -1 belonging to $\langle \mathcal{P}_3 \rangle$. The other additional case takes place, when the eigenvalues $\cos\varphi_i = 0$ of the multiplicity $2\nu''$ are in $\langle \mathcal{P}_{21} \rangle$ and $\langle \mathcal{P}_{22} \rangle$. Besides they corresponds to the cosine value 0 belonging to $\langle \mathcal{P}_4 \rangle$. Also they corresponds to the pair eigenvalues of the multiplicity $\nu'' \sin\varphi_i = +1$ in $\langle \mathcal{P}_{11} \rangle$ and $\sin\varphi_i = -1$ in $\langle \mathcal{P}_{12} \rangle$. The first value of the sine corresponds to the sine value $+1$ belonging to $\langle \mathcal{P}_4 \rangle$. In order to separate all the characteristic eigenspaces, it is necessary to orthogonalize them. After that the partial subspaces $\langle \mathcal{P}_{11} \rangle$, $\langle \mathcal{P}_{21} \rangle$, $\langle \mathcal{P}_3 \rangle$, $\langle \mathcal{P}_4 \rangle$, $\langle \mathcal{P}_{12} \rangle$, $\langle \mathcal{P}_{22} \rangle$ become entirely orthogonal.

Now suppose that other assumptions, taken before, do not hold. If $r_1 + r_2 > n$, then $\langle \mathcal{P}_3 \rangle = \langle im\ A_1 \rangle \cap \langle im\ A_2 \rangle$. Besides, if $r_2 > r_1$, then $\langle \mathcal{P}_4 \rangle = \langle im\ A_2 \rangle \cap \langle ker\ A_1' \rangle$. In according with these new conditions, the signs of unitary sine and cosine eigenvalues in $\langle \mathcal{P}_3 \rangle$ and $\langle \mathcal{P}_4 \rangle$ should be changed. For equirank lineors the subspace $\langle \mathcal{P}_4 \rangle$ is absent!

All the bases used are right ($det\{R\} = +1$). Among them are the original Cartesian base $\tilde{E}$ and the new Cartesian bases in the planes $\langle u_i, v_i \rangle$, i. e., $\tilde{E}_1 = R_W\{\tilde{E}\} = \{I\}$ (they form the binary part of the *trigonometric base*). In the trigonometric base, one may find the contradiagonal values of the sine up to their signs according to (236), then the cosine signs are exactly determined by (237); and vice versa. Both determine completely the absolute value and the sign of the counter-clockwise scalar angle φ_i in $[-\pi/2; +\pi/2]$. This segment is the range of angles for planars or non-oriented lineors.

Analogous reasoning may be realized for distributions of the projective secant and tangent values in the four eigenspaces of the tensor spherical angle between two lineors.

All these tensor projective trigonometric functions, as it is exposed above, in their sine-cosine (176)-(179) and tangent-secant (211)-(212) pairs forms the corresponding pairs of symmetric (orthogonal) and non-symmetric (oblique) eigenprojectors (i. e., four ones) with respect to the image and the kernel for each lineors. Therefore the same trigonometric base (the directed base of the diagonal cosine) is used for canonical forms of orthogonal eigenreflectors (176), (177) and affine ones (211), (212). These forms are

$+Ref\{A_1A_1'\}$ $\qquad$ $+\,Ref\{A_2A_2'\}$ (241)

$\ddots$						
	$+\cos\varphi_i \quad -\sin\varphi_i$ $-\sin\varphi_i \quad -\cos\varphi_i$					
		$\ddots$				
			-1			
				$\ddots$		
					-1	
						$\ddots$

$\langle\mathcal{P}_3\rangle$ (row 4), $\langle\mathcal{P}_4\rangle$ (row 6)

$\ddots$						
	$+\cos\varphi_i \quad +\sin\varphi_i$ $+\sin\varphi_i \quad -\cos\varphi_i$					
		$\ddots$				
			$+1$			
				$\ddots$		
					-1	
						$\ddots$

,

(they are the algebraic sum (237) and (236) provided that $r_1 \le r_2,\ r_1 + r_2 \le n$);

$+Ref\{B\}$ $\qquad$ $+\,Ref\{B'\}$ (242)

$\ddots$				
	$+\sec\ \varphi_i \quad +\tan\varphi_i$ $-\tan\varphi_i \quad -\sec\ \varphi_i$			
		$\ddots$		
			-1	
				$\ddots$

$\langle\mathcal{P}_3\rangle$ (row 4)

$\ddots$				
	$+\sec\ \varphi_i \quad -\tan\varphi_i$ $+\tan\varphi_i \quad -\sec\ \varphi_i$			
		$\ddots$		
			$+1$	
				$\ddots$

,

(they are the algebraic sum (238) and (239) provided that $2r \le n$).

5.6 The trigonometric theory of prime roots $\sqrt{I}$

In this section, we describe connection between the main notions of tensor trigonometry and the theory of prime roots $\sqrt{I}$ (i. e., without nilpotent matrix summand as in (21) or (76) – Part I). Fix an original Cartesian base $\tilde{E}$ in $\langle\mathcal{E}^n\rangle$. In this base any *prime square root of* the matrix I is the reflector (sign-indefinite nonsingular matrix), either symmetric or nonsymmetric – see formulae (176)–(179) and (211), (212).
So, it is $(\sqrt{I})_s = Ref\{Bm\}$, in particular $(\sqrt{I})_s = Ref\{AA'\}$; or $\sqrt{I} = Ref\{Bp\}$.
They can be converted, with the certain modal transformation $T \cdot \{\tilde{E}\} = \tilde{E}_D$, into the dual block-unity D-form of $\sqrt{I} = Ref\{Bm\}$ or $\sqrt{I} = Ref\{Bp\}$:

$$R_W' \cdot \sqrt{I} \cdot R_W = I^{\pm} = \left[\begin{array}{c|c} +I & Z \\ \hline Z & -I \end{array}\right] \begin{array}{l} q^+ \\ q^- \end{array} \quad (q^+ + q^- = n,\ \ q^+ = rank\ B,\ \ q^- = sing\ B).$$

For any trigonometric matrix (i. e., matrix, bound with a tensor angle) its trigonometric rank is defined by the binary structure of the tensor angle (see, for example, in (235)). Here the trigonometric rank 2τ also corresponds to an *index q of the reflector*:

$$2\tau = 2q = 2\min\{q^+, q^-\} = 2\min\{r, n - r\}.$$

Separate *symmetric* roots $(\sqrt{I})_s = (\sqrt{I})'_s$. For a null-normal matrix Bm, for example, $+Ref\{Bm\} = \overleftarrow{Bm} - \overrightarrow{Bm} = (\sqrt{I})_s$. Put, without loss of generality, $Bm = AA'$.

Let $(\sqrt{I})_1$ and $(\sqrt{I})_2$ be a pair of independent symmetric roots. Then, in $\langle \mathcal{E}^n \rangle$, these roots and the orthoreflectors are connected as follows:

$$\left\{ \begin{array}{l} \overleftarrow{A_1A_1'} = \frac{(I+(\sqrt{I})_1)}{2}, \ \overrightarrow{A_1A_1'} = \frac{(I-(\sqrt{I})_1)}{2}, \\ \overleftarrow{A_2A_2'} = \frac{(I+(\sqrt{I})_2)}{2}, \ \overrightarrow{A_2A_2'} = \frac{(I-(\sqrt{I})_2)}{2}. \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} (\sqrt{I})_1 = \overleftarrow{A_1A_1'} - \overrightarrow{A_1A_1'}, \\ (\sqrt{I})_2 = \overleftarrow{A_2A_2'} - \overrightarrow{A_2A_2'}. \end{array} \right\} \quad (243)$$

From this, taking into account (163), (171), (176), and (177), we obtain

$$\left. \begin{array}{lll} \cos\tilde{\Phi}_{12} - \sin\tilde{\Phi}_{12} & = (\sqrt{I})_1 & = +Ref\{A_1A_1'\}, \\ \cos\tilde{\Phi}_{12} + \sin\tilde{\Phi}_{12} & = (\sqrt{I})_2 & = +Ref\{A_2A_2'\}, \end{array} \right\} \quad (244)$$

$$\cos\tilde{\Phi}_{12} = [(\sqrt{I})_1 + (\sqrt{I})_2]/2, \quad \sin\tilde{\Phi}_{12} = [(\sqrt{I})_2 - (\sqrt{I})_1]/2. \quad (245)$$

The homogeneous projectors are equirank, iff $(\sqrt{I})_1$ and $(\sqrt{I})_2$ have the same index, either q^-, or q^+ (as the trigonometric rank for a pair of lineors or null-prime matrix). Remember, that the orthoreflectors $+Ref\{AA'\}$ and $-Ref\{AA'\}$ have their mutually orthogonal mirrors $\langle ker\ A' \rangle$ and $\langle im\ A \rangle$ in the Euclidean space.

Corollaries (for $\langle \mathcal{E}^n \rangle$)

1. *A symmetric root $\sqrt{I}$ defines one-to-one a unique symmetric orthogonal reflector as well as a unique mutual pair of spherically orthogonal projectors and a unique right tensor angle of the same trigonometric rank.*

2. *Any pair of symmetric roots $(\sqrt{I})_1$ and $(\sqrt{I})_2$ defines a unique pair of spherically orthogonal projectors, a unique tensor angle $\tilde{\Phi}_{12}$ and its trigonometric functions.*

3. *If an original Cartesian base $\tilde{E}$ is fixed, then all the matrix notions, according to item* 2, *due to* (243) – (245), *may be converted into compatible monobinary W-forms in a trigonometric Cartesian base $\tilde{E}_1 = R_W\{\tilde{E}\} = \{I\}$.*

Separate *nonsymmetric* prime roots $\sqrt{I} \neq (\sqrt{I})'$. For a null-prime matrix Bp, for example, $+Ref\{Bp\} = \overleftarrow{Bp} - \overrightarrow{Bp} = \sqrt{I} \neq (\sqrt{I})'$. Denote the matrix Bp briefly as B. Then we have the following bond of these roots and oblique reflectors:

$$\left\{ \begin{array}{ll} \overleftarrow{B} = \frac{(I+\sqrt{I})}{2}, & \overrightarrow{B} = \frac{(I-\sqrt{I})}{2}, \\ \overleftarrow{B'} = \frac{(I+(\sqrt{I})')}{2}, & \overrightarrow{B'} = \frac{(I-(\sqrt{I})')}{2}. \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \sqrt{I} = \overleftarrow{B} - \overrightarrow{B}, \\ (\sqrt{I})' = \overleftarrow{B'} - \overrightarrow{B'}. \end{array} \right\} \quad (246)$$

From this, taking into account (198), (203), (211), (212), we obtain

$$\left. \begin{array}{lll} \sec\ \tilde{\Phi}_B - i\tan\tilde{\Phi}_B & = \sqrt{I} & = +Ref\{B\}, \\ \sec\ \tilde{\Phi}_B + i\tan\tilde{\Phi}_B & = (\sqrt{I})' & = +Ref\{B'\}, \end{array} \right\} \quad (247)$$

$$\sec\ \tilde{\Phi}_B = (\sqrt{I} + (\sqrt{I})')/2, \quad i\tan\tilde{\Phi}_B = ((\sqrt{I})' - \sqrt{I})/2. \quad (248)$$

The roots $\sqrt{I}$ and $(\sqrt{I})'$ always have the same trigonometric rank. Remember, that the oblique reflectors $+Ref\{Bp\}$ and $+Ref\{Bp\}$ have their mutually oblique mirrors $\langle ker\ B \rangle$ and $\langle im\ B \rangle$ in the Euclidean space – see for the non-transposed reflector.

Corollaries (for $\langle \mathcal{E}^n \rangle$)

1. *Any nonsymmetric prime root $\sqrt{I}$ defines a unique nonsymmetric (i. e., oblique) reflector as well as a unique mutual pair of spherically oblique projectors.*

2. *Any pair of nonsymmetric prime roots $\sqrt{I}$ and $(\sqrt{I})'$ define a unique pair of spherically oblique projectors, a unique tensor angle $\tilde{\Phi}_B$ with trigonometric functions.*

3. *If a Cartesian base $\tilde{E}$ is fixed, then all the notions (item 2), due to (246)−(248), may be converted into compatible monobinary W-forms in a trigonometric Cartesian base $\tilde{E}_1 = R_W\{\tilde{E}\} = \{I\}$.*

Moreover, the roots $\sqrt{I}$ and $(\sqrt{I})'$ define:

an unique pair of equirank projectors in (243) satisfying condition $det \cos \tilde{\Phi}_{12} \neq 0$;
an unique pair of orthoprojectors $\overleftrightarrow{BB'}$ and $\overleftrightarrow{B'B}$ such that $det \cos \tilde{\Phi}_B \neq 0$;
an unique pair of symmetric roots $(\sqrt{I})_1$ and $(\sqrt{I})_2$ with $det[(\sqrt{I})_1 + (\sqrt{I})_2] \neq 0$.

This follows from relations similar to (186)–(189) and (226).

5.7 Rotational trigonometric functions of motive-type spherical angles

In the sequel, in order to infer some matrix formulae and connected with them inequality we shall use so called *the principle of binarity*. It consists in the following.

The prime real matrices P_1 and P_2 are anticommutative iff they may be represented jointly in their real anticommutative monobinary cell forms W_1 and W_2 in a certain *real local base* (sect. 4.1). If the original affine base is $\tilde{E}$, then here the local base is $\tilde{E}_1 = V_W \cdot \tilde{E} = \{I\}$. The matrices P_1 and P_2 are anticommutative on their common real eigenspaces of dimensions 1 and 2 (see more in sect. 7.2). These forms W_1 and W_2 are a direct sum of their monobinary cells of the identical structure.

Moreover, any analytical function $F(P_1, P_2)$ in the base $\tilde{E}$ may be expressed in the base $\tilde{E}_1$ as $F(W_1, W_2)$. In particular, this realizes for elementary operations of summation and multiplication. The scalar invariants of $F(P_1, P_2)$ are the same invariants for $F(W_1, W_2)$. (In theory of matrices, the analogous *principle of unarity* is applied for analytical functions of several prime commutative matrices with their joint reducing to diagonal forms.) The principle of binarity is based on the fact that original and squares of anticommutative prime matrices P_1 and P_2 commute each with another. Both these principles enable one to generalize analytical operations over simplest cell structures and results onto original matrices and their analytical functions.

Suppose, in particular, in $\langle \mathcal{E}^n \rangle$: $P_1 = \cos \tilde{\Phi}_{12}$, $P_2 = \sin \tilde{\Phi}_{12}$ for the *equirank lineors* A_1 and A_2, according to formulae (163) and (171). Then $\langle \mathcal{P}_4 \rangle = \mathbf{0}$. But non-zero $\langle \mathcal{P}_3 \rangle$ exists iff it exists in canonical cosine form (237) (as positive or negative unity block).

By (176) and (177) for these anticommutative P_1 and P_2 we have the analytical function

$$F(P_1, P_2) = (P_1 + P_2) \cdot (P_1 - P_2) = [+Ref\{A_2A_2'\}] \cdot [+Ref\{A_1A_1'\}] = [-Ref\{A_2A_2'\}] \cdot [-Ref\{A_1A_1'\}].$$

Then there holds $T_W = R_W$, so the W-form of $F(P_1, P_2)$ in the *trigonometric base* $\tilde{E}_1 = R_W \cdot \tilde{E} = \{I\}$ is expressed by the *orthogonal rotational matrix* at the angle $2\Phi_{12}$:

$$\overset{+Ref\{A_2A_2'\}}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} +\cos\varphi_i & +\sin\varphi_i \\ +\sin\varphi_i & -\cos\varphi_i \end{matrix} & \\ & & \ddots \end{bmatrix}} \cdot \overset{+Ref\{A_1A_1'\}}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} +\cos\varphi_i & -\sin\varphi_i \\ -\sin\varphi_i & -\cos\varphi_i \end{matrix} & \\ & & \ddots \end{bmatrix}} = \overset{Rot(+2\Phi_{12})}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} \cos 2\varphi_i & -\sin 2\varphi_i \\ +\sin 2\varphi_i & \cos 2\varphi_i \end{matrix} & \\ & & \ddots \end{bmatrix}},$$

where $\langle \mathcal{P}_3 \rangle$ is the *unity block* $+I$ as $(\pm 1)\cdot(\pm 1) = +1$ for unity cosine part in (237). This 2×2-cell implements rotation at the counter-clockwise angle $+2\varphi_i$ on trigonometric eigenplanes. In $\tilde{E}$, it implements *spherical rotation* at the *motive* tensor angle $+2\Phi_{12}$:

$$Ref\{A_2A_2'\} \cdot Ref\{A_1A_1'\} = (\cos\tilde{\Phi}_{12} + \sin\tilde{\Phi}_{12}) \cdot (\cos\tilde{\Phi}_{12} - \sin\tilde{\Phi}_{12}) =$$
$$= \cos^2\tilde{\Phi}_{12} - \sin^2\tilde{\Phi}_{12} + 2\sin\tilde{\Phi}_{12}\cos\tilde{\Phi}_{12} = \cos^2\Phi_{12} - \sin^2\Phi_{12} + 2i\sin\Phi_{12}\cos\Phi_{12} =$$
$$= \cos 2\Phi_{12} + i\sin 2\Phi_{12} = Rot\ 2\Phi_{12} = [-Ref\{A_2A_2'\}] \cdot [-Ref\{A_1A_1'\}], \quad (249)$$
$$[\pm Ref\{A_1A_1'\}] \cdot [\pm Ref\{A_2A_2'\}] = Rot(-2\Phi_{12}) = Rot\ 2\Phi_{21}. \quad (250)$$

Notation $Rot\ \Phi$ is used for matrix rotational functions of *binary motive type* tensor spherical angles Φ. Such tensor angles do not contain in their notion the tilde symbol necessary for projective tensor angles. The following *united properties* hold for the main sine-cosine pairs of projective and motive tensor angles (see more in sect 5.8): $\cos^2\tilde{\Phi} = \cos^2\Phi$, $\sin^2\tilde{\Phi} = \sin^2\Phi$; and $\sin\tilde{\Phi}\cdot\cos\tilde{\Phi} = i\sin\Phi\cdot\cos\Phi = \cos\Phi\cdot i\sin\Phi = -\cos\tilde{\Phi}\cdot\sin\tilde{\Phi}$. These formulae also illustrate **Rule 1** (see above), but for the motive type trigonometric functions. Note, that $Rot\ \Phi_{12}$ in (249) is a *trigonometric square root* (i. e., as result of the original angle dimidiating in the each binary cell of W-form!):

$$Rot\ \Phi_{12} = \{[\pm Ref\{A_2A_2'\}] \cdot [\pm Ref\{A_1A_1'\}]\}^{1/2}. \quad (251)$$

Formula (249) is interpreted as follows. Orthogonal reflection of $\langle im\ A_1 \rangle$ (or $\langle ker\ A_1' \rangle$) and then of $\langle im\ A_2 \rangle$ (or $\langle ker\ A_2' \rangle$) is equivalent to rotation at the doubled angle between $\langle im\ A_1 \rangle$ and $\langle im\ A_2 \rangle$. It is quite clear when we deal with two vectors or straight lines. The rotational matrix ($\tau = 1$), according to (249), (176), (177), is

$$Rot\ \Phi_{12} = [(I - 2\cdot\overleftarrow{\mathbf{a_2a_2'}})\cdot(I - 2\cdot\overleftarrow{\mathbf{a_1a_1'}})]^{1/2} = [(I - 2\cdot\overrightarrow{\mathbf{a_2a_2'}})\cdot(I - 2\cdot\overrightarrow{\mathbf{a_1a_1'}})]^{1/2} =$$
$$= \left[I - 2\cdot\left(\frac{\mathbf{a_1a_1'}}{\mathbf{a_1'a_1}} + \frac{\mathbf{a_2a_2'}}{\mathbf{a_2'a_2}}\right) + 4\cos^2\varphi_{12}\cdot\frac{\mathbf{a_2a_1'}}{\mathbf{a_1'a_2}}\right]^{1/2}. \quad (252)$$

Here the mirrors are either **a** or a hyperplane $\langle ker\ \mathbf{a}'\rangle$ – orthocomplement of $\langle im\ \mathbf{a}\rangle$,

$$Rot\ \Phi_{12}\cdot\overleftrightarrow{\mathbf{a_1 a_1'}}\cdot Rot\ (-\Phi_{12}) = \overleftrightarrow{\mathbf{a_2 a_2'}},\ \overleftarrow{\mathbf{aa}'} = \frac{\mathbf{aa}'}{\mathbf{a}'\mathbf{a}}, \overleftarrow{\mathbf{a_2 a_1'}} = \frac{\mathbf{a_2 a_1'}}{\mathbf{a_1' a_2}}\ (\overleftarrow{\mathbf{ee}'} = \mathbf{ee}', \overleftarrow{\mathbf{e_2 e_1'}} = \frac{\mathbf{e_2 e_1'}}{\cos\varphi_{\mathbf{12}}}).$$

If these $n\times 1$-vectors are oriented, then the angle φ_{12} in the trigonometric eigenplane of the matrix $Rot\ \Phi$ in the Euclidean space $\langle\mathcal{E}^n\rangle$ varies in $[-\pi;\pi]$.

A tensor rotation is performed in the $2\tau_R$-dimensional subspace ($\tau_R = r_1$ – Figure 2) with respect to its orthocomplement of dimension $n - 2\tau_R$, i. e., as the *generalized rotation axis*. The rotation at angle $+\varphi_i$ on i-th trigonometric eigenplanes realizes in $[-\pi/2;+\pi/2]$. An orthogonal matrix R is a *rotational function* if $det\ R = +1$, it is a *reflector* if $R = R'$. These two properties may be sometimes formally compatible.

Real prime matrices are called *compatible* if their W-forms have the same structure in a common base. In particular, they may be commutative or anticommutative ones. Real normal matrices may be converted into W-forms with rotational transformations of the base, we denote them as R_W. For compatible normal matrices, R_W is same.

The most general variant of formulae (249) and (250) for compatible reflectors is

$$(\cos\tilde{\Phi}_{12} \pm \sin\tilde{\Phi}\ _{12})(\cos\tilde{\Phi}_{34} \pm \sin\tilde{\Phi}\ _{34}) = Rot\ \{\pm\Phi_{12} \pm \Phi_{34}\}.$$

In tensor trigonometry, besides the particular reflectors, so called the *mid-reflector* is very important. For a pair of the given planars their mid-reflector is one between $Ref(-\Phi_{12})$ and $Ref(+\Phi_{12})$ ($r_1 = r_2 = r$) or $Ref(-\Phi_B)$ and $Ref((+\Phi_B))$, i. e., for the middle subspace of tensor angle $\tilde{\Phi}_{12}$ or $\tilde{\Phi}_B$. But it is not attach only to this concrete pair of objects. It is defined for the set of pairs of linear objects having such common mid-reflector. It has *the sign-alternating unity diagonal W-form* congruous to the *cosine diagonal form* (237) in the trigonometric base $\tilde{E}_1$ of the projective angle $\tilde{\Phi}$, i. e., in its trigonometric subspace with the axes as $u_1,\dots,u_\tau$ and $v_1,\dots,v_\tau$. The cosine axes in the zero sine subspace $\langle\mathcal{P}_3\rangle$ are the same with their eigenvalues ± 1; $\langle\mathcal{P}_4\rangle = \mathbf{0}$ as $r_1 = r_2$. According concretely to (171), (172) and formally to (237), the projective cosine is the algebraic sum of two *orthogonal* terms with algebraically positive and negative eigenvalues (Figure 2):

$$\cos\tilde{\Phi}_{12} = \{\cos\tilde{\Phi}_{12}\}^{\oplus} + \{\cos\tilde{\Phi}_{12}\}^{\ominus}, \quad \{\cos\tilde{\Phi}_{12}\}^{\oplus}\cdot\{\cos\tilde{\Phi}_{12}\}^{\ominus} = Z.$$

These summands are singular matrices. Here the mid-reflector mirror is the subspace $\langle im\ \{\cos\tilde{\Phi}_{12}\}^{\ominus}\rangle$, given by axes v_i. According to (176) the mid-reflector is expressed as

$$Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus}\} = \{\overrightarrow{\cos\tilde{\Phi}_{12}}\}^{\ominus} - \{\overleftarrow{\cos\tilde{\Phi}_{12}}\}^{\ominus} = \{\sqrt{I}\}_S = \{R_W\cdot I^{\pm}\cdot R_W'\}. \quad (253)$$

The mirror of this mid-reflector is situated in the middle between two mirrors in (176) and (177) for the tensor angle – see on their structures (241). In order to prove this, we obtain this mid-reflector by two ways: by rotating the 1-st reflector at the angle $\{+\Phi_{12}/2\}$ and by rotating the 2-nd reflector at the angle $\{-\Phi_{12}/2\}$ in the base $\tilde{E}_1$:

$$\underset{Rot\ \Phi_{12}/2}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} \cos\varphi_i/2 & -\sin\varphi_i/2 \\ \sin\varphi_i/2 & \cos\varphi_i/2 \end{matrix} & \\ & & \ddots \end{bmatrix}} \cdot \underset{+\,Ref\{A_1A_1'\}}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} +\cos\varphi_i & -\sin\varphi_i \\ -\sin\varphi_i & -\cos\varphi_i \end{matrix} & \\ & & \ddots \end{bmatrix}} \cdot \underset{Rot'\ \Phi_{12}/2}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} \cos\varphi_i/2 & \sin\varphi_i/2 \\ -\sin\varphi_i/2 & \cos\varphi_i/2 \end{matrix} & \\ & & \ddots \end{bmatrix}} =$$

$$= \underset{Rot'\ \Phi_{12}/2}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} \cos\varphi_i/2 & \sin\varphi_i/2 \\ -\sin\varphi_i/2 & \cos\varphi_i/2 \end{matrix} & \\ & & \ddots \end{bmatrix}} \cdot \underset{+\,Ref\{A_2A_2'\}}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} +\cos\varphi_i & +\sin\varphi_i \\ +\sin\varphi_i & -\cos\varphi_i \end{matrix} & \\ & & \ddots \end{bmatrix}} \cdot \underset{Rot\ \Phi_{12}/2}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} \cos\varphi_i/2 & -\sin\varphi_i/2 \\ \sin\varphi_i/2 & \cos\varphi_i/2 \end{matrix} & \\ & & \ddots \end{bmatrix}} =$$

$$+Ref\{\cos\tilde{\Phi}_{12}\}^{\oplus} = (\sqrt{I})_s \tag{254}$$

$$= \begin{bmatrix} \ddots & & \\ & \begin{matrix} +1 & 0 \\ 0 & -1 \end{matrix} & \\ & & \ddots \end{bmatrix}.$$

The small reverse.

Now we may return to justification of projective trigonometric functions definition in the beginning of the chapter. For this, it is sufficient to consider only the main, i. e., sine-cosine functions pair. The choice of projective sine forms (163), (169) is clear and obvious for a pair of vectors or lineors. But the choice of projective cosine forms (171), (172) is not obvious. The two alternative forms are possible:

$$\cos\tilde{\Phi}_{12} = \overrightarrow{A_2A_2'} - \overleftarrow{A_1A_1'} == I - \overleftarrow{A_1A_1'} - \overleftarrow{A_2A_2'}$$
$$\cos\tilde{\Phi}_B = \overleftarrow{B'B} - \overrightarrow{BB'} = I - \overleftarrow{BB'} - \overleftarrow{B'B}$$

In the case, we have both eigen orthoreflectors, for example, for a pair of lineors as

$$\left.\begin{aligned} \overleftarrow{A_1A_1'} - \overrightarrow{A_1A_1'} &= Ref\{A_1A_1'\} = -\cos\tilde{\Phi}_{12} - \sin\tilde{\Phi}_{12} = \cos\tilde{Z}_1, \\ \overleftarrow{A_2A_2'} - \overrightarrow{A_2A_2'} &= Ref\{A_2A_2'\} = -\cos\tilde{\Phi}_{12} + \sin\tilde{\Phi}_{12} = \cos\tilde{Z}_2; \end{aligned}\right\} \text{(variant 1)}$$

$$\left.\begin{aligned} \overrightarrow{A_1A_1'} - \overleftarrow{A_1A_1'} &= Ref\{A_1A_1'\} = +\cos\tilde{\Phi}_{12} + \sin\tilde{\Phi}_{12} = \cos\tilde{Z}_1, \\ \overrightarrow{A_2A_2'} - \overleftarrow{A_2A_2'} &= Ref\{A_2A_2'\} = +\cos\tilde{\Phi}_{12} - \sin\tilde{\Phi}_{12} = \cos\tilde{Z}_2; \end{aligned}\right\} \text{(variant 2)}$$

where the mirrors types are $\langle ker\ A'\rangle$ and $\langle im\ A\rangle$. Whether the variants are correct? For example, at active rotations, as (254), the cosine and sine in the variants appear in the rotational matrices according to formulae (249)–(252). However, in these variants, the rotation (254) as $+2\varphi_i$ on trigonometric eigenplanes is realized formally only at the *not* counter-clockwise angle. Hence, the both other cosine forms are not correct!

For compatible trigonometric matrices, in addition to (250), (251), (254) and (226):

$$Ref\{A_2A_2'\} = Rot\ \Phi_{12} \cdot Ref\{A_1A_1'\} \cdot Rot\ (-\Phi_{12}) = Rot\ 2\Phi_{12} \cdot Ref\{A_1A_1'\}. \quad (255)$$

$$Rot(+\Phi_{12}) \cdot \{\cos\tilde{\Phi} \pm \sin\tilde{\Phi}\} \cdot Rot(-\Phi_{12}) = Rot(+2\Phi_{12}) \cdot \{\cos\tilde{\Phi} \pm \sin\tilde{\Phi}\} =$$
$$= \{\cos\tilde{\Phi} \pm \sin\tilde{\Phi}\} \cdot Rot(-2\Phi_{12}) = \{\cos(\tilde{\Phi} \pm 2\tilde{\Phi}_{12}) \pm \sin(\tilde{\Phi} \pm 2\tilde{\Phi}_{12})\}.$$

And add to (254): $Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus} = Rot\ \Phi_{12} \cdot Ref\{A_1A_1'\} = Rot\ (-\Phi_{12}) \cdot Ref\{A_2A_2'\}$.

$$\left.\begin{array}{l} Ref\{A_2A_2'\} = Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus} \cdot Ref\{A_1A_1'\} \cdot Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus}, \\ Ref\{A_1A_1'\} = Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus} \cdot Ref\{A_2A_2'\} \cdot Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus}. \end{array}\right\} \quad (256).$$

Rule 2. *Compatible spherical rotational matrices commute. In their multiplications the tensor argument angles of motive type form an algebraic sum.*

Rule 3. *In multiplication of a rotation and a symmetric reflector, if they are compatible, the rotation is transferred the reflector with the change of its tensor angle sign.* (The rules may be inferred with applying the principle of binarity as above.)

Corollaries

1. *The types of tensor angle in eigenreflectors (i. e., when bound with the angle) and in rotational functions are different!!! In first case, it is projective. In second case, it is motive. But in classical scalar form of these angles, this difference is absent!*

2. *Compatible active rotational transformation of a reflector as a 2-valent tensor at an angle Φ is equivalent to its rotation as an 1-valent tensor at the angle 2Φ.*

Put $A_2 = R_{12}A_1$ (with $det\ R_{12} = 1$) in (98) and (99), then

$$\overleftrightarrow{A_2A_2'} = R_{12} \cdot \overleftrightarrow{A_1A_1'} \cdot R_{12}', \qquad Ref\{A_2A_2'\} = R_{12} \cdot Ref\{A_1A_1'\} \cdot R_{12}'.$$

Denote as the complete set of all such matrices. The matrix $Rot\ \Phi_{12}$, defined by (251), has the trigonometric subspace of *the minimal* dimension $2\tau_R$ in $\langle R_{12}\rangle$. It is trigonometrically compatible with the reflectors $Ref\{A_1A_1'\}$, $Ref\{A_2A_2'\}$, $Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus}$. Generally, there holds $\langle R_{12}\rangle \equiv \langle Rot\ \Phi_{12} \cdot Rot\ \Theta_{12}\rangle$ or $\langle R_{12}\rangle \equiv \langle Rot\ \Theta_{12} \cdot Rot\ \Phi_{12}\rangle$. Here and further Φ is a spherical angle of the *principal* rotation, Θ is a spherical angle of the *secondary* rotation (or so-called *orthospherical angle*, i. e., angle of compatible rotation orthogonal with respect to the rotations $Rot\ \Phi$). Principal rotations often are called *boost*. $\langle Rot\ \Theta\rangle$ forms the subgroup of general rotations group. In non-Euclidean Geometries of the spherical type, the principal angles Φ play a main motive role.

In the *motive version*, the compatible rotations of two types satisfy relations

$$\left.\begin{array}{l} Rot\ \{\pm\Phi_{12}\} \cdot Ref\{\cos\tilde{\Phi}\}^{\ominus} \cdot Rot\ \{\pm\Phi_{12}\} = Ref\{\cos\tilde{\Phi}\}^{\ominus}, \\ Rot'\ \Theta_{12} \cdot Ref\{\cos\tilde{\Phi}\}^{\ominus} \cdot Rot\ \Theta_{12} = Ref\{\cos\tilde{\Phi}\}^{\ominus} = \\ = Rot\ \Theta_{12} \cdot Ref\{\cos\tilde{\Phi}\}^{\ominus} \cdot Rot'\ \Theta_{12}, \end{array}\right\} \quad (257)$$

where in the main reflector, in particular, $\tilde{\Phi} = \tilde{\Phi}_{12}$. Transferring through the reflector, the principal rotation changes its angle sign and is annihilated; but the secondary rotation is transferring through both unity parts of the reflector without changes!

In the *projective version*, the orthogonal reflectors of two types satisfy relations

$$\left.\begin{array}{l} Ref\{\cos\tilde{\Phi}\}^{\ominus}\cdot Ref\{Bm\}(\Phi)\cdot Ref\{\cos\tilde{\Phi}\}^{\ominus} = Ref\{Bm\}(\overset{\angle}{\Phi}_{12}),\\ Ref\{\cos\tilde{\Phi}\}^{\ominus}\cdot Ref\{Bm\}(\Theta)\cdot Ref\{\cos\tilde{\Phi}\}^{\ominus} = Ref\{Bm\}(\Theta).\\ (Ref\{Bm\}(\Phi) = \pm(\cos\tilde{\Phi}\mp\sin\tilde{\Phi}),\ Ref\{Bm\}(\Theta) = \pm(\cos\tilde{\Theta}\mp\sin\tilde{\Theta}).) \end{array}\right\} \quad (258)$$

Note the simplest case: $Ref\{Bm\}(\Phi)\rightarrow Ref\{Bm\}(-\Phi)$ iff $Ref\{\cos\tilde{\Phi}\}^{\ominus}\in\langle\{I^{\pm}\}\rangle$. Relations similar to (257) and (258) are the basis for the *quasi-Euclidean geometry and the trigonometry in it* with the *reflector tensor of a quasi-Euclidean space* $\{\sqrt{I}\}_S$ independently introduced similar to mid-reflector (253). In quasi-Euclidean geometry, a reflector tensor is not metrical, it does not determine internal and external products, it merely determines admissible transformations – rotations and reflections of two types (principal and secondary). Tensor angles corresponding to the reflector tensor have compatible orientations. The *simplest (diagonal!) W-form* $\{I^{\pm}\}$ of a reflector tensor in (253) and here corresponds to the *coaxially oriented* linear space. A reflector-tensor and an Euclidean quadratic metric determine a spherical trigonometry; and vice versa!

Relations (257, 258) define an *external non-Euclidean spherical geometry of index q*. The latter is a *spherical geometry in a hyperspace of constant positive radius R*. This geometry is realized on a special *hyperspheroid* embedded into the *quasi-Euclidean space* $\langle\mathcal{Q}^{n+q}\rangle$ determined by an independent set reflector-tensor and Euclidean metric.

5.8 The sine, cosine, secant, and tangent of a motive type tensor angle

The paired rotational matrices R and R' ($detR = +1$) – see (249), (250) consist of the *commutative* tensor sine and cosine of a binary *motive type* tensor angle (Φ_{12} or Φ_B):

$$Rot\ \Phi = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cos\varphi_i & -\sin\varphi_i & \\ & +\sin\varphi_i & \cos\varphi_i & \\ \hline & & & \ddots \end{array}\right] = R,\quad Rot\ (-\Phi) = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cos\varphi_i & +\sin\varphi_i & \\ & -\sin\varphi_i & \cos\varphi_i & \\ \hline & & & \ddots \end{array}\right] = R',\quad (259)$$

$$\cos\Phi = \cos'\Phi = (Rot\ \Phi + Rot'\ \Phi)/2 = [Rot\ (+\Phi) + Rot\ (-\Phi)]/2, \quad (260)$$

$$i\sin\Phi = -(i\sin\Phi)' = (Rot\ \Phi - Rot'\ \Phi)/2 = [Rot\ (+\Phi) - Rot\ (-\Phi)]/2. \quad (261)$$

The *realificated motive sine* is a *real valued skewsymmetric matrix* with the eigenvalues $\mu_j = \pm i\sin\varphi_j$, but $\sin\Phi$ is a *true motive sine.* The motive secant is clearly defined as

$$\sec\ \Phi = \sec'\ \Phi = \cos^{-1}\Phi. \quad (262)$$

Tensor motive tangent is defined through D-forms (see more obviously in sect 5.10.):

$$\{\tan\Phi\}_D = \{\sin\Phi\}_D\cdot\{\sec\ \Phi\}_D = \{\sec\Phi\}_D\cdot\{\sin\Phi\}_D\rightarrow(\tan\Phi = \tan'\Phi). \quad (263)$$

$$Rot\ \Phi \cdot Rot'\ \Phi = \sin^2\Phi + \cos^2\Phi = I = \cos^2\Xi + \sin^2\Xi \quad \text{(Ptolemy Invariant)}. \tag{264}$$

$$\sec^2\Phi - \tan^2\Phi = I = \mathrm{cosec}^2\,\Xi - \cot^2\Xi. \tag{265}$$

$$\left.\begin{array}{l} \sin\Phi\cdot\cos\Phi = \cos\Phi\cdot\sin\Phi = \sin\tilde{\Phi}\cdot\cos\tilde{\Phi} = -\cos\tilde{\Phi}\cdot\sin\tilde{\Phi}, \\ \sec\Phi\cdot\tan\Phi = \tan\Phi\cdot\sec\Phi = \tan\tilde{\Phi}\cdot\sec\tilde{\Phi} = -\sec\tilde{\Phi}\cdot\tan\tilde{\Phi}. \end{array}\right\} \tag{266}$$

For the cosine and sine of a motive rotation angle in $A_2 = Rot\ \Phi \cdot A_1$, obviously, we have $\langle\mathcal{P}_4\rangle = \mathbf{0}$; $dim\langle\mathcal{P}_3\rangle$ (as *unity block* $+I$, see in sect. 5.7) depends on the relation between n and $rankA$. The dimension is either $(n-2r)$, or $(2r-n)$, or $\langle\mathcal{P}_3\rangle = \mathbf{0}$.

Fix an original Cartesian base $\tilde{E}$ in $\langle\mathcal{E}^n\rangle$. The canonical W-forms for a real orthogonal rotational matrix at the motive angle Φ (or Θ) and for its cosine and sine in the trigonometric base of diagonal cosine $\tilde{E}_1 = R_W \cdot \tilde{E} = \{I\}$ are following (if $2r < n$):

$$Rot\ \Phi = \cos\Phi + i\sin\Phi = \exp(i\Phi) = Rot'\ (-\Phi) = Rot^{-1}(-\Phi) = \tag{267}$$

$$= \underset{\cos\Phi}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi_j & 0 & \\ & 0 & \cos\varphi_j & \\ & & & \ddots & \\ & & & & \boxed{+1} \end{bmatrix}} + i \underset{\sin\Phi}{\begin{bmatrix} \ddots & & & \\ & 0 & i\sin\varphi_j & \\ & -i\sin\varphi_j & 0 & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}}.$$

The tensor sine and angle eigenvalues 0 correspond to the subspace $\langle\mathcal{P}_3\rangle$.

Describe the canonical forms of a motive angle and its motive functions. At first, we use the complex-valued base of the sine and angle D-form, then return to the original *real-valued* trigonometric base of the diagonal cosine. Such identical modal transformation gives the canonical form of a motive angle Φ in the trigonometric base:

$$\overset{\tilde{E}_D}{\Rightarrow} \cos \underset{D(\Phi)}{\begin{bmatrix} \ddots & & & \\ & +\varphi_j & 0 & \\ & 0 & -\varphi_j & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}} + i\sin \underset{D(\Phi)}{\begin{bmatrix} \ddots & & & \\ & +\varphi_j & 0 & \\ & 0 & -\varphi_j & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}} \overset{\tilde{E}_D^{-1}}{\Rightarrow}$$

$$\overset{\tilde{E}_D^{-1}}{\Rightarrow} \cos \underset{\Phi}{\begin{bmatrix} \ddots & & & \\ & 0 & +i\varphi_j & \\ & -i\varphi_j & 0 & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}} + i\sin \underset{\Phi}{\begin{bmatrix} \ddots & & & \\ & 0 & +i\varphi_j & \\ & -i\varphi_j & 0 & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}} =$$

$$= \exp \underset{i\Phi}{\begin{bmatrix} \ddots & & & \\ & 0 & -\varphi_j & \\ & +\varphi_j & 0 & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}}.$$

The formulae for motive angles follow in addition to (164), (170) for projective ones (with B and B' according to (213), (214) or as independent $n \times n$-lineors of rank r):

$$\Phi_{12} = -(\Phi_{12})' = -\Phi_{21}, \qquad \Phi_B = -(\Phi_B)' = -\Phi'_B. \tag{268}$$

$$\overset{\Phi}{\cos\begin{bmatrix} \ddots & & & \\ & 0 & +i\varphi_j & \\ & -i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}} = \overset{D(\Phi)}{\cos\begin{bmatrix} \ddots & & & \\ & +\varphi_j & 0 & \\ & 0 & -\varphi_j & \\ & & & \ddots \end{bmatrix}} = \overset{\cos\Phi}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi_j & 0 & \\ & 0 & \cos\varphi_j & \\ & & & \ddots \end{bmatrix}},$$

$$\overset{\Phi}{i\sin\begin{bmatrix} \ddots & & & \\ & 0 & +i\varphi_j & \\ & -i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}} = \overset{i\sin\Phi}{\begin{bmatrix} \ddots & & & \\ & 0 & -\sin\varphi_j & \\ & +\sin\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}}.$$

Note, that after an change in (259) of the angle Φ by its complement $\Xi = \Pi/2 - \Phi$ with the use of formulae (175), the new matrix-function of Φ gives a rotation at Ξ:

$$\overline{Rot}\ \Phi = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \sin\varphi_i & -\cos\varphi_i & \\ & +\cos\varphi_i & \sin\varphi_i & \\ \hline & & & \ddots \end{array}\right] = Rot\ \Xi = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cos\xi_i & -\sin\xi_i & \\ & +\sin\xi_i & \cos\xi_i & \\ \hline & & & \ddots \end{array}\right]!$$

5.9 Relations between projective and motive angles and functions

From (236)–(239), and also (277), (278) – see below, we obtain (*in common bases*):

$$Ref\{\cos\tilde{\Phi}\}^{\ominus}\cdot(i\tilde{\Phi}) = \Phi = (-i\tilde{\Phi})\cdot Ref\{\cos\tilde{\Phi}\}^{\ominus}, \quad \tilde{\Phi}^2 = \Phi^2, \tag{269}$$

$$Ref\{\cos\tilde{\Phi}\}^{\ominus}\cdot\begin{Bmatrix} +\cos\tilde{\Phi} \\ -\sin\tilde{\Phi} \\ +\sec\ \tilde{\Phi} \\ -i\tan\tilde{\Phi} \end{Bmatrix} = \begin{Bmatrix} +\cos\Phi \\ +i\sin\Phi \\ +\sec\ \Phi \\ +\tan\Phi \end{Bmatrix} = \begin{Bmatrix} +\cos\tilde{\Phi} \\ +\sin\tilde{\Phi} \\ +\sec\ \tilde{\Phi} \\ +i\tan\tilde{\Phi} \end{Bmatrix}\cdot Ref\{\cos\tilde{\Phi}\}^{\ominus}. \tag{270}$$

<u>**Rule 4**</u>. *Any even degrees of tensor angles of projective and motive types as well as their same name tensor trigonometric functions are equal.* ($\langle\mathcal{P}_4\rangle = \mathbf{0}$, $r_1 = r_2$).

In real Cartesian bases, $\tilde{\Phi}$ and $i\Phi$ are real symmetric and antisymmetric bivalent tensors. Find the *complex local trigonometric base* $\tilde{E}_0$ for installing complex *pseudo-hyperbolic* analogues $\{i\varphi\}_c$ of real spherical angles $\{\varphi\}_r$ as the diagonal square root of mid-reflector or reflector tensor (254), gotten by the modal transformation of $E_1 = \{I\}$:

$$\tilde{E}_0 = \left[R'_W\cdot\sqrt{Ref\{\cos\tilde{\Phi}\}^{\ominus}}\cdot R_W\right]\cdot\tilde{E}_1 = (\sqrt{I^{\pm}})_D\cdot\tilde{E}_1 = \begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & i & \\ & & & \ddots \end{bmatrix}\cdot\tilde{E}_1 = R_c\cdot\{I\} = \{R_c\}. \tag{271}$$

Recall (see sect. 5.5, 5.7), that $\tilde{E}_1 = R_W \cdot \tilde{E} = \{I\}$ is the *real local trigonometric base* for W-forms, $\tilde{E}$ is the original Cartesian base. The complex local base $\tilde{E}_0$, unlike the base $\tilde{E}_1$, has imaginary *ordinate* axes, what correspond to the algebraically negative projective cosine eigenvalues ($u_j \to u_j, v_j \to iv_j$). All the real spherical notions are translated into $\tilde{E}_0 = \{R_c\}$ as the pseudohyperbolic ones by modal transformation R_c. Below this is exposed clearly in (272) only for the projective and motive tensor angles. Substituting a base $\tilde{E}_1$ for $\tilde{E}_0$ does not change the cosines and the secants; the angles, their sines and tangents are transformed into the pseudohyperbolic analogues. Further we use indexes "r" and "c" for notions in the real and complex bases $\tilde{E}_1$, $\tilde{E}_0$.

$$R_c^{-1} \cdot \{\tilde{\Phi}\}_r \cdot R_c = \{\tilde{\Phi}\}_c \equiv \{\Phi\}_r \equiv i\{-i\tilde{\Phi}\}_c \Leftrightarrow R_c^{-1} \cdot \{i\Phi\}_r \cdot R_c = -\{i\Phi\}_c \equiv \{-i\tilde{\Phi}\}_r \equiv \{-i\Phi\}_c, \quad (272)$$

$$\{\sin\tilde{\Phi}\}_c \equiv \{\sin\Phi\}_r \equiv \{i\sinh(-i\tilde{\Phi}\}_c \Leftrightarrow \{i\sin\Phi\}_c \equiv i\{\sin\tilde{\Phi}\}_r \equiv \{\sinh(i\Phi)\}_c, \quad (273-274)$$

$$\{i\tan\tilde{\Phi}\}_c \equiv i\{\tan\Phi\}_r \equiv \{\tanh(i\tilde{\Phi})\}_c \Leftrightarrow \{\tan\Phi\}_c \equiv \{\tan\tilde{\Phi}\}_r \equiv \{i\tanh(-i\Phi)\}_c. \quad (275-276)$$

$$\{\tilde{\Phi}\}_r = \begin{bmatrix} \ddots & & & \\ & 0 & +\varphi_j & \\ & +\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{\tilde{\Phi}\}_r{}' \;\to\; \{\tilde{\Phi}\}_c = \begin{bmatrix} \ddots & & & \\ & 0 & +i\varphi_j & \\ & -i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{\tilde{\Phi}\}_c{}^*, \quad (277)$$

$$\{i\Phi\}_r = \begin{bmatrix} \ddots & & & \\ & 0 & -\varphi_j & \\ & +\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = -\{i\Phi\}_r{}' \;\to\; \{-i\Phi\}_c = \begin{bmatrix} \ddots & & & \\ & 0 & -i\varphi_j & \\ & -i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{i\Phi\}_c{}^*. \quad (278)$$

<u>Corollaries</u> $-\{\tilde{\Phi}\}'_c = \{\tilde{\Phi}\}_c \equiv \{\Phi\}_r = -\{\Phi\}'_r \Leftrightarrow \{\Phi\}'_c = \{\Phi\}_c \equiv \{\tilde{\Phi}\}_r = \{\tilde{\Phi}\}'_r$!

Canonical forms of spherical and imaginary pseudohyperbolic trigonometric functions:

$$\{\cos(\pm\tilde{\Phi})\}_r = \begin{bmatrix} \ddots & & & \\ & +\cos\varphi_j & 0 & \\ & 0 & -\cos\varphi_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & +\cosh i\varphi_j & 0 & \\ & 0 & -\cosh i\varphi_j & \\ & & & \ddots \end{bmatrix} = \{\cosh(\pm i\tilde{\Phi})\}_c, \quad (279)$$

$$\{\cos(\pm\Phi)\}_r = \begin{bmatrix} \ddots & & & \\ & +\cos\varphi_j & 0 & \\ & 0 & +\cos\varphi_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & +\cosh i\varphi_j & 0 & \\ & 0 & +\cosh i\varphi_j & \\ & & & \ddots \end{bmatrix} = \{\cosh(\pm i\Phi)\}_c, \quad (280)$$

$$\{\sec(\pm\tilde{\Phi})\}_r = \begin{bmatrix} \ddots & & & \\ & +\sec\varphi_j & 0 & \\ & 0 & -\sec\varphi_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & +\text{sech } i\varphi_j & 0 & \\ & 0 & -\text{sech } i\varphi_j & \\ & & & \ddots \end{bmatrix} = \{\text{sech}(\pm i\tilde{\Phi})\}_c, \quad (281)$$

$$\{\sec(\pm\Phi)\}_r = \begin{bmatrix} \ddots & & & \\ & +\sec\varphi_j & 0 & \\ & 0 & +\sec\varphi_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & +\text{sech } i\varphi_j & 0 & \\ & 0 & +\text{sech } i\varphi_j & \\ & & & \ddots \end{bmatrix} = \{\text{sech}(\pm i\Phi)\}_c, \quad (282)$$

$$\{\sin\tilde{\Phi}\}_r = \begin{bmatrix} \ddots & & & \\ & 0 & +\sin\varphi_j & \\ & +\sin\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & 0 & +\sinh i\varphi_j & \\ & -\sinh i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{i\sinh(-i\tilde{\Phi})\}_c, \qquad (283)$$

$$\{i\sin\Phi\}_r = \begin{bmatrix} \ddots & & & \\ & 0 & -\sin\varphi_j & \\ & +\sin\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & 0 & -\sinh i\varphi_j & \\ & -\sinh i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{\sinh(-i\Phi)\}_c, \qquad (284)$$

$$\{i\tan\tilde{\Phi}\}_r = \begin{bmatrix} \ddots & & & \\ & 0 & -\tan\varphi_j & \\ & +\tan\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & 0 & -\tanh i\varphi_j & \\ & -\tanh i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{\tanh(-i\tilde{\Phi})\}_c, \qquad (285)$$

$$\{\tan\Phi\}_r = \begin{bmatrix} \ddots & & & \\ & 0 & +\tan\varphi_j & \\ & +\tan\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & 0 & +\tanh\ i\varphi_j & \\ & -\tanh\ i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{i\tanh(-i\Phi)\}_c. \qquad (286)$$

In the next chapter, we shall use these complex canonical pseudohyperbolic forms for the clear introducing real-valued motive and projective hyperbolic tensor angles, trigonometric functions, and reflectors in real pseudo-Euclidean space. *Cosines and secants are real-valued notions, and therefore, they are invariants of* R_c!

With Moivre's and Euler's formulae for the rotational matrix and angles, we have:

$$\begin{gathered} Rot\{m\Phi\} = \cos\{m\Phi\} + i\sin\{m\Phi\} = Rot^m\Phi = \\ = \cosh\{i\cdot m\Phi\} + \sinh\{i\cdot m\Phi\} = \exp\{i\cdot m\Phi\} \rightarrow \\ \rightarrow i\cdot\{m\Phi\} = \ln\ Rot\{m\Phi\} \rightarrow i\Phi = \ln\ Rot\ \Phi \rightarrow \Phi = -i\ln\ Rot\ \Phi. \qquad (287) \end{gathered}$$

This give motive tensor angle from a rotation tensor! The bond $\Phi \leftrightarrow \tilde{\Phi}$ is in (272)!

$$\{Rot^m\Phi\}_r = \begin{bmatrix} \ddots & & & & \\ & +\cos m\varphi_j & -\sin m\varphi_j & & \\ & +\sin m\varphi_j & +\cos\varphi_j & & \\ & & & \ddots & \\ & & & \boxed{1} & \\ & & & & \ddots \end{bmatrix} = \exp\begin{bmatrix} \ddots & & & & \\ & 0 & -m\varphi_j & & \\ & +m\varphi_j & 0 & & \\ & & & \ddots & \\ & & & \boxed{0} & \\ & & & & \ddots \end{bmatrix},$$

$$\{Rot^m\Phi\}_c = \begin{bmatrix} \ddots & & & & \\ & +\cos m\varphi_j & -i\sin m\varphi_j & & \\ & -i\sin m\varphi_j & +\cos\varphi_j & & \\ & & & \ddots & \\ & & & \boxed{1} & \\ & & & & \ddots \end{bmatrix} = \exp\begin{bmatrix} \ddots & & & & \\ & 0 & -im\varphi_j & & \\ & -im\varphi_j & 0 & & \\ & & & \ddots & \\ & & & \boxed{0} & \\ & & & & \ddots \end{bmatrix}.$$

The value $m = 1/2$ gives trigonometric square root (251) of the rotational matrix.

5.10 Deformational trigonometric functions and cross projecting

Similarly (249) and due to the principle of binarity, consequent multiplication of two *oblique* eigenreflectors for a pair of equirank lineors (planars) as in (211-214) determines tangent-secant motive transformation – the *spherical deformational matrix* function of the same tensor angle, as example, for planars $\langle im\ B\rangle$, $\langle im\ B'\rangle$ ($\langle ker\ B\rangle$, $\langle ker\ B'\rangle$):

$$\underset{\pm Ref\{B'\}}{\begin{bmatrix} \ddots & & & \\ & +\sec\ \varphi_j & -\tan\varphi_j & \\ & +\tan\varphi_j & -\sec\ \varphi_j & \\ & & & \ddots \end{bmatrix}} \underset{\pm\ Ref\{B\}}{\begin{bmatrix} \ddots & & & \\ & +\sec\ \varphi_j & +\tan\varphi_j & \\ & -\tan\varphi_j & -\sec\ \varphi_j & \\ & & & \ddots \end{bmatrix}} = \underset{Def\ \alpha_B}{\begin{bmatrix} \ddots & & & \\ & +\sec\ \alpha_j & +\tan\alpha_j & \\ & +\tan\alpha_j & +\sec\ \alpha_j & \\ & & & \ddots \end{bmatrix}}.$$

(i. e., $\alpha_B \neq 2\Phi_B$). Besides, remember, that Φ is a principal spherical motive angle.

<u>**Rule 5**</u>. *Deformational matrix functions only as the trigonometrically compatible are commutative, but their angles-arguments do not form an algebraic sum.*

In general, these matrices contain the diagonal-unity block $+I$ corresponding to zero sine subspace $\langle\mathcal{P}_3\rangle$. They, as functions, perform a deformation at some motive spherical angle $+\ \alpha_B$. Its matrix form is only similar to (249) and (250):

$$Ref\{B'\}\cdot Ref\{B\} = (\sec\ \tilde{\Phi}_B + i\tan\tilde{\Phi}_B)\cdot(\sec\ \tilde{\Phi}_B - i\tan\tilde{\Phi}_B) =$$

$$= \sec^2\tilde{\Phi}_B + \tan^2\tilde{\Phi}_B + 2i\tan\tilde{\Phi}_B\cdot\sec\ \tilde{\Phi}_B =$$

$$= \sec^2\Phi_B + \tan^2\Phi_B + 2\tan\ \Phi_B\cdot\sec\ \Phi_B = Def\ \alpha_B = Def'\alpha_B, \qquad (288)$$

$$Ref\{B\}\cdot Ref\{B'\} = Def^{-1}\alpha_B = Def(-\alpha_B). \qquad (289)$$

Notation Def is used for the deformational matrix functions of a motive-type spherical tensor angle. For them **Rule 2** does not work in entire. Deformational matrix functions are based on motive tangents and secants. (The total analogy with spherical rotations is obviously absent in the case only of a spherical kind of angles.) The binary (in trigonometric part) tensor deformation as well as rotation is executed in the trigonometric subspace (Figure 2) with respect to its spherically orthogonal complement.

The tensor secant and tangent were introduced preliminary in sect. 5.8 by (262) and (263). Now we may give their quite natural definitions in terms of the spherical deformational matrix similarly to (260) and (261):

$$\sec\ \Phi = (Def\ \Phi + Def^{-1}\Phi)/2 = [Def\ \Phi + Def(-\Phi)]/2, \qquad (290)$$

$$\tan\Phi = (Def\ \Phi - Def^{-1}\Phi)/2 = [Def\ \Phi - Def(-\Phi)]/2. \qquad (291)$$

The tensor cosecant and cotangent are the inverse or quasi-inverse sine and tangent.

The canonical W-forms of the real-valued deformational matrix is implemented in the same real-valued trigonometric base ($-\pi/2 < \varphi_j < \pi/2$):

$$Def\ \Phi = Def'\Phi = sec\ \Phi + \tan\Phi = \begin{bmatrix} \ddots & & & & \\ & \sec\ \varphi_j & \tan\varphi_j & & \\ & \tan\varphi_j & \sec\ \varphi_j & & \\ & & & \ddots & \\ & & & & \boxed{1} & \\ & & & & & \ddots \end{bmatrix}, \tag{292}$$

$$Def^{-1}\Phi = Def(-\Phi) = \sec\ \Phi - \tan\Phi. \tag{293}$$

The canonical forms for the rotational and deformational matrix functions of a pseudohyperbolic angle in the complex-valued trigonometric base are the following (for these definitions, formulae (277)-(286) may be used):

$$\{Rot\ \Phi\}_c = \cos\Phi + \{-i\sin\Phi\}_c = \begin{bmatrix} \ddots & & & & \\ & \cosh(-i\varphi)_j & +\sinh(-i\varphi)_j & & \\ & +\sinh(-i\varphi)_j & \cosh(-i\varphi)_j & & \\ & & & \ddots & \\ & & & & \boxed{1} & \\ & & & & & \ddots \end{bmatrix} =$$

$$= \cosh(-i\Phi)_c + \sinh(-i\Phi)_c, \tag{294}$$

$$\{Rot\ \Phi\}_c^{-1} = \{Rot(-\Phi)\}_c = \cos\Phi + \{i\sin\Phi\}_c = \cosh(-i\Phi)_c - \sinh(-i\Phi)_c; \tag{295}$$

$$\{Def\ \Phi\}_c = \sec\ \Phi + \{\tan\Phi\}_c = \begin{bmatrix} \ddots & & & & \\ & \mathrm{sech}(-i\varphi)_j & -\tanh(-i\varphi)j & & \\ & +\tanh(-i\varphi)_j & \mathrm{sech}(-i\varphi)_j & & \\ & & & \ddots & \\ & & & & \boxed{1} & \\ & & & & & \ddots \end{bmatrix} =$$

$$= \mathrm{sech}(-i\Phi)_c + \tanh(-i\Phi)_c, \tag{296}$$

$$\{Def\ \Phi\}_c^{-1} = \{Def(-\Phi)\}_c = \sec\ \Phi - \{\tan\ \Phi\}_c = \mathrm{sech}(-i\Phi)_c - \tanh(-i\Phi)_c. \tag{297}$$

For the rotational and deformational matrices, their determinants as well as determinants of their binary cells are equal to 1. Any deformational matrix is symmetric and positive definite. Rotation of its 2×2-cells (on the trigonometric eigenplanes) at angles $\varphi_j = \pm\pi/4$ transforms these cells into diagonal ones. The eigenvalues of the matrix cells are pairs $\mu_{2j} = \sec\ \varphi_j + \tan\varphi_j > 0$, $\mu_{2j+1} = \sec\ \varphi_j - \tan\varphi_j = \mu_{2j}^{-1} > 0$, and here not necessity in the values $\mu_k = +1$ inside the matrix unity block $\langle\mathcal{P}_3\rangle$.

In order to clarify the sense of the binary deformation function, represent it in the new base spherically rotated by modal transformation in each cells of W-form at angles $\varphi_j = \pm\pi/4$. For example, expose the rotation of some cell at angles $\varphi_j = +\pi/4$.

Namely:

$$\begin{bmatrix} \ddots & & & \\ & \sec\ \varphi_j & \tan\varphi_j & \\ & \tan\varphi_j & \sec\ \varphi_j & \\ & & & \ddots \end{bmatrix} =$$

$$= \begin{bmatrix} \ddots & & & \\ & \cos\pi/4 & -\sin\pi/4 & \\ & \sin\pi/4 & \cos\pi/4 & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & & \\ & \sec\ \varphi_j + \tan\varphi_j & 0 & \\ & 0 & \sec\varphi_j - \tan\varphi_j & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & & \\ & \cos\pi/4 & \sin\pi/4 & \\ & -\sin\pi/4 & \cos\pi/4 & \\ & & & \ddots \end{bmatrix}.$$

Now it is seen that the modal spherical deformation in canonical form (292) on the trigonometric eigenplane consists in:
1) extension of the coordinate grid along the principal diagonal (the 1-st and 3-rd quadrants) with coefficient $\mu = \sec\ \varphi + \tan\varphi > 0$,
2) contraction of the coordinate grid along the secondary diagonal with coefficient $\mu^{-1} = \sec\ \varphi - \tan\varphi > 0$.
Similarly to the real-valued binary structure (149) for a complex number, the following binary unique representation of an arbitrary real *positive* number by 2×2 deformational matrix in terms of a spherical angle $(-\pi/2 < \varphi < \pi/2)$ holds with two eigenvalue:

$$\mu = \sec\ \varphi + \tan\varphi > 0,\ \mu^{-1} = \sec\ \varphi - \tan\varphi > 0. \qquad (298)$$

From here we have $\sec\ \varphi = (\mu + \mu^{-1})/2,\quad \tan\varphi = (\mu - \mu^{-1})/2$. Numbers (298) are equivalent to analogous ones $\exp(+\gamma)$, $\exp(-\gamma)$, what will be clear in next chapter.

One more interpretation of a binary deformation is respected to so called *cross bases.* They may be use in relativistic transformations of geometric objects in Minkowskian space-time. Consider two Cartesian bases $\tilde{E}_i$ and $\tilde{E}_j$ and the rotational transformation $\tilde{E}_i = Rot(-\Phi_{ij})\tilde{E}_j$. Cartesian coordinates of a vector $\mathbf{a}$ in the two bases $\tilde{E}_j$ and $\tilde{E}_i$ are related as at *passive modal transformations* by the angle $+\Phi_{ij}$:

$$\mathbf{a}^{(i)} = Rot\ \Phi_{ij}\mathbf{a}^{(j)} = \begin{bmatrix} \ddots & & & \\ & \cos\varphi_t & -\sin\varphi_t & \\ & \sin\varphi_t & \cos\varphi_t & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_1^{(j)} \\ x_2^{(j)} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \cos\varphi_t x_1^{(j)} - \sin\varphi_t x_2^{(j)} \\ \sin\varphi_t x_1^{(j)} + \cos\varphi_t x_2^{(j)} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x_1^{(i)} \\ x_2^{(i)} \\ \vdots \end{bmatrix}.$$

In 2×2-cells, the base $\tilde{E}_i$ is the result of rotating $\tilde{E}_j$ at the clockwise angles $+\varphi_t$. Introduce *cross bases* $\tilde{E}_{i,j}$ with mixed axes $\langle x_1^{(i)}, x_2^{(j)}\rangle$ and $\tilde{E}_{j,i}$ with mixed axes $\langle x_1^{(j)}, x_2^{(i)}\rangle$. These both bases are related by the *cross modal transformation*:

$$\tilde{E}_{i,j} = Def(-\Phi_{ij})\tilde{E}_{j,i}. \qquad (299)$$

In t-th cells, so called *cross coordinates* of a vector $\mathbf{a}$ in the cross bases $\tilde{E}_{i,j}$ and $\tilde{E}_{j,i}$ are related as at *passive cross modal transformations* by the angle $+\Phi_{ij}$:

$$\mathbf{a}^{(i,j)} = Def(+\Phi_{ij})\mathbf{a}^{(j,i)} = \begin{bmatrix} \ddots & & & \\ & \sec\ \varphi_t & \tan\varphi_t & \\ & \tan\varphi_t & \sec\ \varphi_t & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_1^{(j,i)} \\ x_2^{(i,j)} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x_1^{(i,j)} \\ x_2^{(j,i)} \\ \vdots \end{bmatrix}. \qquad (300).$$

Then the cross coordinates of vector $\mathbf{a}^{(i,j)}$ are determined here by *cross projecting* with the use of deformational matrix-function of a principal motive angle compatible with a reflector tensor of the space (see sect. 5.7). In a quasi-Euclidean space, coordinates of a linear object, obtained with Cartesian projecting, correspond to trigonometric invariants of types (182) or (264). Contrary, its cross coordinates, obtained with cross projecting, correspond to trigonometric *one-step pseudo-invariant* of types (208) or (265) on the basis of sine-tangent spherical-hyperbolic analogy – see further in Ch. 6.

5.11 Special transformations of orthogonal and oblique eigenprojectors

In an Euclidean space, there exists an one-to-one correspondence between a centralized planar and a symmetric projector of the same rank. There also exists an one-to-one correspondence between the planar and its orthocomplement. Any planar may be transformed into each other one of the same rank with tensor rotation as well as with tensor mid-reflector (mid-reflector give only the single motive angle Φ!). Formulae for such transformations may be derived, for example, of (256), (226), (176) and (177) or with direct applying the principle of binarity.

$$\overleftrightarrow{A_2A_2'} = Rot\ \Phi_{12} \cdot \overleftrightarrow{A_1A_1'} \cdot Rot'\ \Phi_{12} = Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus} \cdot \overleftrightarrow{A_1A_1'} \cdot Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus}, \quad (301)$$

$$\overleftrightarrow{B'B} = Rot\ \Phi_B \cdot \overleftrightarrow{BB'} \cdot Rot'\ \Phi_B = Ref\{\cos\tilde{\Phi}_B\}^{\ominus} \cdot \overleftrightarrow{BB'} \cdot Ref\{\cos\tilde{\Phi}_B\}^{\ominus}. \quad (302)$$

These are rotation and reflection transformations of 2-valent orthogonal tensors inside of the symbolic octahedron (Figure 1). Use the octahedron for illustration. The diagonal PQ generates isosceles triangles PZQ and PIQ, where $\angle PZQ \equiv \angle PIQ \equiv \Phi_B$.

Moreover, in an Euclidean space, there exists due to (217)–(220) an one-to-one correspondence between a pair of *equirank* centralized planars $\langle im\ A_1\rangle$, $\langle im\ A_2\rangle$ (then $k(A_1A_2', r) = det(A_1'A_2) \neq 0$) and a pair of oblique eigenprojectors $\overleftarrow{B}$, $\overleftarrow{B'}$ (as in (213) $B = A_1A_2'$). Then $\overleftarrow{B}$ and $\overleftarrow{B'}$ ($\overrightarrow{B}$ and $\overrightarrow{B'}$) are transformed into each other with tensor deformation as well as with tensor mid-reflector. Formulae for such transformations (as formulae (301), (302)) may be derived too with the principle of binarity.

$$\overset{\rightleftarrows}{B'} = Def\ \Phi_B \cdot \overset{\rightleftarrows}{B} \cdot Def(-\Phi_B) = Ref\{\cos\tilde{\Phi}_B\}^{\ominus} \cdot \overset{\leftrightarrows}{B} \cdot Ref\{\cos\tilde{\Phi}_B\}^{\ominus}. \quad (303)$$

(These non-similarity and similarity with 1st and 2nd parts of (302) are quite logical.)

Following formulae are similar to rotational prototypes (255) and (256):

$$Ref\{B'\} = Def\ \Phi_B \cdot Ref\{B\} \cdot Def(-\Phi_B) =$$

$$= Ref\{\cos\tilde{\Phi}_B\}^{\ominus} \cdot Ref\{B\} \cdot Ref\{\cos\tilde{\Phi}_B\}^{\ominus}, \quad (304)$$

$$Ref\{\cos\tilde{\Phi}_B\}^{\ominus} = Def\ \Phi_B \cdot Ref\{B\} = Def(-\Phi_B) \cdot Ref\{B'\}. \quad (305)$$

If the original matrix B is null-prime, then, for example, from formulae (186)-(189) and relation $\cos\tilde{\Phi}_B \cdot \sec\ \tilde{\Phi}_B = I$ one may get the mutual modal transformations:

$$\left\{ \begin{matrix} \overleftrightarrow{B'B} \\ \overset{\rightleftarrows}{B'} \end{matrix} \right\} = \cos\tilde{\Phi}_B \cdot \left\{ \begin{matrix} \overleftrightarrow{BB'} \\ \overset{\rightleftarrows}{B} \end{matrix} \right\} \cdot \sec\ \tilde{\Phi}_B = \sec\ \tilde{\Phi}_B \cdot \left\{ \begin{matrix} \overleftrightarrow{BB'} \\ \overset{\rightleftarrows}{B} \end{matrix} \right\} \cdot \cos\tilde{\Phi}_B. \tag{306}$$

(The formulae may be checked also with the use of eigenprojectors multiplication table in sect. 5.2.) Formulae indicated above represent the modal transformations found by different manners, but *the results are the same.* Express all the eigenprojectors in terms of corresponding projective trigonometric functions pairs according to (176)–(179):

$$\left.\begin{aligned} \overleftarrow{A_1A_1'} &= (I + \cos\tilde{\Phi} - \sin\tilde{\Phi})/2 = \overleftarrow{BB'}, \\ \overrightarrow{A_1A_1'} &= (I - \cos\tilde{\Phi} + \sin\tilde{\Phi})/2 = \overrightarrow{BB'}, \\ \overleftarrow{A_2A_2'} &= (I + \cos\tilde{\Phi} + \sin\tilde{\Phi})/2 = \overleftarrow{B'B}, \\ \overrightarrow{A_2A_2'} &= (I - \cos\tilde{\Phi} - \sin\tilde{\Phi})/2 = \overrightarrow{B'B}, \end{aligned}\right\} \tag{307}$$

$$\left.\begin{aligned} \overleftarrow{B} &= (I + \sec\ \tilde{\Phi} - i\,\mathrm{tg}\,\tilde{\Phi})/2 = \overleftarrow{A_1A_2'}, \\ \overrightarrow{B} &= (I - \sec\ \tilde{\Phi} + i\,\mathrm{tg}\,\tilde{\Phi})/2 = \overrightarrow{A_1A_2'}, \\ \overleftarrow{B'} &= (I + \sec\ \tilde{\Phi} + i\,\mathrm{tg}\,\tilde{\Phi})/2 = \overleftarrow{A_2A_1'}, \\ \overrightarrow{B'} &= (I - \sec\ \tilde{\Phi} - i\,\mathrm{tg}\,\tilde{\Phi})/2 = \overrightarrow{A_2A_1'}. \end{aligned}\right\} \tag{308}$$

These expressions show that *the principle of binarity is valid for projectors too.* There exists a bijection between the set of eigen orthoprojectors and the set of symmetric idempotent matrices of the same size and rank. Iff the matrix B is null-prime, then $det\cos\tilde{\Phi} \neq 0$, and there exists a bijection between the pairs $\langle\overleftrightarrow{BB'}, \overleftrightarrow{B'B}\rangle$ and $\langle\overset{\rightleftarrows}{B}, \overset{\rightleftarrows}{B'}\rangle$.

Represent orthoprojectors in the trigonometric W-form according to (307). Principle of binarity enable one to evaluate the modal matrices for constructing D-forms. For example, consider this for orthoprojector $\overleftarrow{BB'}$. In i-cells of matrices, there holds:

$$\underset{Rot\ \Phi_B/2}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi/2 & -\sin\varphi/2 & \\ & \sin\varphi/2 & \cos\varphi/2 & \\ & & & \ddots \end{bmatrix}} \cdot \frac{1}{2} \underset{\overleftarrow{BB'}}{\begin{bmatrix} \ddots & & & \\ & 1+\cos\varphi & -\sin\varphi & \\ & -\sin\varphi & 1-\cos\varphi & \\ & & & \ddots \end{bmatrix}} \cdot \underset{Rot'\Phi_B/2}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi/2 & \sin\varphi/2 & \\ & -\sin\varphi/2 & \cos\varphi/2 & \\ & & & \ddots \end{bmatrix}} =$$

$$= \underset{D(\overleftarrow{BB'})}{\begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & 0 & \\ & & & \ddots \end{bmatrix}}, \text{ i. e., } V_{col}^{-1} \cdot \overleftrightarrow{BB'} \cdot V_{col} = D(\overleftrightarrow{BB'}).$$

This matrix is expressed in the original orthogonal base $\tilde{E}$ as $\overleftarrow{BB'}$, but in D-form they is expressed as above in the base:

$$\tilde{E}_D = V_{col} \cdot \tilde{E} = Rot\ (-\Phi_B/2) \cdot \tilde{E} = Rot\ (-\Phi_B/2) \cdot R'_W \tilde{E}_1 = \{Rot\ (-\Phi_B/2)\}, \quad (309)$$

(here $R_W \cdot \tilde{E} = \tilde{E}_1 = \{I\}$ is the base of W-forms).

The following orthogonal eigenvector-columns of the same modal matrix correspond to the subspaces $\langle im\ B\rangle$ and $\langle ker\ B\rangle$:

$$\mathbf{b}_{i_I} = R_W \cdot \begin{bmatrix} \vdots \\ 0 \\ +\cos\varphi_i/2 \\ -\sin\varphi_i/2 \\ 0 \\ \vdots \end{bmatrix}, \quad \mathbf{d}_{i_I} = R_W \cdot \begin{bmatrix} \vdots \\ 0 \\ +\sin\varphi_i/2 \\ +\cos\varphi_i/2 \\ 0 \\ \vdots \end{bmatrix}.$$

By analogy, ones find the modal matrix for getting the base for the eigen orthoprojector $\overleftarrow{B'B}$ diagonal form, i. e., for $D(\overleftrightarrow{B'B}) = V_{col}^{-1} \cdot \overleftrightarrow{B'B} \cdot V_{col}$. This base is

$$\tilde{E}_D = V_{col} \cdot \{\tilde{E}\} = Rot\ (+\Phi_B/2) \cdot R_W \{\tilde{E}\} = R_W \cdot \{R'_W \cdot Rot\ (+\Phi_B/2) \cdot R_W\}\{\tilde{E}\}. \quad (310)$$

The following orthogonal eigenvector-columns of the other modal matrix, gotten by (310), correspond to the subspaces $\langle im\ B'\rangle$ and $\langle ker\ B'\rangle$:

$$\mathbf{b}_{i_{II}} = R_W \cdot \begin{bmatrix} \vdots \\ 0 \\ +\cos\varphi_i/2 \\ +\sin\varphi_i/2 \\ 0 \\ \vdots \end{bmatrix}, \quad \mathbf{d}_{i_{II}} = R_W \cdot \begin{bmatrix} \vdots \\ 0 \\ -\sin\varphi_i/2 \\ +\cos\varphi_i/2 \\ 0 \\ \vdots \end{bmatrix}.$$

Modal matrices for constructing D-forms of mutual oblique eigenprojectors will be derived in Chapter 6 with the use of spherical-hyperbolic analogy. Here, we present preliminary two expressions in terms of arithmetic roots of the same deformational matrix, though they have no *spherical* trigonometric sense:

$$\{R'_W \cdot \sqrt{Def\ \Phi_B}\} \cdot \overleftrightarrows{B} \cdot \{\sqrt{Def\ (-\Phi_B)}\} \cdot R_W\} = D(\overleftrightarrows{B}), \quad (311)$$

$$\{R'_W \cdot \sqrt{Def\ (-\Phi_B)}\} \cdot \overleftrightarrows{B'} \cdot \{\sqrt{Def\ \Phi_B}\} \cdot R_W\} = D(\overleftrightarrows{B'}). \quad (312)$$

Thus we can see much common in various modal transformations of the mutual eigenprojectors and eigenreflectors from one into another with trigonometric rotational and deformational modal matrices. This is usually obviously, when the operations are executed in their same bases of W-forms. What's more, in the middle of the modal transformations we have their diagonal forms. In particular, for mutual eigenreflectors we get their mid-reflector.

5.12 Tensor spherical trigonometric functions with a frame axis

Consider the set of centralized homogeneous motions in Euclidean or quasi-Euclidean spaces (see sect. 5.7). They are determined by spherical rotational matrix functions *Rot* Φ. Rotational matrices with the minimal trigonometric subspace for homogeneous motions of a vector, a straight line, and a hyperplane in an Cartesian base have the unique trigonometric 2×2-cell (see (251) and (252)). Similar trigonometric matrix functions are called further as *elementary*. Notation *rot* Φ, *rot* Θ is used for them as the particular cases of *Rot* Φ, *Rot* Θ. Elementary trigonometric matrices are used further for description of principal spherical rotations and reflections in an quasi-Euclidean space $\langle \mathcal{Q}^{n+q} \rangle$ with set diagonal reflector tensor $\{I^{\pm}\}$ of the index $q = 1$ (sect. 5.7). The elementary rotations have one eigen scalar rotation angle φ and the unique rotation frame axis. The very important variant, when the *frame axis* is $\langle x_{n+1} \rangle$ in $\langle \mathcal{Q}^{n+1} \rangle$. In this especial case, matrices for principal elementary rotations in the so-called *E-form* have the following canonical structure in the universal Cartesian base $\tilde{E}_{1u} = \{I\}$:

$$\{rot(\pm\Phi)\}_{3\times 3}$$

$1-(1-\cos\varphi)\cos^2\alpha_1$	$-(1-\cos\varphi)\cos\alpha_1\cos\alpha_2$	$\mp\sin\varphi\cos\alpha_1$
$-(1-\cos\varphi)\cos\alpha_1\cos\alpha_2$	$1-(1-\cos\varphi)\cos^2\alpha_2$	$\mp\sin\varphi\cos\alpha_2$
$\pm\sin\varphi\cos\alpha_1$	$\pm\sin\varphi\cos\alpha_2$	$\cos\varphi$

, (313)

$$\{rot(\pm\Phi)\}_{(n+1)\times(n+1)}$$

$I_{n\times n} - (1-\cos\varphi)\cdot \mathbf{e}_\alpha \mathbf{e}'_\alpha$	$\mp\sin\varphi\cdot\mathbf{e}_\alpha$
$\pm\sin\varphi\cdot\mathbf{e}'_\alpha$	$\cos\varphi$

, $\quad (\mathbf{e}_\alpha \mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha \mathbf{e}'_\alpha}).$ (314)

The coordinates of the matrices are expressed, as usual, with respect to the right Cartesian base $\tilde{E}_{1u}$. The *oriented* straight line $\langle x_{n+1} \rangle$ is the frame (polar) axis for the eigen rotation angle φ, the angle is positive for $rot\{+\Phi\}$, and has the directional cosines $\cos\alpha_k$, $k = 1, \ldots, n$ in the base $\langle x_1, \ldots, x_n \rangle$ – the frame axis orthocomplement.

At first, prove formula (313). Find a rotational transformation of the complement base $\langle x_1, x_2 \rangle$ into new same base $\langle x'_1, x'_2 \rangle$ such that the axis $\langle x'_1 \rangle$, $\mathbf{e}_\alpha = (\cos\alpha_1, \cos\alpha_2)$ (where $\cos^2\alpha_1 + \cos^2\alpha_2 = 1$), and the frame axis $\langle x_3 \rangle$ should be coplanar. This transformation is the spherical rotation matrix at a certain tensor angle β_{12}. If $n = 2$, then it has the scalar eigen angle α_1, and the rotational matrix demanded is

$$rot\ \beta_{12}$$

$\cos\alpha_1$	$-\sin\alpha_1$	0
$+\sin\alpha_1$	$\cos\alpha_1$	0
0	0	1

.

This matrix function executes the rotation on the plane $\langle x_1, x_2 \rangle$ at the angle α_1.

Further, in this new 3-dimensional base $\tilde{E}$ we use the elementary principal rotational matrix function $rot\ \Phi$, but in the 2×2-cell corresponding to the plane $\langle x_1', x_3 \rangle$, with following condition: if the frame axis is $\langle x_1' \rangle$, then the angle of rotation is counter-clockwise; if the frame axis is $\langle x_3 \rangle$, then this angle is clockwise. So, the last form of this elementary spherical rotational matrix is

$$\{rot(\pm\Phi)\}$$

$\cos\varphi$	0	$-\sin\varphi$
0	1	0
$\sin\varphi$	0	$\cos\varphi$

. (315)

Then we transform the matrix in E-form applying the inverse base rotation

$$\{rot(\pm\Phi)\}_{3\times 3} = rot\ \beta_{12} \cdot \{rot(\pm\Phi)\} \cdot rot\ \beta_{12}\,'.$$

The result is rotational matrix function (313) with the frame axis $\langle x_3 \rangle$ for the motive tensor angle Φ in 3-dimensional Cartesian base $\tilde{E}_{1u}$.

General formula (314) is inferred similarly. Find a rotational transformation of $\langle x_1, \ldots, x_n \rangle$ into $\langle x_1', \ldots, x_n' \rangle$ such that the axis $\langle x_1' \rangle$, the directional cosines vector $\mathbf{e}_\alpha = \{\cos\alpha_k\}$ ($\sum_{k=1}^{n} \cos^2\alpha_k = 1$), and the frame axis $\langle x_{n+1} \rangle$ should be coplanar. Use consequently tensor angles of the radius-vector rotation with their spherical coordinates: β_{12} in the plane $\langle x_1, x_2 \rangle$, $\beta_{1'3}$ in the plane $\langle x_1', x_3 \rangle$, ..., $\beta_{1''\ldots'n}$ in the plane $\langle x_1'' \ldots ', x_n \rangle$. Due to the trigonometric nature of the transformations, we have the following formulae:

$$\left.\begin{aligned} \cos\beta_{12} &= \cos\alpha_1 / \sqrt{\cos^2\alpha_1 + \cos^2\alpha_2}, \\ \cos\beta_{1'3} &= \sqrt{\cos^2\alpha_1 + \cos^2\alpha_2} / \sqrt{\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3}, \\ &\vdots \\ \cos\beta_{1''\ldots'n} &= \sqrt{\cos^2\alpha_1 + \cdots + \cos^2\alpha_{n-1}} = \sin\alpha_n. \end{aligned}\right\}$$

The consequent rotations are executed with the matrices $rot\ \beta_{12}$, $rot\ \beta_{1'3}$, ... :

$rot\ \beta_{12}$

$\cos\beta_{12}$	$-\sin\beta_{12}$	Z
$\sin\beta_{12}$	$\cos\beta_{12}$	
Z'		I_{n-1}

,

$rot\ \beta_{1'3}$

$\cos\beta_{1'3}$	0	$-\sin\beta_{1'3}$	Z
0	1	0	
$\sin\beta_{1'3}$	0	$\cos\beta_{1'3}$	
Z'			I_{n-2}

, ...

The result is the base of the simplest 2×2-cell form for the elementary rotation:

$$\tilde{E} = rot\ \beta \cdot \tilde{E}_1,$$

where $rot\ \beta = rot\ \beta_{12} \cdot rot\ \beta_{1'3} \cdots rot\ \beta_{1''\ldots'n}$.

Then construct the 2-dimensional form for this elementary rotation in the hyperplane $\langle x'_1, \ldots, x'_n, x_{n+1}\rangle$ with respect to the base $\tilde{E}$:

$$\{rot\ \Phi\}$$

$$\begin{array}{|c|c|c|} \hline \cos\varphi & \mathbf{0}' & -\sin\varphi \\ \hline \mathbf{0} & I_{n-1} & \mathbf{0} \\ \hline \sin\varphi & \mathbf{0}' & \cos\varphi \\ \hline \end{array}\ . \tag{316}$$

Further we transform the matrix in E-form applying the inverse base rotation

$$\{rot\ \Phi\}_{(n+1)\times(n+1)} = rot\ \beta \cdot \{rot\ \Phi\}_{can} \cdot rot'\ \beta.$$

<u>The result</u> is rotational matrix function (314) with the frame axis $\langle x_{n+1}\rangle$ for the motive tensor angle Φ in $(n+1)$-dimensional Cartesian base $\tilde{E}_{1u}$.

Similarly, if the angle of elementary rotation is negative (as an angle in the same trigonometric plane), then there holds

$$\{rot(-\Phi)\}_{(n+1)\times(n+1)} = rot\ \beta \cdot \{rot(-\Phi)\}_{can} \cdot rot'\ \beta.$$

Any elementary trigonometric matrices represented in their E-forms have always the minimal trigonometric rank $\tau = 1$ respecting to a principal tensor angle Φ (sect. 5.5), as their W-form includes only one principal 2×2-cell! Other 2×2-cells are absent.

If one deals with point objects (given by radius-vectors) or hyperplanes (i. e., for them $r = 1$ or $s = 1$), then their continuous modal transformations are completely determined by the elementary rotational matrices. They either may be given, or may be determined for two the objects by their eigenreflectors according to (251). Expose these modal transformations in quasi-Euclidean space in terms of the elementary matrices with the *straight order* of two pure types rotations. They are spherical $rot\ \Phi_{12}$ and orthospherical $rot\ \Theta_{12}$. In this general transformation, the matrices is set initially in the original base $\tilde{E}_1$ Then we have really

$$\tilde{E}_2 = R_{12} \cdot \tilde{E}_1 = rot\ \Phi_{12} \cdot rot\ \Theta_{12} \cdot \tilde{E}_1 =$$

$$= \{rot\ \Phi_{12} \cdot rot\ \Theta_{12} \cdot rot(-\Phi_{12})\} \cdot rot\ \Phi_{12} \cdot \tilde{E}_1. \tag{317}$$

In this interpretation, after realization of the principal motion $rot\ \Phi_{12}$ (*boost*) the secondary orthospherical rotation in the base $\tilde{E}_{1s} = rot\ \Phi_{12} \cdot \tilde{E}_1$ (in the brackets) is realized. The coordinates of indicated above geometric objects are transforming by passive way (as for 1-valent tensors) with the reverse order of modal transformations:

$$\mathbf{u}^{(2)} = rot(-\Theta_{12}) \cdot rot(-\Phi_{12}) \cdot \mathbf{u}^{(1)}. \tag{318}$$

The similar orders of pure types modal rotations are using in two-step and multistep motions of geometric objects (see more in Ch. 11).

In addition to exposed for rotational matrices, finally we consider briefly analogous elementary spherical deformational matrix functions $def\ \Phi$ as (292). The deformational matrices with the minimal trigonometric subspace for homogeneous deformation of a vector, a straight line, and a hyperplane in an Cartesian base have too the unique trigonometric 2×2-cell. Notation $def\ \Phi$ is used for them as the particular case of $Def\ \Phi$. Elementary deformations have also one eigen scalar deformation angle φ and accordingly the same unique deformation frame axis. The more important variant, if the frame axis is $\langle x_{n+1}\rangle$ in $\langle \mathcal{Q}^{n+1}\rangle$. Then the matrices in the special Cartesian base $\tilde{E}_{1u} = \{I\}$ have the canonical structure in E-form:

$\{def(\pm\Phi)\}_{3\times 3}$

$1 + (\sec\varphi - 1)\cos^2\alpha_1$	$(\sec\varphi - 1)\cos\alpha_1\cos\alpha_2$	$\pm\tan\varphi\cdot\cos\alpha_1$
$(\sec\varphi - 1)\cos\alpha_1\cos\alpha_2$	$1 + (\sec\varphi - 1)\cos^2\alpha_2$	$\pm\tan\varphi\cdot\cos\alpha_2$
$\pm\tan\varphi\cdot\cos\alpha_1$	$\pm\tan\varphi\cdot\cos\alpha_2$	$\sec\varphi$

, (319)

$\{def(\pm\Phi)\}_{(n+1)\times(n+1)}$

$I_{n\times n} + (\sec\varphi - 1)\cdot \mathbf{e}_\alpha \mathbf{e}'_\alpha$	$\pm\tan\varphi\cdot\mathbf{e}_\alpha$
$\pm\tan\varphi\cdot\mathbf{e}'_\alpha$	$\sec\varphi$

$$, \quad (\mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha})\ . \tag{320}$$

The coordinates of the matrices are expressed, as usual, with respect to the right Cartesian base $\tilde{E}_{1u}$. The *oriented* straight line $\langle x_{n+1}\rangle$ is the frame (polar) axis for the angle φ of deformation, this angle is positive for $def\ {+\Phi}$ and has the directional cosines $\cos\alpha_k$, $k = 1, \ldots, n$, with respect to the base $\langle x_1, \ldots, x_n\rangle$ of $\langle x_{n+1}\rangle$ orthocomplement. The canonical E-forms (319), (320) are inferred by similar way.

In quasi-Euclidean space $\langle \mathcal{Q}^{n+1}\rangle$ *not axes-oriented* with the set quasi-Cartesian base $\{\mathbf{e}_k\} = R\tilde{E}_1$ and the selected ordinate $\mathbf{e}_{n+1}$, its reflector tensor is defined as follows:

$$\{\sqrt{I}\}_S = R'\{I^{\mp}\}R = \overrightarrow{\mathbf{e_{n+1}e'_{n+1}}} - \overleftarrow{\mathbf{e_{n+1}e'_{n+1}}} = -ref\{\mathbf{e_{n+1}e'_{n+1}}\} = I - 2\cdot\mathbf{e_{n+1}e'_{n+1}},\ \ (q=1), \tag{321}$$

where $\mathbf{e_{n+1}}$ is also the axis $\langle x_{n+1}\rangle$ and the *orthogonal reflector mirror* – see (176). (In the most general case, $n \times q$ quasi-orthogonal matrix Rq of (129) may be used.)

Thus this chapter represented fundamentals of Tensor Trigonometry in its Euclidean and quasi-Euclidean versions, which are realized in their same spaces. These spaces have the quadratic Euclidean metric. In any quasi-Euclidean space, its reflector tensor may be given either in the *sign-alternating unity forms* $\{I^{\pm}\}$ $(q \le n)$ and $\{I^{\mp}\}$ $(n < q)$ or in the *general form* $\{\sqrt{I}\}_S = \{R_W I^{\pm} R'_W\}$. In these spaces, a reflector tensor generates the continuous group of motions including the *set of principal spherical rotations* and the *subgroup of secondary orthospherical rotations*, and in addition the *set of principal orthogonal reflectors* connected with the eigenprojectors. Note, that the n-dimensional Euclidean geometry, when $q = 0$ and the q-dimensional anti-Euclidean geometry, when $n = 0$, are two extreme cases of the general quasi-Euclidean geometry with unity $\{I\}$ and antiunity $\{-I\}$ reflector tensors. So, the reflector tensor with the given metric are main attributes of these spaces.

Chapter 6

Pseudo-Euclidean tensor and scalar trigonometry as a basis

6.1 The hyperbolic tensor angles, trigonometric functions, and reflectors

Modal transformation (271) gives rise to pseudohyperbolic angles and their functions. Angles $\{-i\Phi\}_c$ have hyperbolic form. *Pseudo-hyperbolic trigonometry* is realizable in the *complex binary (real-imaginary) pseudo-Euclidean space* $\langle \mathcal{P}^{n+q} \rangle_c$. Structure of this space is determined by the mid-reflector or generally by the set *reflector tensor* $\{I^{\pm}\}$ of original $\langle \mathcal{Q}^{n+q} \rangle$ (sect. 5.7), the scalar product is invariant in these isometric spaces:

$$\mathbf{x}'\mathbf{x} = (R_c \mathbf{z}_{01})' \cdot (R_c \mathbf{z}_{01}) = [(\sqrt{I^{\pm}})\mathbf{z}_{01}]' \cdot [(\sqrt{I^{\pm}})\mathbf{z}_{01}] = \mathbf{z}'_{01}\{\sqrt{I^{\pm}}\}^2 \mathbf{z}_{01} = \mathbf{z}'_{01}\{I^{\pm}\}\mathbf{z}_{01},$$

where $\mathbf{z}_{01} = R_c^{-1} \cdot \mathbf{x}$ in $\tilde{E}_{01} = R_c \cdot \tilde{E}_1$, according to (271); $\tilde{E}_1 = \{I\}$. Thus in $\langle \mathcal{P}^{n+q} \rangle_c$ we have $\{I^{\pm}\} = R'_c \cdot R_c = R_c^2 = \{\sqrt{I^{\pm}}\}_D^2$ as the *metric tensor*. With respect to the original Cartesian base $\tilde{E}$ the latter may have the form $\{R_W \cdot I^{\pm} \cdot R'_W\} = \{\sqrt{I}\}_S$. Hence, in $\langle \mathcal{P}^{n+q} \rangle_c$ a reflector tensor and a metric tensor are equivalent! Importance of the complex pseudo-Euclidean space consists in simple following transition off spherical notions into hyperbolic ones. This is realized with the use of intermediate angles – pseudoanalogues (277), (278), for example, in the case of motive angles by two ways:

$$\Phi \leftrightarrow -i\Phi \leftrightarrow \Gamma, \quad \varphi_j \leftrightarrow -i\varphi_j \leftrightarrow \gamma_j, \; (\mathbf{x} \; in \; \tilde{E}_1 \leftrightarrow \mathbf{z}_{01} \; in \; \tilde{E}_{01} \leftrightarrow \mathbf{u} \; in \; \tilde{E}_1), \tag{322}$$

$$\Gamma \leftrightarrow +i\Gamma \leftrightarrow \Phi, \quad \gamma_j \leftrightarrow i\gamma_j \leftrightarrow \varphi_j, \; (\mathbf{u} \; in \; \tilde{E}_1 \leftrightarrow \mathbf{z}_{02} \; in \; \tilde{E}_{02} \leftrightarrow \mathbf{x} \; in \; \tilde{E}_1). \tag{323}$$

This transition between imaginary and real angles is called *spherical–hyperbolic analogy of abstract type* with preserving binary spaces structure and reflector tensor. Applying abstract analogy (322) to relations (277)–(286) and (294)–(297), one obtains the hyperbolic analogs of angles, trigonometric functions and reflectors in a *new* real-valued pseudo-Euclidean space $\langle \mathcal{P}^{n+q} \rangle$ with the same metric reflector tensors $\{\sqrt{I}\}_S$, and their W-forms in the base of diagonal cosine $\tilde{E}_1$. In $\tilde{E} = R_W^{-1} \cdot \tilde{E}_1$ we have the following.

$$R_W \cdot \begin{bmatrix} \ddots & & & \\ & \cosh\gamma_j & \sinh\gamma_j & \\ & \sinh\gamma_j & \cosh\gamma_j & \\ & & & \ddots \end{bmatrix} \cdot R'_W = \cosh\Gamma + \sinh\Gamma =$$

$$= Roth\;(+\Gamma) = Roth'\;\Gamma = \exp\;(+\Gamma), \tag{324}$$

$$Roth\;(-\Gamma) = \cosh\;\Gamma - \sinh\;\Gamma = Roth^{-1}\;\Gamma = \exp(-\Gamma). \tag{325}$$

$$\Rightarrow \; Roth\;(+\Gamma) \cdot Roth\;(-\Gamma) = \exp\;(+\Gamma) \cdot \exp(-\Gamma) = \cosh^2\Gamma - \sinh^2\Gamma = I.$$

This is the hyperbolic rotational matrix function of the motive angle Γ (or $-\Gamma$).

$$R_W \cdot \begin{bmatrix} \ddots & & & \\ & \operatorname{sech}\gamma_j & -\tanh\gamma_j & \\ & +\tanh\gamma_j & \operatorname{sech}\gamma_j & \\ & & & \ddots \end{bmatrix} \cdot R'_W = \operatorname{sech}\Gamma + i\tanh\Gamma = Defh\,(+\Gamma), \quad (326)$$

$$Defh\,(-\Gamma) = \operatorname{sech}\Gamma - i\tanh\,\Gamma = Defh^{-1}\,\Gamma = Defh'\,\Gamma. \quad (327)$$

$$\Rightarrow\; Defh\,(+\Gamma)\cdot Defh\,(-\Gamma) = Defh\,\Gamma \cdot Defh'\,\Gamma = \operatorname{sech}^2\Gamma + \tanh^2\Gamma = I.$$

This is the hyperbolic deformational matrix function of the motive angle Γ (or $-\Gamma$).

$$R_W \cdot \begin{bmatrix} \ddots & & & \\ & \cosh\gamma_j & \pm\sinh\gamma_j & \\ & \mp\sinh\gamma_j & -\cosh\gamma_j & \\ & & & \ddots \end{bmatrix} \cdot R'_W = \cosh\tilde{\Gamma} \mp \sinh\tilde{\Gamma}, \quad (328)$$

$$R_W \cdot \begin{bmatrix} \ddots & & & \\ & \operatorname{sech}\gamma_j & \mp\tanh\gamma_j & \\ & \mp\tanh\gamma_j & -\operatorname{sech}\gamma_j & \\ & & & \ddots \end{bmatrix} \cdot R'_W = \operatorname{sech}\tilde{\Gamma} \mp \tanh\tilde{\Gamma}. \quad (329)$$

They are the hyperbolic orthogonal and oblique reflectors with the projective angle $\tilde{\Gamma}$.

In pseudo-Euclidean trigonometry, the *mid-reflector* by analogy to definition (253), with the maximal trigonometric rank of trigonometrically compatible angles, is identical to a reflector tensor determined the *non-coaxially oriented* pseudo-Euclidean space:

$$Ref\;\{\cosh\tilde{\Gamma}\}^{\ominus} = \{\sqrt{I}\}_S = \{R_W \cdot I^{\pm} \cdot R'_W\}, \quad (\tau = \tau_{max} = q). \quad (330)$$

(*The tensor in W-form* $\{I^{\pm}\}$ *and the matrix* R_W *are not trigonometrically compatible*!)

Apply the principle of binarity and take into account (271) and (324), the result are the following conditions of annihilation similar to (257) for secondary orthospherical rotations $Rot\;\Theta$ and for quasi-Euclidean principal rotations as

$$Roth\;\Gamma \cdot \{\sqrt{I}\}_S \cdot Roth\;\Gamma = \{\sqrt{I}\}_S.$$

The modal transformation R_c^{-1} converts hyperbolic angles and functions into pseudo-spherical ones. Angles $\{i\Gamma\}_c$ have spherical form. *Pseudo-spherical trigonometry* is realizable in isometric to original $\langle \mathcal{P}^{n+q} \rangle$ a *complex quasi-Euclidean space* $\langle \mathcal{Q}^{n+q} \rangle_c$. The scalar product is invariant in both these isometric spaces:

$$\mathbf{u}' \cdot \{I^{\pm}\} \cdot \mathbf{u} = (R_c\mathbf{u})'(R_c\mathbf{u}) = \mathbf{z}'_{02} \cdot \mathbf{z}_{02}.$$

Such space $\langle \mathcal{Q}^{n+q} \rangle_c$ is a complex isomorphism of real pseudo-Euclidean one. It was introduced by H. Poincaré in 1905 [47] as the two-dimensional model of a relativistic space-time with the Lorentz transformations group called so too by Poincaré.

Abstract analogy (323) applied to the pseudo-spherical angles and functions gives finally the original spherical notions in a quasi-Euclidean space. **The whole closed cycle (322)–(323) with abstract spherical–hyperbolic analogy is described.**

The analogy with spherical formulae (269) and (270) connects hyperbolic projective and motive angles and their functions *in common bases* in terms of mid-reflectors:

$$-i\tilde{\Gamma}_{12} \cdot Ref\ \{\cosh \tilde{\Gamma}_{12}\}^{\ominus} = \Gamma_{12} = Ref\ \{\cosh \tilde{\Gamma}_{12}\}^{\ominus} \cdot i\tilde{\Gamma}_{12}, \quad (\ \{\tilde{\Gamma}_{12}\}^2 = \{\Gamma_{12}\}^2\ ).$$

Abstract analogy in (322), (323) give no a quantitative relation between real-valued spherical and hyperbolic angles or functions. Such relations may be determined if an one-to-one concrete correspondence in the universal Cartesian base $\tilde{E}_1 = \{I\}$ between argument angles is fixed. Spherical and hyperbolic angles, functions and transformations with the isomorphic correspondence in all eigen quasiplanes and pseudoplanes in $\tilde{E}_1$ may be represented clearly at the general trigonometric diagram (Figure 3).

6.2 Covariant concrete (or specific) spherical–hyperbolic analogy

Note, that the ranges of spherical sines and hyperbolic tangents as well as spherical tangents and hyperbolic sines of angles in the base $\tilde{E}_1 = \{I\}$ are identical. Put in $\tilde{E}_1$:

$$\sin\varphi \equiv \tanh\gamma, \qquad \tan\varphi \equiv \sinh\gamma. \tag{331}$$

Then, on the basis of (331), argument angles are connected by the following equalities:

$$\gamma = \gamma(\varphi) = \operatorname{artanh}(\sin\varphi) = \operatorname{arsinh}(\tan\varphi) = \ln(\sec\varphi + \tan\varphi),$$

$$\varphi = \varphi(\gamma) = \arctan(\sinh\gamma) = \arcsin(\tanh\gamma) = -i\ln(\operatorname{sech}\,\gamma + i\tanh\gamma).$$

The differentials and derivatives will be further very useful in instantaneous bases $\tilde{E}_m$:

$$d\varphi(\gamma) = \operatorname{sech}\gamma\ d\gamma, \quad d\gamma(\varphi) = \sec\varphi\ d\varphi,$$

$$\frac{d\gamma(\varphi)}{d\varphi} = \sec\varphi \equiv \left(\frac{d\varphi(\gamma)}{d\gamma}\right)^{-1} = \operatorname{sech}^{-1}\gamma = \cosh\gamma \to \cos\varphi \equiv \operatorname{sech}\,\gamma.$$

According to the trigonometric diagram at Figure 3, the main values of spherical angles are in $[-\pi/2;\ +\pi/2]$, as in Ch. 5. For this range of the angles their cosines and sines are nonnegative, thus formulae (331) may be supplemented by two analogs:

$$\cos\varphi \equiv \operatorname{sech}\,\gamma \geq 0, \qquad \sec\ \varphi \equiv \cosh\gamma \geq 0. \tag{332}$$

The range $[-\pi/2;\ \pi/2]$ of spherical angles is sufficient for trigonometric transformations (rotations, deformations) of lineors and 2-valent tensors. Identities (331) generate *sine–tangent spherical–hyperbolic analogy of the concrete (or specific) type.* It is represented in its tensor variant also in $\tilde{E}_1$ (with particular identities in binary cells) as follows:

$$\left.\begin{array}{l} \sin\Phi \equiv \tanh\Gamma,\ \tan\Phi \equiv \sinh\Gamma, \\ \cos\Phi \equiv \operatorname{sech}\,\Gamma,\ \sec\ \Phi \equiv \cosh\Gamma, \end{array}\right\} \varphi_j \in [-\pi/2;\ +\pi/2],\ \gamma_j \in (-\infty; +\infty). \tag{333}$$

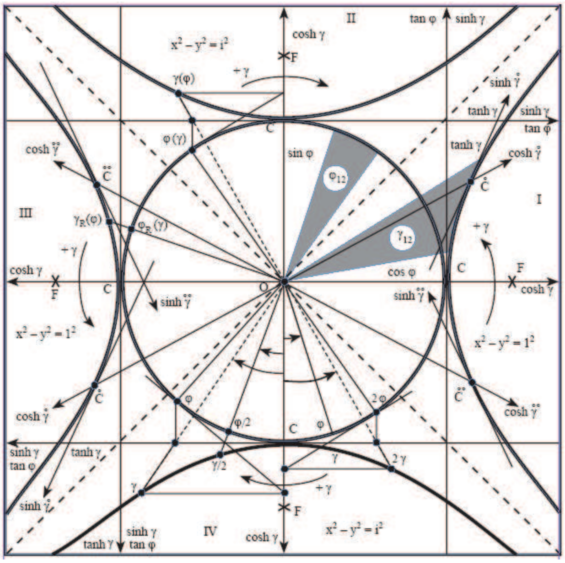

Figure 3. The trigonometric diagram with spherical–hyperbolic analogies in an eigen plane – pseudoplane with respect to the right universal base $\tilde{E}_1$. (The angle φ is spherical, the angle γ is hyperbolic.)

Here we use the following notations:
I, II, III, IV are the hyperbolic quadrants of a pseudoplane with conjugate hyperbolae (I, III and II, IV) and hyperbolic angles dividing by the two asymptotic diagonals.
$\overset{\circ}{\gamma}$ and $\overset{\circ\circ}{\gamma}$ are the positive and negative angles of hyperbolic rotations determined along hyperbolae, they are shown in I and III.
$\varphi(\gamma)$ and $\gamma(\varphi)$ are the examples of specific sine-tangent spherical–hyperbolic analogy, they are shown in II; for hyperbolae focus: $\gamma_F = \gamma(\pi/4) = \omega \approx 0.881$ rad.
$\varphi_R(\gamma)$ and $\gamma_R(\varphi)$ are the examples of specific tangent-tangent spherical–hyperbolic analogy, they are shown in III.
Besides, bisection and duplication of an hyperbolic angle with respect to the base $\tilde{E}_1$ with the use of these analogies are shown in the left and right parts of IV.

The specific sine-tangent analogy with respect to the base $\tilde{E}_1 = \{I\}$ may be widen onto all types of trigonometric matrix functions:

$$Roth\ \Gamma \equiv Def\ \Phi, \tag{334}$$

$$Rot\ \Phi \equiv Defh\ \Gamma. \tag{335}$$

Relations between both motive tensor angles in the base $\tilde{E}_1$, are the following:

$$\Gamma = \Gamma(\Phi) = \ln Def\ \Phi, \qquad i\Phi = i\Phi(\Gamma) = \ln Defh\ \Gamma. \tag{336}$$

They follow from (333)–(335). The original unity base $\tilde{E}_1$ serves here for simultaneous representations of connected angles Γ and Φ as well as their trigonometric functions for the correct realization of the *concrete (or specific) spherical–hyperbolic analogy* !

If the mid-reflector for Γ_B is used as a reflector tensor, then hyperbolic reflectors (328), (329) are hyperbolic analogies of spherical ones (178), (179) and (211), (212):

$$Ref\{BB'\} = \text{sech}\ \tilde{\Gamma}_B - \tanh \tilde{\Gamma}_B,\ Ref\{B'B\} = \text{sech}\ \tilde{\Gamma}_B + \tanh \tilde{\Gamma}_B; \quad (337), (338)$$

$$Ref\{B\} = \cosh\ \tilde{\Gamma}_B - i \sinh \tilde{\Gamma}_B,\ Ref\{B'\} = \cosh\ \tilde{\Gamma}_B + i \sinh \tilde{\Gamma}_B. \quad (339), (340)$$

One of interesting applications of the concrete spherical–hyperbolic analogy is correct introducing *parallel angles* in non-Euclidean geometries (see more in Ch.1A). The parallel angle of Lobachevsky $\Pi(a)$ is not a motion angle in hyperbolic geometries, because it has spherical nature. But the complementary spherical angle $\varphi = \pi/2 - \Pi(a)$ is the "parallel angle" too and the motion angle in geometries with spherical principal motions in any admissible bases. The angle $\gamma(\varphi)$, expressed here according to (331), is the "parallel angle" too and the motion angle in geometries with hyperbolic principal motions in any admissible bases. From the connection of $\Pi(a)$ and $\gamma(\varphi)$, *taking into account (331)*, the Lobachevsky formula $\Pi(a) = 2 \arctan[\exp(-\gamma)]$ follows, *but only in the universal base* $\tilde{E}_1$. Hence, it may be used only for connection of $\Pi(a)$ with one-step or summarized multistep collinear hyperbolic motions angle. For direct inferring connection of $\Pi(a)$ and γ, the *contravariant sine-cotangent analogy* is used (so, see in [69, s. 10.6]), correctly only in the base $\tilde{E}_1$. Hence, in this concrete analogy, the angle γ as covariant and the angle $\Pi(a)$ as contravariant vary in different directions!

Spherical–hyperbolic analogy of the two types (abstract and concrete in the base $\tilde{E}_1$) generates the following *quart-circle* of motive matrix functions transformations:

$$\begin{array}{ccccccc} Rot\ (i\Gamma) & \equiv & Defh\ (-i\Phi) & \Leftrightarrow & Roth\ \Gamma & \equiv & Def\ \Phi \\ \Updownarrow & & & & & & \Updownarrow \\ Rot\ \Phi & \equiv & Defh\ \Gamma & \Leftrightarrow & Roth\ (-i\Phi) & \equiv & Def\ (i\Gamma). \end{array} \quad (341)$$

Rules 2 and 3 of sect. 5.7 stay valid for trigonometrically compatible hyperbolic rotational matrices and orthogonal reflectors. The hyperbolic variant of rule 3 is **Rule 6**. Rules 3 and 6 are *foundation for principal rotations.* All the rules hold for scalar trigonometric functions and transformations in pseudoplane. In particular,

$$\prod_{j=1}^{m} (\sec\ \varphi_j \pm \tan \varphi_j)^{h_j} \equiv \prod_{j=1}^{m} (\cosh \gamma_j \pm \sinh \gamma_j)^{h_j} = \exp\left(\sum_{j=1}^{m} \pm h_j \gamma_j\right) =$$

$$= \exp \gamma = \cosh \gamma + \sinh \gamma \equiv \sec\ \varphi + \tan \varphi, \quad \varphi \in [-\pi/2; \pi/2],\ \text{(see Ch. 5.10)}.$$

The sine-tangent analogy generates hyperbolic orthogonal forms of affine projectors, quasi-inverse matrices, and reflectors considered before, if the mid-reflector for Γ_B is used as a reflector tensor. Then hyperbolic relations are similar to spherical ones (249):

$$Ref\{B'\} \cdot\ Ref\{B\} = (\cosh \tilde{\Gamma}_B + \sinh \tilde{\Gamma}_B)(\cosh \tilde{\Gamma}_B - \sin \tilde{\Gamma}_B) = Roth\ 2\Gamma_B. \quad (342)$$

This means that reflection $\{(\sqrt{I})'_h(\sqrt{I})_h\}$, where $(\sqrt{I})_h \neq (\sqrt{I})'_h\}$ is the prime non-symmetric root, is the double hyperbolic rotation similar to (251). Rotational matrix *Roth* Γ_B is a *trigonometric* hyperbolic square root of the *symmetric* matrix in square brackets similar to spherical one in (251); but in this case it is also an *arithmetic* root

$$Roth\ \Gamma_B = [(\pm Ref\{B\})' \cdot (\pm Ref\{B\})]_S^{1/2} = [Roth\ 2\Gamma_B]_S^{1/2}. \quad (343)$$

If $\mathbf{a_1}, \mathbf{a_2}$ are in common non-oriented vectors or planars of rank 1 and $\mathbf{a_1 a_2} \neq 0$, then they determine *the elementary rotational hyperbolic matrix* with $\tau = 1$:

$$Roth\ \Gamma_{12} = \left[(I - 2\overleftarrow{\mathbf{a_2 a_1'}})(I - 2\overleftarrow{\mathbf{a_1 a_2'}})\right]^{1/2} = \tag{344}$$

$$= \left[I - 2\left(\frac{\mathbf{a_1 a_2'}}{\mathbf{a_2' a_1}} + \frac{\mathbf{a_2 a_1'}}{\mathbf{a_1' a_2}}\right) + 4\cosh^2\gamma_{12} \cdot \frac{\mathbf{a_2 a_2'}}{\mathbf{a_2' a_2}}\right]^{1/2},$$

where

$$\overleftarrow{\mathbf{a_2 a_1'}} = \frac{\mathbf{a_2 a_1'}}{\mathbf{a_1' a_2}}, \quad \overleftarrow{\mathbf{a_1 a_2'}} = \frac{\mathbf{a_1 a_2'}}{\mathbf{a_2' a_1}}.$$

(Thus, $\mathbf{a_1} = \mathbf{e_1}, \quad \mathbf{a_2} = \mathbf{e_2}, \rightarrow \mathbf{e_2' e_1} = \mathbf{e_1' e_2} = \cos\varphi_{12} \equiv \mathrm{sech}\ \gamma_{12}, \quad \mathbf{e_2 e_1'} = \overleftarrow{\mathbf{e_2 e_1'}}$.) Recall, that $\{\overleftarrow{\mathbf{a_2 a_1'}}\}$ is a projector into $\langle im\ \mathbf{a_2}\rangle$ parallel to $\langle ker\ \mathbf{a_1'}\rangle \equiv \langle im\ \mathbf{a_1}\rangle^{\perp}$.

The tensor *spherical angle* α_B in (288) is evaluated quantitatively with the use of (336):

$$\left.\begin{array}{l} Def\ \Phi_B \equiv Roth\ \Gamma_B, \\ Def\ \alpha_B \equiv Roth\ 2\Gamma_B \end{array}\right\} \rightarrow \ \alpha_B = -i\ln\{Defh\ [2\ln\ (Def\ \Phi_B)]\}.$$

The sine-tangent analogy leads to the following four expressions for the mid-reflector:

$$Ref\{\cos\tilde{\Phi}\}^{\ominus} = Ref\{sec\ \tilde{\Phi}\}^{\ominus} \equiv Ref\{\cosh\tilde{\Gamma}\}^{\ominus} = Ref\{sech\ \tilde{\Gamma}\}^{\ominus}. \tag{345}$$

Right multiply the matrices in quart circle (341) by the mid-reflector, we obtain the *similar quart circle for the reflectors.* Repeat this operation once more and we return to their original motive type. Relations between projective angles similar to (336) may also be easily derived. Definitions of projective hyperbolic angles and functions may be obtained from the spherical ones with the use of sine-tangent analogy, if the mid-reflector (345) for Γ_B is used as the pseudo-Euclidean space reflector tensor.

Application of spherical modal transformations (256), (303) and (304) gives the similar hyperbolic relations:

$$\left.\begin{array}{r} Ref\{B'\} = Roth\ \Gamma_B \cdot Ref\{B\} \cdot Roth\ (-\Gamma_B) = \\ = Ref\{\cosh\tilde{\Gamma}_B\}^{\ominus} \cdot Ref\{B\} \cdot Ref\{\cos\tilde{\Gamma}_B\}^{\ominus}, \\ \overleftrightarrows{B'} = Roth\ \Gamma_B \cdot \overleftrightarrows{B} \cdot Roth\ (-\Gamma_B) = Ref\{\cosh\tilde{\Gamma}_B\}^{\ominus} \cdot \overleftrightarrows{B} \cdot Ref\{\cos\tilde{\Gamma}_B\}^{\ominus}. \end{array}\right\} \tag{346}$$

Consider the set $\langle T_B\rangle \equiv \langle Roth\ \Gamma_B{\cdot}Rot\ \Theta_B\rangle$ of modal rotational matrices performing operations (346). Here the matrix $Roth\ \Gamma_B$ determined by (343) has the trigonometric subspace of the minimal dimension among all matrices of $\langle T_B\rangle$. In particular, it enables one to evaluate the rotation variant of modal matrices for transforming affine projectors into D-forms, i. e., developing further relations (311), (312):

$$\left.\begin{array}{l} R_W' \cdot\ Roth\ (\Gamma_B/2) \cdot \overleftrightarrows{B} \cdot Roth\ (-\Gamma_B/2) \cdot R_W = D\{\overleftrightarrows{B}\}, \\ R_W' \cdot\ Roth\ (-\Gamma_B/2) \cdot \overleftrightarrows{B'} \cdot Roth\ (\Gamma_B/2) \cdot R_W = D\{\overleftrightarrows{B'}\}. \end{array}\right\} \tag{347}$$

As in the quasi-Euclidean geometry (see sect. 5.7), in pseudo-Euclidean one only for matrices of two types: *Roth* Γ (principal hyperbolic rotations), *Rot* Θ (orthospherical rotations formed the subgroup of general rotations group) the following relations hold

$$\left.\begin{array}{l} Roth\ \{\pm\Gamma_{12}\}\cdot Ref\{\cosh\tilde{\Gamma}\}^{\ominus}\cdot Roth\ \{\pm\Gamma_{12}\} = Ref\{\cosh\tilde{\Gamma}\}^{\ominus}, \\ Rot'\ \Theta_{12}\cdot Ref\{\cosh\tilde{\Gamma}\}^{\ominus}\cdot Rot\ \Theta_{12} = Ref\{\cosh\tilde{\Gamma}\}^{\ominus} = \\ = Rot\ \Theta_{12}\cdot Ref\{\cosh\tilde{\Gamma}\}^{\ominus}\cdot Rot'\ \Theta_{12}. \end{array}\right\} \quad (348)$$

In the *projective version* the hyperbolic reflectors of two types at rotation of Γ_{12}, as in (346) or (256), satisfy relations

$$\left.\begin{array}{l} Ref\{\cos\tilde{\Gamma}\}^{\ominus}\cdot Ref\{\cos\tilde{\Gamma}_{12}\}^{\ominus}\cdot Ref\{\cos\tilde{\Gamma}\}^{\ominus} = Roth\ \overset{\angle}{\Gamma}_{12}, \\ Ref\{\cos\tilde{\Gamma}\}^{\ominus}\cdot Ref\{\cos\tilde{\Theta}_{12}\}^{\ominus}\cdot Ref\{\cos\tilde{\Gamma}\}^{\ominus} = Ref\{\cos\tilde{\Theta}_{12}\}^{\ominus}. \end{array}\right\} \quad (349)$$

Relations (348, 349) are pseudo-Euclidean analogs of quasi-Euclidean ones (257, 258). (348) and (349) are the basis for the *pseudo-Euclidean geometry and its trigonometry* with mid-reflector (345) as a set reflector tensor of a pseudo-Euclidean space introduced independently, as well as for *external non-Euclidean hyperbolic geometry of index q*. The latter is a *hyperbolic geometry in a hyperspace of constant negative curvature*. This geometry is realized on special hyperboloids of radii $\pm R$ embedded into the *pseudo-Euclidean space* $\langle\mathcal{P}^{n+q}\rangle$ determined by a reflector tensor as the metric tensor too.

6.3 The reflector tensor in quasi- and pseudo-Euclidean interpretations

Applications of hyperbolic and spherical matrices of these two principal types in tensor trigonometry need in theoretical justification including a choice of basis metric spaces, admissible transformations and bases. Fix an initial arithmetic (affine) space with the original coordinate base $\tilde{E}_1 = \{I\}$ of vector-columns. Introduce in this space in quite independent way a reflector tensor for beginning in its general form as $\{\sqrt{I}\}_S$ (see in sect. 5.7). Generally it determines in the base $\tilde{E}_1 = \{I\}$ the *non-coaxial* orientation of these basis spaces and rotations of these three types defined in (257) and (348):

in the space $\langle\mathcal{Q}^{n+q}\rangle \equiv \langle\mathcal{E}^n\rangle \boxplus \langle\mathcal{E}^q\rangle \equiv CONST$ principal spherical rotations $\langle Rot\ \Phi\rangle$

$$Rot\ \Phi\cdot\{\sqrt{I}\}_S\cdot Rot\ \Phi = \{\sqrt{I}\}_S = Rot\ (-\Phi)\cdot\{\sqrt{I}\}_S\cdot Rot\ (-\Phi);$$

in the space $\langle\mathcal{P}^{n+q}\rangle \equiv \langle\mathcal{E}^n\rangle \boxtimes \langle\mathcal{E}^q\rangle \equiv CONST$ principal hyperbolic rotations $\langle Roth\ \Gamma\rangle$

$$Roth\ \Gamma\cdot\{\sqrt{I}\}_S\cdot Roth\ \Gamma = \{\sqrt{I}\}_S = Roth\ (-\Gamma)\cdot\{\sqrt{I}\}_S\cdot Roth\ (-\Gamma);$$

in the spaces $\langle\mathcal{Q}^{n+q}\rangle$ and $\langle\mathcal{P}^{n+q}\rangle$ secondary orthospherical rotations $\langle Rot\ \Theta\rangle$

$$Rot'\ \Theta\cdot\{\sqrt{I}\}_S\cdot Rot\ \Theta = \{\sqrt{I}\}_S = Rot\ \Theta\cdot\{\sqrt{I}\}_S\cdot Rot'\ \Theta.$$

The *binary quasi-Euclidean space* $\langle \mathcal{Q}^{n+q} \rangle$ (see initially in the end of Ch. 5.7) is determined by an Euclidean quadratic metric and the reflector tensor $\{\sqrt{I}\}_S$. They define the admissible modal transformations, in particular, transformations of bases:

$$\tilde{E}_2 = Rot\ \Phi \cdot Rot\ \Theta \cdot \tilde{E}_1\ or\ \tilde{E}_3 = Rot\ \Theta \cdot Rot\ \Phi \cdot \tilde{E}_1. \tag{350}$$

$\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$ are called *rotationally connected quasi-Cartesian bases.* Quasi-Euclidean trigonometry are realized in spaces $\langle \mathcal{Q}^{n+q} \rangle$, with respect here to right quasi-Cartesian bases such as (350). The transformations in (350) form the *proper Q-group.*

The *binary pseudo-Euclidean space* $\langle \mathcal{P}^{n+q} \rangle$ (see more in Chs. 10 and 11) is determined by a pseudo-Euclidean quadratic metric and the reflector tensor $\{\sqrt{I}\}_S$. They define the admissible modal transformations, in particular, transformations of bases:

$$\tilde{E}_2 = Roth\ \Gamma \cdot Rot\ \Theta \cdot \tilde{E}_1\ or\ \tilde{E}_3 = Rot\ \Theta \cdot Roth\ \Gamma \cdot \tilde{E}_1. \tag{351}$$

$\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$ are called *rotationally connected pseudo-Cartesian bases.* Pseudo-Euclidean trigonometry are realized in spaces $\langle \mathcal{P}^{n+q} \rangle$, with respect here to right pseudo-Cartesian bases such as (351). The transformations in (351) form the proper *Lorentz group.*

Introduce the right so-called **universal bases** including the original base $\tilde{E}_1 = \{I\}$:

$$\langle \tilde{E}_{Iu} \rangle \equiv \langle Rot\ \Theta \cdot \tilde{E}_1 \rangle \quad (\tilde{E}'_{Iu}\tilde{E}_{iu} = I,\ \tilde{E}'_{Iu}\{\sqrt{I}\}_S\tilde{E}_{Iu} = \{\sqrt{I}\}_S,\ det\ \tilde{E}_{Iu} = +1). \tag{352}$$

The transformations $\langle Rot\ \Theta \rangle$ form the *orthospherical subgroup*, what is the intersection of the *Q-group* and the *Lorentz group*, but only with respect to universal bases $\langle \tilde{E}_{Iu} \rangle$!

A reflector tensor and a choice of the principal trigonometry from its two kinds (spherical or hyperbolic) determine the spaces quadratic metric (Euclidean or pseudo-Euclidean); and vice versa! The two complete sets of admissible motions in the spaces contain the subsets of $Q-$ or Lorentz transformations, what stipulates the spaces isotropy, and the parallel translations, what stipulates the spaces homogeneity!

The base $\tilde{E}_1 = \{I\}$ is the simplest universal base by its form. Universal bases enable one to jointly realize quasi-Euclidean and pseudo-Euclidean trigonometries on the basis of concrete spherical–hyperbolic analogy, but only with *one-step* motions. Note, in particular, that in STR (special theory of relativity) physical *one-step* motions with respect to relatively fixed Observer are described in terms of universal bases.

Consider how a reflector tensor acts on matrices eigenprojectors in both spaces. Let B be a null-prime matrix, used initially in an affine space $\langle \mathcal{A}^n \rangle$. Introduce in the space the reflector tensor as the mid-reflector of the tensor angle for the matrix B in two following variants (with introducing metrics for external and internal products):

$$\{\sqrt{I}\}_S = Ref\ \{\cos \tilde{\Phi}_B\}^{\ominus} \equiv Ref\ \{\cosh \tilde{\Gamma}_B\}^{\ominus}. \tag{353}$$

We got quasi- and pseudo-Euclidean spaces. The identity sign is true only in $\langle \tilde{E}_{Iu} \rangle$.

In the first case, the symmetric projectors $\overleftarrow{BB'}$ and $\overrightarrow{BB'}$ are spherically orthogonal each to another in Euclidean and quasi-Euclidean spaces with a metric tensor $\{I^+\}$, i. e., reflector tensor (353) does not determine here internal and external products.

In the second case, the non-symmetric projectors $\overleftarrow{B}$ and $\overrightarrow{B}$ are hyperbolically orthogonal each to another in a pseudo-Euclidean space with *metric* reflector tensor $\{I^{\pm}\}$ according to (353). This fact follows taking into account last formula in (346):

$$(\overleftarrow{B})' Ref\ \{\cosh \tilde{\Gamma}_B\}^{\ominus} \overrightarrow{B} = Ref\ \{\cosh \tilde{\Gamma}_B\}^{\ominus} \cdot \overleftarrow{B} \cdot \overrightarrow{B} = Z.$$

Consequently, B^- (sect. 2.1) is a *hyperbolically orthogonal quasi-inverse matrix* with respect to reflector tensor (353) if the matrix B is null-prime. Also, in this case, the direct sum $\langle im\ B\rangle \oplus \langle ker\ B\rangle$ is hyperbolically orthogonal. Then for non-symmetric projectors $\overleftarrow{B}$ and $\overrightarrow{B}$, their eigen subspaces corresponding to the eigenvalues 0 and 1 are hyperbolically orthogonal too.

Equirank projectors $\overleftarrow{B}$ and $\overleftarrow{B'}$ as well as $\overrightarrow{B}$ and $\overrightarrow{B'}$ and their planars are transformed into each other with hyperbolic rotation in (346) as hyperbolically orthogonal 2-valent tensors and tensor objects. Projectors $\overleftarrow{B}$ and $\overrightarrow{B}$ hyperbolically orthogonally project into $\langle im\ B\rangle$ and $\langle ker\ B\rangle$. Projective formulae (186)–(197) are transformed. In the symbolic octahedron (Figure 1) the diagonal RS generates two pseudo-isosceles triangles RZS and RIS with equal hyperbolic angles $\angle RZS \equiv \angle RIS \equiv \Gamma_B$. These facts are responses to the introduction of reflector tensor (353) in the hyperbolic form.

With hyperbolic analogues of Moivre and Euler formulae, we have – see (287):

$$Roth\{m\Gamma\} = \cosh\{m\Gamma\} + \sinh\{m\Gamma) = Roth^m\ \Gamma =$$

$$= \exp\{m\Gamma\} \to \{m\Gamma\} = \ln\ Roth\{m\Gamma\} \to \Gamma = \ln\ Roth\ \Gamma.$$

This give motive tensor angle from a rotation tensor! In the cellular form, there hold:

$$\{Rot^m\Gamma\} = \begin{bmatrix} \ddots & & & & & \\ & \cosh m\gamma_j & \sinh m\gamma_j & & & \\ & \sinh m\gamma_j & \cosh \gamma_j & & & \\ & & & \ddots & & \\ & & & & \boxed{1} & \\ & & & & & \ddots \end{bmatrix} = \exp \begin{bmatrix} \ddots & & & & & \\ & 0 & m\gamma_j & & & \\ & m\gamma_j & 0 & & & \\ & & & \ddots & & \\ & & & & \boxed{0} & \\ & & & & & \ddots \end{bmatrix},$$

Here the value $m = 1/2$ gives arithmetic and trigonometric square root (343) of the rotational matrix. Properties of this matrix, no depending on the rotation angle, are the same as ones of a deformational spherical matrix. This matrix is symmetric and positive definite, its eigenvalues are

$\mu_{2j} = \cosh \gamma_j + \sinh \gamma_j > 0$, $\mu_{2j+1} = \mu_{2j}^{-1} = \cosh \gamma_j - \sinh \gamma_j > 0$, and also may be in addition $\mu_k = +1$.

Any pair of positive numbers x and x^{-1} may be uniquely represented in terms of a scalar hyperbolic angle, in particular, as a 2×2-matrix (see sect. 5.10).

Note, that in order to establish compatibility of certain transformations (rotations, reflections) or mid-reflector for some tensor angle with the space reflector tensor in both tensor trigonometries on the basis of quadratic metrics, one may use for this defining relations (257), (258) and (348), (349) as the corresponding criterions.

6.4 Scalar trigonometry in a pseudoplane

A diagonal reflector tensor $\{I^{\pm}\}$ produces a *coaxially oriented* pseudo-Euclidean space $\langle \mathcal{P}^{n+q} \rangle$, which has binary structure and admissible to it the pseudo-Cartesian bases $\tilde{E}$. Represent the hyperbolic rotational matrix *Roth* Γ at a level of the j-th 2×2 cell in W-form (324) with respect to the trigonometric base $\tilde{E}_1 = \{I\}$, where the rotation realizes along the *characteristic quadrohyperbola* of coupled hyperbolae (Figure 3). In the j-th eigen pseudoplane, two axes – ordinate and abscissa are the eigenvectors $\mathbf{u}$ and $\mathbf{v}$ for the D-forms of cosh Γ (with $\pm \cosh \gamma$) and $\{I^{\pm}\}$ (with ± 1); two asymptotes of the quadrohyperbola with respect to any admissible base $\tilde{E}$ are the main and lateral *invariant diagonals* – lines with zero quadratic metric. Hence, these two asymptotes for all similar quadrohyperbolae are invariant under hyperbolic rotations of the base. If a pseudo-Euclidean space dimension n is greater than 2, then the diagonals correspond to an *invariant dividing hypersurface.* At $q = 1$, $n > 2$, it is an asymptotic hypersurface for the embedded *hyperboloids* I an II (pseudospheres of radii $R = \pm 1$ and $R = \pm i$) – see more in Ch. 12. If the j-th pseudoplane cuts such hyperboloids, then, on it in the base $\tilde{E}_1$, the rotation in its elementary form $\{roth\ \Gamma\}_{2\times 2}$ (see in sect. 6.5) is performed along the quadrohyperbola. In four hyperbolic quadrants, *positive scalar angles* γ_j *are measured here in direction to the main invariant diagonal, and vice versa.*

A hyperbolic angle γ is measured through the pseudo-Euclidean length of a unity hyperbola arc in the pseudoplane with the trigonometric base $\tilde{E}_1$ with its pseudo-Cartesian axes $\{x, y\}$ – Figure 3. The length are real-valued in quadrant I, III as $R\gamma$ and imaginary-valued in quadrant II, IV as $iR\gamma$. As the real parameter, its pseudo-Euclidean length is noted further as λ (the Euclidean length of the arc is greater):

$$\lambda = R \int_{\gamma_1}^{\gamma_2} \sqrt{(d \sinh \gamma)^2 - (d \cosh \gamma)^2} = R(\gamma_2 - \gamma_1) < R \int_{\gamma_1}^{\gamma_2} \sqrt{(d \sinh \gamma)^2 + (d \cosh \gamma)^2}.$$

The area of a hyperbolic sector is $S = R^2(\gamma_2 - \gamma_1)/2$. In these four quadrants the *radius-vectors of pseudocurvature* $\pm R$ or $\pm iR$ is hyperbolically orthogonal to the hyperbola tangent at the point of tangency in an admissible base $\tilde{E}$ and $d\lambda = Rd\gamma$. These vector and tangent determine local hyperbolically connected coordinate axes. The focus of the hyperbola corresponds to the *especial hyperbolic angle* $\omega \approx 0.881$ *rad*:

$$\sinh \omega = 1, \quad \cosh \omega = \sqrt{2}, \quad \tanh \omega = \sqrt{2}/2, \quad \coth \omega = \sqrt{2}. \tag{354}$$

By sine-tangent analogy, $\varphi(\omega) = \pi/4$, $\gamma(\pi/4) = \omega$. Thus the angle or the number ω is the hyperbolic analogue of the angle or the number $\pi/4$. We shall often use the angle ω in the sequel. For example, $\sin(\pi/4 \pm i\omega) = 1 \pm (\sqrt{2}/2)i; \cos(\pi/4 \pm i\omega) = 1 \mp (\sqrt{2}/2)i$.

Sine-tangent analogy (331)–(333) is a basic concrete analogy in the monograph given a lot of interesting results. (It appears even at the cross projection – Ch. 4A!) There exist infinitely many kinds of concrete analogies. Consider briefly some of them.

(Sine-cotangent analogy with *contrary* angles-analogs changes was mentioned above.)

Introduce concrete *tangent-tangent analogy* with respect to the universal base $\tilde{E}_1$ too with the following another condition (see Figure 3, quadrant III):

$$\tan\varphi_R \equiv \tanh\gamma \;\rightarrow\; \varphi_R = \varphi_R(\gamma) = \arctan(\tanh\gamma), \quad (-\pi/4 \le \varphi_R \le +\pi/4). \tag{355}$$

The angle-analog φ_R and the angle γ are determined by the same radius-vector. That is why the angle $\varphi_R(\gamma)$ here is called *visual*. Sometimes, the angle φ_R is used for *descriptivity* in STR (what is correct only with respect to the universal bases $\langle\tilde{E}_{Iu}\rangle$!). This mapping leads to other relations between spherical and hyperbolic functions:

$$\left.\begin{array}{ll} \sin\varphi_R \equiv \sinh\gamma/\sqrt{\cosh 2\gamma}, & \cos\varphi_R \equiv \cosh\gamma/\sqrt{\cosh 2\gamma}, \\ \sinh\gamma \equiv \sin\varphi_R/\sqrt{\cos 2\varphi_R}, & \cosh\gamma \equiv \cos\varphi_R/\sqrt{\cosh 2\varphi_R}. \end{array}\right\} \tag{356}$$

$$\varphi_R(\gamma) < \varphi(\gamma) < \gamma, \quad (if\ \gamma > 0). \tag{357}$$

For example, $\varphi_R(\omega) \approx 35^\circ$, $\gamma_R(\pi/4) = \infty$. In general, all the visually obvious concrete spherical–hyperbolic analogies are reduced to identities similar to

$$\tan(k_1\varphi/2) \equiv \tanh(k_2\gamma/2) \Leftrightarrow \left\{\begin{array}{ll} \sin(k_1\varphi) = \tanh(k_2\gamma), & \tan(k_1\varphi) = \sinh(k_2\gamma), \\ \cos(k_1\varphi) = \text{sech}\ (k_2\gamma), & \text{sec}\ (k_1\varphi) = \cosh(k_2\gamma), \end{array}\right.$$

$-\pi/4 \le k_1\varphi/2 \le \pi/4$. In practice, four variants are important:
1) $k_1 = k_2 = 1$ (this corresponds to (331)),
2) $k_1 = k_2 = 2$ (this corresponds to (355)),
3) $k_1 = 1,\ k_2 = 2$,
4) $k_1 = 2,\ k_2 = 1$.
Joint application of (1) and (2) gives pure geometric (using a compass and a ruler only) duplication and bisection of a hyperbolic angle with respect to the base $\tilde{E}_1$ (Figure 3):

$$\left.\begin{array}{l} a)\ \tan\varphi \equiv \tanh\gamma,\ \ \varphi_R = \varphi_R(\gamma),\ \ \varphi = 2\varphi_R \ \ \rightarrow\ \ \tan\varphi \equiv \sinh 2\gamma; \\ b)\ \tan\varphi \equiv \sinh\gamma,\ \ \varphi = \varphi(\gamma),\ \ \varphi_R = 2\varphi/2 \ \ \rightarrow\ \ \tan\varphi_R \equiv \tanh\gamma/2. \end{array}\right\} \tag{358}$$

$$|\varphi_R(\gamma)| < |\varphi(\gamma)| < 2|\varphi_R(\gamma)|.$$

Indeed, if $\cos\varphi \equiv sech\ \gamma$ and $\cos(2\varphi_R) \equiv sech\ (2\gamma)$, then $\cos\varphi > \cos(2\varphi_R)$; but if $\tan\varphi \equiv \sinh\gamma$ and $\tan\varphi_R \equiv \tanh\gamma$, then $|\tan\varphi| > |\tan\varphi_R|$.

Sine-tangent concrete analogy is especially important in different kinds of tensor trigonometry and their application in non-Euclidean geometries and theory of relativity (see Appendix). For example, this analogy together with abstract one establish the direct relations between all trigonometric transformations in quart circle (341).

The sign-alternating reflector tensor structure $\{I^{\pm}\}$ permits to represent in pseudo-Euclidean space $\langle\mathcal{P}^{n+q}\rangle$ very various geometric objects with different quadratic metrics: Euclidean, anti-Euclidean and pseudo-Euclidean. These possibilities are determined by the numbers n and q. Practically in $\langle\mathcal{P}^{n+q}\rangle$ pseudo-Euclidean linear object are most interesting for the tensor trigonometry and its applications.

Simplest pseudo-Euclidean geometric objects may be represented in a pseudoplane. However generally, lengths of intervals, hyperbolic arcs in $\langle \mathcal{Q}^{n+q} \rangle_c$ are either imaginary, or zero, or real, what is determined by their quasi-Euclidean quadratic metric. But in $\langle \mathcal{P}^{n+q} \rangle$, thanks to metric tensor $\{I^{\pm}\}$ adopted in the theory of relativity, the imaginary intervals are transformed into real-valued time-like ones, the real intervals correspond to space-like ones in pseudo-Euclidean space. Accordingly, all imaginary hyperbolic function after acting abstract analogy (322) are transformed into real-valued ones too!

Sine-tangent analogy gives all trigonometric formulae for a right pseudo-Euclidean triangle ABC (plane scalar trigonometry began with solving a right triangle!). Its real-valued legs a and b lie in two different hyperbolic quadrants (suppose $a \leq b$). Then the finite principal angle γ at the vertex A is contrary to the leg $a < b$. Denote the real-valued pseudo-hypotenuse as g. Universal pseudo-Euclidean Pythagorean Theorem is $g^2 = b^2 - a^2$, because $a \leq b$. If the angle γ is in hyperbolic quadrant I (Figure 3), then the triangle ABC is exterior, g is outside of two invariant (or isotropic, or light in relativistic physics) diagonals. If $|a| = |b|$, then γ is infinite, g is situated onto the invariant diagonal with zero length. If the angle γ is in hyperbolic quadrant II, then the triangle is interior, g is inside of two invariant diagonals. Further, choose for determinacy the *exterior triangle* ABC. Its legs a and b belong to distinct eigen subspaces of reflector tensor with eigenvalues -1 and $+1$, i. e., they are time-like and space-like segments. In order to infer trigonometric formulae for the right triangle ABC, preliminary it is necessary consider locations and behavior of all its hyperbolic angles and sides with Euclidean analogs according to concrete sine–tangent analogy in $\tilde{E}_1$.

The *acute principal angle* γ at the vertex A is contrary to the leg a and adjacent to the leg b. Positive scalar values of the angle and its spherical analog $\varphi(\gamma)$ are measured in direction to the main invariant diagonal off the leg $b = AC$ (i. e., *Cartesian* axis x).

The *acute complementary angle* υ is defined here correctly as the angle at the vertex B between the pseudo-hypotenuse $g = AB$ and the internal isotropic diagonal passing through the vertex B. And its spherical analog is $(\pi/4 - \varphi_R)$. Positive scalar values of the angle υ and its spherical analog are measured also in direction to the isotropic diagonal off the pseudo-hypotenuse (see visually on the book Cover with *Einsteinian two opposite light rays*, in Ch. 12 at Figure 4 and in Appendix at Figure 1A, Ch. 3A).

The *obtuse opposite infinite angle* β at the vertex B is contrary to the leg b and adjacent to the leg a. Geometrically it consists of the acute angle υ and the *local infinite hyperbolic angle* $\delta = +\infty$ (it is as if the *geometric sum* $\upsilon + \delta$). The angle $\delta = \infty$ (its analog $\varphi_R = \pi/4$) is disposed between the main invariant diagonal passing through the vertex B and the leg $a = BC$ (parallel to the *time-like* axis y). The vertex B is common for υ and the local infinite angle δ. If $|a| = |b|$, then $\gamma = \beta = \delta = +\infty, \ \upsilon = 0$.

The *right angle* ν is disposed between the legs a and b within of both hyperbolic quadrants I and II. The angle is equal to zero in hyperbolic metric, because it consists of two infinite antithetical angles $+\delta$ and $-\delta$ ($\varphi_R = \pm\pi/4$) in these hyperbolic quadrants (directions of these angles measurement are to one side, i. e., of $b = AC$ to $a = BC$).

The principal acute angle $\gamma(\varphi)$ varies in the interval $\gamma \in (0; +\infty)$, its argument varies in the interval $\varphi \in [0;\ +\pi/2]$, but the visual angle φ_R varies in $[0;\ +\pi/4]$. The angle $\gamma(\varphi)$ can not increase more. When the angle $\gamma(\varphi)$ is increasing in its interval, the complementary acute angle $\upsilon(\xi)$ is decreasing in the interval $[+\infty; 0]$, its argument is decreasing in the interval $\xi \in [+\pi/2; 0]$, where $\xi = \pi/2 - \varphi$. The opposite obtuse angle varies from its initial maximal value $+2\delta$ up to its final minimal value $+\delta$.

Sine-tangent analogy determines an one-to-one correspondence between three hyperbolic angles (γ, υ, $\delta = \infty$) and spherical analogs (φ, ξ, $d = \pi/2$), and also for three sides of the right triangle, as in the pseudoplane and as in the quasiplane, with respect to the universal base $\tilde{E}_1$. Under this map the first *Euclidean* axis (with the leg b) is invariant, now as the first Cartesian axis; the main invariant diagonal is transformed into the second Cartesian axis (under the angle $\varphi(\delta) = \pi/2$); the leg $a = CB$ is rotated to the left at spherical angle $\varphi(\gamma)$ into the new leg $a_E = CB'$, i. e., up to its contact with the central circle of radius $g_E = g = \sqrt{b^2 - a^2}$ at the point of tangency B', now as the new vertex of the triangle $AB'C$ in the quasiplane; the pseudo-hypotenuse $g = AB$ is transformed into the new leg $AB' = g_E$ with the same length. Now the principal angle $\varphi(\gamma)$ at the vertex A is contrary to the rotated leg $a_E = a$, the complementary angle $\xi(\upsilon)$ at the vertex C is contrary to the new leg $AB' = g_E = g$, and the new right angle $d = \pi/2$ (from the local infinite angle δ) at the vertex B' is contrary to the new hypotenuse $AC = b_E = b$. The quasi-Euclidean Pythagorean theorem is $b^2 = g^2 + a^2$. We have two pseudo-Euclidean Pythagorean theorems with hypotenuses g and legs a; and we may see clearly also covariant sine-tangent and contravariant sine-cotangent concrete spherical–hyperbolic analogies! (See their *tensor forms* in Ch. 12.) There hold

$$\left.\begin{array}{c} \sinh\gamma = a/g \equiv \tan\varphi, \quad \tan\xi = g/a \equiv \sinh\upsilon \to \sinh\gamma\cdot\sinh\upsilon = 1, \\ \cosh\gamma = b/g \equiv \sec\varphi, \quad \sin\xi = g/b \equiv \tanh\upsilon \to \cosh\gamma\cdot\tanh\upsilon = 1, \\ \cosh\gamma\cdot\tanh\upsilon = \cosh\upsilon\cdot\tanh\gamma = 1 = \operatorname{sech}\,\gamma\cdot\coth\upsilon = \operatorname{sech}\,\upsilon\cdot\coth\gamma; \\ (\gamma,\upsilon = 0 \Leftrightarrow \upsilon,\gamma = \pm\infty) \quad \Leftrightarrow \quad (\varphi,\xi = 0 \Leftrightarrow \xi,\varphi = \pm\pi/2). \end{array}\right\} \qquad (359)$$

$$\left.\begin{array}{c} \sinh\gamma = \operatorname{csch}\,\upsilon = a/g = \tan\varphi = \cot\ \xi,\ [\sinh(\gamma,\upsilon) = \operatorname{csch}\,(\upsilon,\gamma)], \\ \cosh\gamma = \coth\ \upsilon = b/g = \sec\ \varphi = \csc\ \xi,\ [\pm\cosh(\gamma,\upsilon) = \coth\,(\upsilon,\gamma)], \\ \tanh\gamma = \operatorname{sech}\,\upsilon = |a|/|b| = \sin\varphi = \cos\xi,\ [\tanh(\gamma,\upsilon) = \pm\ \operatorname{sech}\,(\upsilon,\gamma)]; \\ (\sinh\gamma = \sinh\upsilon = 1 \ \Leftrightarrow\ \cosh\omega = \coth\omega = \sqrt{2} \ \Leftrightarrow\ \gamma = \upsilon = \omega\ !) \end{array}\right\} \qquad (360)$$

$$\cosh^2(\gamma,\upsilon) - \sinh^2(\gamma,\upsilon) = +1 = \coth^2(\upsilon,\gamma) - \operatorname{csch}^2(\upsilon,\gamma) - \text{two invariants for } \{I^{\pm}\}!$$

$$\tanh^2(\gamma,\upsilon) + \operatorname{sech}^2(\gamma,\upsilon) = +1 - \text{one-step quasi-invariant for} \{I^{+}\}.$$

$$-1 < \tanh(\gamma+\upsilon) \equiv \sin\sigma < +1 \ \Leftrightarrow\ (-\infty = -\delta < \gamma+\upsilon < +\delta = +\infty). \qquad (361)$$

For complementary angles γ, υ ($\varphi, \xi = \Pi(a)$, but $d\xi = -d\varphi$), we have these formulae:

$$d\upsilon = -d\gamma/\sinh\gamma, d\gamma = -d\upsilon/\sinh\upsilon;\ \tan\xi/2 \equiv \tanh\upsilon/2, \sin\xi \equiv \tanh\upsilon \to (331)\text{ etc.};$$

$$\gamma,\upsilon = \ln\coth(\upsilon/2,\gamma/2) \equiv \ln\cot(\xi/2,\varphi/2) \ \leftrightarrow\ \exp(-\gamma,-\upsilon) = \tanh(\upsilon/2,\gamma/2).$$

After an change in (324) or (362) of the angle Γ by its complement Υ with the use of formulae (360), the new matrix-function of Γ gives a rotation at Υ (see sect. 5.8 too):

$$\overline{Roth}\ \Gamma = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \coth\gamma_i & \operatorname{csch}\gamma_i & \\ & \operatorname{csch}\gamma_i & \coth\gamma_i & \\ \hline & & & \ddots \end{array}\right] = Roth\ \Upsilon = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cosh\upsilon_i & \sinh\upsilon_i & \\ & \sinh\upsilon_i & \cosh\upsilon_i & \\ \hline & & & \ddots \end{array}\right]!$$

Two invariant relations (see above) correspond to these two types of rotations!

6.5 Elementary tensor hyperbolic trigonometric functions with frame axes

Consider matrices of quart circle (341). If a certain matrix structure in this quart circle is known, then other ones (spherical and hyperbolic) may be quickly evaluated with the use of abstract and concrete spherical–hyperbolic analogy. So, from spherical rotational structures (313), (314) or deformational structure (319), (320) obtained in Ch. 5 in canonical E-forms the analogous structures for hyperbolic matrices follow:

$\{roth\ (\pm\Gamma)\}_{4\times 4}$

$1+(\cosh\gamma-1)\cos^2\alpha_1$	$(\cosh\gamma-1)\cos\alpha_1\cos\alpha_2$	$(\cosh\gamma-1)\cos\alpha_1\cos\alpha_3$	$\pm\sinh\gamma\cos\alpha_1$
$(\cosh\gamma-1)\cos\alpha_1\cos\alpha_2$	$1+(\cosh\gamma-1)\cos^2\alpha_2$	$(\cosh\gamma-1)\cos\alpha_2\cos\alpha_3$	$\pm\sinh\gamma\cos\alpha_2$
$(\cosh\gamma-1)\cos\alpha_1\cos\alpha_3$	$(\cosh\gamma-1)\cos\alpha_2\cos\alpha_3$	$1+(\cosh\gamma-1)\cos^2\alpha_3$	$\pm\sinh\gamma\cos\alpha_3$
$\pm\sinh\gamma\cos\alpha_1$	$\pm\sinh\gamma\cos\alpha_2$	$\pm\sinh\gamma\cos\alpha_3$	$\cosh\gamma$

(362)

$\{roth\ (\pm\Gamma)\}_{(n+1)\times(n+1)}$

$I_{n\times n}+(\cosh\gamma-1)\cdot\mathbf{e}_\alpha\mathbf{e}'_\alpha$	$\pm\sinh\gamma\cdot\mathbf{e}_\alpha$
$\pm\sinh\gamma\cdot\mathbf{e}'_\alpha$	$\cosh\gamma$

$$(\mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}). \qquad (363)$$

$\{defh\ (\pm\Gamma)\}_{4\times 4}$

$1+(\operatorname{sech}\gamma-1)\cos^2\alpha_1$	$(\operatorname{sech}\gamma-1)\cos\alpha_1\cos\alpha_2$	$(\mathit{sech}\ \gamma-1)\cos\alpha_1\cos\alpha_3$	$\mp\tanh\gamma\cos\alpha_1$
$(\operatorname{sech}\gamma-1)\cos\alpha_1\cos\alpha_2$	$1+(\operatorname{sech}\gamma-1)\cos^2\alpha_2$	$(\operatorname{sech}\gamma-1)\cos\alpha_2\cos\alpha_3$	$\mp\tanh\gamma\cos\alpha_2$
$(\operatorname{sech}\gamma-1)\cos\alpha_1\cos\alpha_3$	$(\operatorname{sech}\gamma-1)\cos\alpha_2\cos\alpha_3$	$1+(\operatorname{sech}\gamma-1)\cos^2\alpha_3$	$\mp\tanh\gamma\cos\alpha_3$
$\pm\tanh\gamma\cos\alpha_1$	$\pm\tanh\gamma\cos\alpha_2$	$\pm\tanh\gamma\cos\alpha_3$	$\operatorname{sech}\gamma$

(364)

$\{defh\ (\pm\Gamma)\}_{(n+1)\times(n+1)}$

$I_{n\times n}+(\operatorname{sech}\gamma-1)\cdot\mathbf{e}_\alpha\mathbf{e}'_\alpha$	$\mp\tanh\gamma\cdot\mathbf{e}_\alpha$
$\pm\tanh\gamma\cdot\mathbf{e}'_\alpha$	$\operatorname{sech}\gamma$

$$(\mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}). \qquad (365)$$

Indicated 4×4 E-forms (362), (364) with frame axes as hyperbolic analogs of (313), (319) may be also inferred directly from their original 2×2-cells (324), (326) as the same analogs of (259), (292) with the scheme similar to (315), (316).

An inversion of E-forms (363), (365) of elementary rotational and deformational matrices consists in application of the simplest operation $\mathbf{e}_\alpha \to (-\mathbf{e}_\alpha)$ equivalent to $rot\ \Pi\cdot\mathbf{e}_\alpha = -\mathbf{e}_\alpha$. Then there holds: $\Gamma\to(-\Gamma)$. Generally, rotational change of an universal base $rot\ \Theta\cdot\tilde{E}_1 = \tilde{E}_{1u}$ leads only to change of the directional cosines unity vector: $rot'\ \Theta_{n\times n}\cdot\mathbf{e}_\alpha = rot\ (-\Theta_{n\times n})\cdot\mathbf{e}_\alpha = \mathbf{e}_{\alpha'}$ within the same Euclidean subspace.

Chapter 7

Trigonometric interpretation of matrices anticommutativity

7.1 Commutativity of prime matrices

Two *biorthogonal* matrices $B_1B_2 = B_2B_1 = Z$ are commutative and anticommutative simultaneously: $B_1B_2 = B_2B_1 = -B_2B_1 = Z$. By the reason, they always are singular: $r_1 + r_2 \leq n$. Due to commutativity, prime biorthogonal matrices P_1, P_2 may be converted into their D-forms D_1, D_2 in a certain common base, where $D_1D_2 = Z$. Consequently, these multiplication relations may be analyzed from trigonometric point of view enough only for prime *nonsingular* matrices (they have no biorthogonal blocks).

Commutative prime matrices P_1 and P_2 are diagonalized in a certain common base:

$$\overset{D(P_1)}{\begin{bmatrix} \ddots & & & \\ & a_j & & \\ & & a_k & \\ & & & \ddots \end{bmatrix}}, \quad \overset{D(P_2)}{\begin{bmatrix} \ddots & & & \\ & b_j & & \\ & & b_k & \\ & & & \ddots \end{bmatrix}}, \quad D(P) = V_{col}^{-1} P V_{col}.$$

The diagonal structure of these forms and consequently commutativity of the matrices are invariant under the following modal transformations of any, say the (j,k)-th, 2×2-cell, that are compatible with this cell and represented in the affine tree W-forms:

$$\overset{W_1}{\begin{bmatrix} \ddots & & & \\ & \pm c & 0 & \\ & 0 & \mp c & \\ & & & \ddots \end{bmatrix}}, \quad \overset{W_2}{\begin{bmatrix} \ddots & & & \\ & 0 & \mp d & \\ & \pm d^{-1} & 0 & \\ & & & \ddots \end{bmatrix}}, \quad \overset{W_3}{\begin{bmatrix} \ddots & & & \\ & 0 & \pm id & \\ & \pm id^{-1} & 0 & \\ & & & \ddots \end{bmatrix}}.$$

The first of them is similar to reflection, it merely changes *pairly* directions of the coordinate axes (with their deformation). The second and third of them are similar to rotation, it permutes *pairly* the diagonal elements as well as coordinate axes (with their compression-stretching). All compositions of such transformations of the tree types form the complete set of modal matrices with respect to the invariant D-form given. All eigenvalues of P_1 and P_2 are supposed to be distinct, otherwise the set should be widen, it should contain base changes in the intersection of P_1 and P_2 eigen subspaces with multiple eigenvalues.

The three affine types of modal matrices indicated above give rise to their admissible *trigonometric* W-forms in $\langle \mathcal{E}^n \rangle$ (i. e., at $d = 1$):

$$\begin{array}{ccc} \mathit{Ref} & \mathit{Rot}\ (\pm\Pi/2) \equiv \mathit{Def}\ (\pm\Delta) & \mathit{Roth}\ (\pm\, i\Pi/2) \equiv \mathit{Defh}\ (\pm\, i\Delta) \\ \begin{bmatrix} \ddots & & & \\ & \pm 1 & 0 & \\ & 0 & \mp 1 & \\ & & & \ddots \end{bmatrix}, & \begin{bmatrix} \ddots & & & \\ & 0 & \mp 1 & \\ & \pm 1 & 0 & \\ & & & \ddots \end{bmatrix}, & \begin{bmatrix} \ddots & & & \\ & 0 & \pm i & \\ & \pm i & 0 & \\ & & & \ddots \end{bmatrix}. \end{array} \tag{366}$$

Consequently, the following trigonometric rule is valid: for commutative prime matrices $P_1, P_2, \ldots$ the bases of $D(P_1)$, $D(P_2), \ldots$ may differ in $\langle \mathcal{E}^n \rangle$ only by indicated in (366) compatible tensor reflections and rotations at spherical angle-arguments $k \cdot \Pi/2$ or pseudohyperbolic angle-arguments $\pm k \cdot i\Pi/2$, $(k = 0, \pm 1, \pm 2, \ldots)$.

Moreover, the two rotational modal matrices may be replaced with applying concrete spherical-hyperbolic analogy (see in Ch. 6) by the deformational pseudospherical or hyperbolic matrices $\mathit{Def}(i\Delta)$ or $\mathit{Defh}\ \Delta$, what have the infinite angle-arguments respectively $i\Delta$ or Δ (*nonperiodic*). However, this exotic variant may be considered only in universal bases.

7.2 Anticommutativity of prime matrices pairs

If a pair of prime matrices P_1 and P_2 are anticommutative, i. e., $P_1P_2 = -P_2P_1$, then

$$P_1^2P_2 = P_2P_1^2, \quad P_1P_2^2 = P_2^2P_1, \quad P_1^2P_2^2 = P_2^2P_1^2.$$

Suppose that the pair of anticommutative prime matrices P_1, P_2 have no biorthogonal blocks (see sect. 7.1). Thus, in first, sizes of these nonsingular matrices are even and, in second, the matrices and their multiplications are nonsingular. According to the principle of binarity (sect. 5.7), they may be converted into the compatible W-forms in a certain common base $\tilde{E} = V_W\{\tilde{E}_1\}$ with the result:

$$\begin{array}{cc} W(P_1) & W(P_2) \\ \begin{bmatrix} \ddots & & & & & \\ & \cdot & \cdot & & & \\ & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \\ & & & \cdot & \cdot & \\ & & & & & \ddots \end{bmatrix}, & \begin{bmatrix} \ddots & & & & & \\ & \cdot & \cdot & & & \\ & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \\ & & & \cdot & \cdot & \\ & & & & & \ddots \end{bmatrix}. \end{array} , \quad W(P_i) = V_W^{-1} P_i V_W, \quad i = 1, 2.$$

Execute such modal transformation V_W of $W(P_1)$ and $W(P_2)$ *together*, in order to convert P_1 into its diagonal form. In the new common base, P_1 and P_2 as before are anticommutative. Now the property is valid iff their compatible j-th 2×2-cells have diagonal and contradiagonal forms (it is proved by the action $D(P_1)P_2 = -P_2D(P_1)$:

$$\begin{bmatrix} \ddots & & & \\ & +a & 0 & \\ & 0 & -a & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & 0 & b_{12} & \\ & b_{21} & 0 & \\ & & & \ddots \end{bmatrix}. \tag{367}$$

If the matrix P_2 rather than P_1 is diagonalized, then 2×2-cells in the new base are

$$\begin{bmatrix} \ddots & & & \\ & 0 & a_{12} & \\ & a_{21} & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & +b & 0 & \\ & 0 & -b & \\ & & & \ddots \end{bmatrix}. \tag{368}$$

In addition in the both cases there holds: $a = \sqrt{a_{12}a_{21}},\ b = \sqrt{b_{12}b_{21}}$ at all indices j. (The special case when both the matrices may be in contradiagonal forms – see later.)

Indeed, for the variant $\Pi_1 = P_1 \cdot P_2$, in general case, we have:

$$\Pi_1 = \begin{bmatrix} \ddots & & & \\ & a_1 & 0 & \\ & 0 & a_2 & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & & \\ & p_{11} & p_{12} & \\ & p_{21} & p_{22} & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & a_1p_{11} & a_1p_{12} & \\ & a_2p_{21} & a_2p_{22} & \\ & & & \ddots \end{bmatrix}.$$

And, for the variant $\Pi_2 = P_2 \cdot P_1 = -\Pi_1$, in general case, we have:

$$\Pi_2 = \begin{bmatrix} \ddots & & & \\ & p_{11} & p_{12} & \\ & p_{21} & p_{22} & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & & \\ & a_1 & 0 & \\ & 0 & a_2 & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & a_1p_{11} & a_2p_{12} & \\ & a_1p_{21} & a_2p_{22} & \\ & & & \ddots \end{bmatrix}.$$

We set P_1 in its diagonal form, and then it is necessary to find the form of P_2.
Obviously, we have the initial conditions: $a_1 \neq 0, a_2 \neq 0$ (as well as $b_1 \neq 0, b_2 \neq 0$).
Further, there hold:
$a_1p_{11} = -a_1p_{11},\ a_2p_{22} = -a_2p_{22} \to p_{11} = p_{22} = 0,$
$a_1p_{12} = -a_2p_{12},\ a_2p_{21} = -a_1p_{21} \to a_1 = -a_2 = +a;\ p_{12} \neq 0, p_{21} \neq 0.$
Analogously, for diagonal elements of P_2 there hold: $b_1 = -b_2 = +b$.
After permutation of a_j in (367), for its two contradiagonal elements there holds:
$\det\ P_1 = -a^2 = -a_{12} \cdot a_{21}$. Analogously, there holds: $\det\ P_2 = -b^2 = -b_{12} \cdot b_{21}$!

The covariant column matrix converting the contradiagonal form in (367) or (368) into D-form may be evaluated, for example, with the use of results in sect. 2.2. This modal matrix may be represented in the following general *affine trigonometric form*, for example, for contradiagonal form of P_2 in (367) as its j-th 2×2-cell:

$$W_{col}^{-1} \cdot W(P) \cdot W_{col} =$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & +\frac{\sqrt{2}}{2}\sqrt{\frac{b_{12}}{b_{21}}} \\ -\frac{\sqrt{2}}{2}\sqrt{\frac{b_{21}}{b_{12}}} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\sqrt{\frac{b_{12}}{b_{21}}} \\ +\frac{\sqrt{2}}{2}\sqrt{\frac{b_{21}}{b_{12}}} & \frac{\sqrt{2}}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} +\sqrt{b_1 b_2} & 0 \\ 0 & -\sqrt{b_1 b_2} \end{bmatrix} = \begin{bmatrix} +b & 0 \\ 0 & -b \end{bmatrix} = D(P), \tag{369}$$

$$W_{col} = \{Rot\ \pi/4\}_{af} = W^{-1} \cdot \{Rot\ \pi/4\} \cdot W, \tag{370}$$

$$D(P) = W_{col}^{-1} \cdot W(P) \cdot W_{col} = W_{col}^{-1} \cdot V_W^{-1} \cdot P \cdot V_W \cdot W_{col} = V_{col}^{-1} \cdot P \cdot V_{col}. \tag{371}$$

Here $det\{Rot\ \pi/4\}_{af} = 1$, $\mu_{1,2} = \cos\pi/4 \pm i \sin\pi/4$. Formula (370) determines a spherical rotational matrix in a certain *affine base*. In particular, in the real Cartesian base, this matrix is $Rot\ \pi/4$; in complex binary Cartesian base (271), it is $Roth\ (-i\pi/4)$. Besides, due to (366)–(368), the diagonal and contradiagonal W-structures are preserved under the base rotations and reflections of their W-forms as in (366), i. e., at compatible right tensor angles.

Consider most important special cases of normal matrices anticommutativity what are related to the tensor trigonometry in $\langle \mathcal{E}^n \rangle$. In general, $a_{12} = \pm a_{21}$, $b_{12} = \pm b_{21}$, and then $V_W = R_W$. Suppose that $P_1 = M_1, P_2 = M_2$ are anticommutative real-valued *normal* matrices (or *complex-valued adequately normal* ones – sect. 4.2). They may be either symmetric (S), or skew-symmetric (K). Three trigonometric variants (S_1 and S_2, S and K, K_1 and K_2) are exposed with the use of (367) and (368). One else variant corresponds to the case when the matrices S and K may be together in contradiagonal forms. (But it is combination of two simple variants.) All these variants are:

A) $a_{12} = a_{21} = +a$, $b_{12} = b_{21} = +b$; $P_1 = S_1$, $P_2 = S_2$, $S_1 \cdot S_2 = -S_2 \cdot S_1$. This corresponds in (183) to $S_1 = \cos\tilde{\Phi}$, $S_2 = \sin\tilde{\Phi}$ ($a^2 + b^2 = 1$, $S_1^2 + S_2^2 = I$). Then

$$V_{col} = R_W \cdot \begin{bmatrix} \ddots & & & \\ & \sqrt{2}/2 & -\sqrt{2}/2 & \\ & +\sqrt{2}/2 & \sqrt{2}/2 & \\ & & & \ddots \end{bmatrix} = Rot\ \pi/4 \cdot R_W.$$

B) $a_{12} = a_{21} = +a$, $-b_{12} = +b_{21} = +b/i$; $P_1 = S$, $P_2 = K$, $S \cdot K = -K \cdot S$. This corresponds in (209) to $S = \sec\tilde{\Phi}$, $K = i\tan\tilde{\Phi}$ ($a^2 - b^2 = 1$, $S^2 - K^2 = I$). Then

$$V_{col} = R_W \cdot \begin{bmatrix} \ddots & & & \\ & \sqrt{2}/2 & -i\sqrt{2}/2 & \\ & -i\sqrt{2}/2 & \sqrt{2}/2 & \\ & & & \ddots \end{bmatrix} = Roth\ i\pi/4 \cdot R_W, \text{(see in scheme (322))}.$$

We have in (204) $S = \cos\tilde{\Phi}$, $K = i\tan\tilde{\Phi}$; and the unusual pair $S = \cos\tilde{\Phi}$, $K = i\sin\Phi$ (in the last case: $\cos\tilde{\Phi}\sin\Phi = (\cos\tilde{\Phi}\sin\Phi)' = \sin'\Phi\cos\tilde{\Phi} = -\sin\Phi\cos\tilde{\Phi}$).

C) $a_{12} = a_{21} = +a = ic$, $-b_{12} = +b_{21} = +b/i$; $K_1 \cdot K_2 = -K_2 \cdot K_1$; $-c^2 - b^2 = 1$, $-K_1^2 - K_2^2 = I$. This variant is given for completeness.

D) Begin with conditions from (B), then transform the base $\tilde{E}$ for both the matrices by *Rot* $\pi/4$. The matrix P_1 and the matrix P_2 (invariant to this rotation) have two *different contradiagonal forms* with the entries $a_{12} = a_{21} = +a,$, $-b_{12} = +b_{21} = +b/i$. This corresponds in (204) to $S = \sin\tilde{\Phi}$, $K = i\tan\tilde{\Phi}$ (or $S = \sin\tilde{\Phi}$, $K = i\sin\Phi$). The bases of such anticommutative trigonometric matrices *in their diagonal forms* are differed by amalgamated rotation *Roth* $i\pi/4 \cdot$ *Rot* $\pi/4$ or *Rot* $\pi/4 \cdot$ *Roth* $i\pi/4$ (or by the tensor angles algebraic sum). For the matrices there hold $a_{12}b_{21} = -a_{21}b_{12}$.

The main result in the trigonometric forms is the following.

1. *Nonsingular prime matrices P_1, P_2 are anticommutative iff bases of D-forms are connected by compatible rotations or reflections at tensor angles $\pm\pi/4$ or / and $\pm i\pi/4$.*

2. *Sizes of nonsingular anticommutative prime matrices P_1, P_2 are even.*

3. *Anticommutative singular prime matrices P_1, P_2 have compatible biorthogonal blocks, what may be converted into biorthogonal D-forms in their common sub-base.*

Note, as in the end of sect. 7.1, that the rotation angles $\pm\pi/4$ and $\pm i\pi/4$ correspond to the deformational angle $\pm\omega$ or $\pm i\omega$ (*nonperiodic*) in universal bases – see in Ch. 6.

* * *

Further consider some trigonometric examples of complex-valued Hermitean normal matrices N_1, N_2 corresponding to examples A, B, C, exposed above. We have

$$b_1 = \rho_1(\cos\beta_1 + i\sin\beta_1), \quad b_2 = \rho_2(\cos\beta_2 + i\sin\beta_2), \quad \rho_1 > 0, \rho_2 > 0, \beta_1, \beta_2 \in [0; 2\pi],$$

$$b = \sqrt{b_1 b_2} = \sqrt{\rho_1\rho_2}\exp[i(\beta_1 + \beta_2)/2],$$

$$\sqrt{b_2/b_1} = \sqrt{\rho_2/\rho_1}\exp(i\beta_{12}), \quad \sqrt{b_1/b_2} = \sqrt{\rho_1/\rho_2}\exp(-i\beta_{12}), \quad \beta_{12} = \beta_2 - \beta_1.$$

As above, variants $b_{12} = \pm b_{21}$, $a_{12} = \pm a_{21}$, $V_W = R_W$ are possible. However, more complicated cases $|b_{12}| = |b_{21}| = \rho_b$, $|a_{12}| = |a_{21}| = \rho_a$; $V_W = U_W$ are possible too.

Let $P_1 = N_1$, $P_2 = N_2$ be anticommutative *Hermitean normal* matrices. Here they may be Hermitean or skew-Hermitean, this corresponds to three anticommutative pairs: H_1 and H_2, H and Q, Q_1 and Q_2. The affine spherical unitary modal matrix V_{col} is

$$\begin{bmatrix} \ddots & & & \\ & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\cdot\exp(-i\beta_{12}) & \\ & +\frac{\sqrt{2}}{2}\cdot\exp(+i\beta_{12}) & \frac{\sqrt{2}}{2} & \\ & & & \ddots \end{bmatrix} = \left\{ Exp\frac{-i\beta_{12}}{2}\cdot Rot\ \pi/4\cdot Exp\frac{+i\beta_{12}}{2} \right\}, \tag{372}$$

(it is more general rotational complex modal matrix, then ones used above), where

$$Exp(+i\beta_{12}/2) = \{Rot\ (+\beta_{12}/2)\}_c = U_W\cdot \begin{bmatrix} \ddots & & & \\ & \exp(+i\beta_{12}/2) & 0 & \\ & 0 & \exp(-i\beta_{12}/2) & \\ & & & \ddots \end{bmatrix}\cdot U_W^*.$$

If $\beta_{12} = \beta_2 - \beta_1 = \pi/2$, then the modal matrix is *Roth* $i\pi/4$ in variant (B) above. It corresponds to the *complex-valued binary Cartesian base* – see (287) in sect. 5.9. More generally, formula (372) expresses *Rot* $\pi/4$ in a Hermitean orthogonal base with imaginary shift at the angle $i\beta_{12}$ in formulae (367) and (368):

$$\overset{N_1}{\begin{bmatrix} \ddots & & & \\ & +\rho_a\exp[i(\alpha_1+\alpha_2)/2] & 0 & \\ & 0 & -\rho_a\exp[i(\alpha_1+\alpha_2)/2] & \\ & & & \ddots \end{bmatrix}}, \overset{N_2}{\begin{bmatrix} \ddots & & & \\ & 0 & \rho_b\exp(i\beta_1) & \\ & \rho_b\exp(i\beta_2) & 0 & \\ & & & \ddots \end{bmatrix}};$$

$$\begin{bmatrix} \ddots & & & \\ & 0 & \rho_a\exp(i\alpha_1) & \\ & \rho_a\exp(i\alpha_2) & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & +\rho_a\exp[i(\beta_1+\beta_2)/2] & 0 & \\ & 0 & -\rho_a\exp[i(\beta_1+\beta_2)/2] & \\ & & & \ddots \end{bmatrix}.$$

For the pair $N_1\cdot N_2 = -N_2\cdot N_1$ three important special cases as above are possible.

A) $\beta_{1j}+\beta_{2j} = \alpha_{1j}+\alpha_{2j} = 0$. Then N_1 and N_2 are the anticommutative Hermitean matrices $P_1 = H_1$, $P_2 = H_2$. In the special case $a_j^2 + b_j^2 = 1$, then these matrices are the projective Hermiteized cosine and sine, and $H_1^2 + H_2^2 = I$, $H_1\cdot H_2 = -H_2\cdot H_1$.

B) $\beta_{1j}+\beta_{2j} = \pi$, $\alpha_{1j}+\alpha_{2j} = 0$. Then N_1 and N_2 are the anticommutative Hermitean and skew-Hermitean matrices $P_1 = H$, $P_2 = Q$. In the special case $a_j^2 - b_j^2 = 1$, then these matrices are the projective Hermiteized secant and skew-Hermiteized tangent, and $H^2 + Q^2 = I$, $H\cdot Q = -Q\cdot H$.

C) $\beta_{1j}+\beta_{2j} = \alpha_{1j}+\alpha_{2j} = \pi$. Then N_1 and N_2 are the anticommutative skew-Hermitean matrices $P_1 = Q_1$, $P_2 = Q_2$., and $-Q_1^2 - Q_2^2 = I$.

Thus all most important types of anticommutative prime matrices types are described.

Chapter 8

Trigonometric spectra and trigonometric inequalities

8.1 Trigonometric spectrum of a null-prime matrix

Matrix characteristic coefficients of higher orders, as well as eigenprojectors, are prime singular matrices with a unique eigenvalue (see Ch. 1 and 2). Consider a null prime matrix B with its coefficient $K_2(B,r)$ of the highest order r and angle $\tilde{\Phi}_B$. Represent $K_2(B,r)$ as an algebraic orthogonal sum over eigen trigonometric subspaces of $\tilde{\Phi}_B$:

$$K_2(B,r) = \sum_{i=1}^{r-\nu'} \overrightarrow{S_i} \cdot K_2(B,r) \cdot \overrightarrow{S_i} + \overrightarrow{S_m} \cdot K_2(B,r) \cdot \overrightarrow{S_m}, \tag{373}$$

where $\overrightarrow{S_i} = \overrightarrow{\cos^2 \tilde{\Phi}_B - \cos^2 \varphi_i \cdot I}$ is the orthogonal projector into the i-th trigonometric eigen plane $\langle \mathcal{P}_i \rangle$ – see (240), $\overrightarrow{S_m} = \overrightarrow{\cos \tilde{\Phi}_B - I}$ is the orthogonal projector into the subspace $\langle \mathcal{P}_m \rangle \equiv \langle im\ B \rangle \cap \langle im\ B' \rangle$ of dimension ν' (see Figure 2). *Here* $\nu'' = 0$ *as the matrix* B *is null-prime!* The orthoprojectors form too the complete algebraic sum;

$$\sum_{i=1}^{r-\nu'} \overrightarrow{S_i} + \overrightarrow{S_m} + \overrightarrow{S_q} = I,$$

where $\overrightarrow{S_q} = \overrightarrow{\cos \tilde{\Phi}_B + I}$ is the orthogonal projector into the subspace $\langle \mathcal{P}_q \rangle \equiv \langle ker\ B \rangle \cap \langle ker\ B' \rangle$ of dimension $n - 2r + \nu'$ (Figure 2). The entire sum of these dimensions $2(r - \nu') + \nu' + (n - 2r + \nu') = n$ is equal to dimension of the whole Euclidean space. In the direct sum, according to the principle of binarity (see sect. 5.7), we have the following. The coefficient $K_2(B,r)$ in the subspace $\langle \mathcal{P}_i \rangle$ is a singular matrix of rank 1 and of size 2×2, the coefficient $K_2(B,r)$ in the subspace $\langle \mathcal{P}_m \rangle$ is a nonsingular matrix of size $\nu' \times \nu'$, and the coefficient $K_2(B,r)$ in the space $\langle \mathcal{P}_q \rangle$ is the zero $(n-2r+\nu') \times (n - 2r + \nu')$-matrix. Thus,

$$K_2(B,r) = \sum_{i=1}^{r-\nu'} \boxplus\ B_i^{2\times 2} \boxplus det\ B_m^{\nu'\times\nu'} \cdot I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{374}$$

where mark $\boxplus$ stands for *direct orthogonal summation*; $r - \nu' \geq 0,\ n - 2r + \nu' \geq 0$ and, consequently, there hold:

$$2r - n \leq \nu' \leq r. \tag{375}$$

If B is a null-normal matrix (see sect. 2.4), then formula (374) is the simplest:

$$K_2(B,r) = det\ B_m^{r\times r} \cdot I^{r\times r} \boxplus Z^{(n-r)\times(n-r)}.$$

We used especial notation beginning with formula (374):

$B_i^{2\times 2}$ for a 2×2-matrix of rank 1, its highest matrix coefficient is, according to (29), the matrix itself, its highest scalar coefficient is the trace of the matrix;

$B_m^{\nu'\times\nu'}$ stands for a $\nu'\times\nu'$-matrix of rank ν', its highest matrix and scalar coefficients are $det\ B_m^{\nu'\times\nu'} \cdot I^{\nu'\times\nu'}$ and $det\ B_m^{\nu'\times\nu'}$ respectively;

$Z^{(n-2r+\nu')\times(n-2r+\nu')}$ is the zero matrix of indicated size not intersecting with $B_i^{2\times 2}$.

The total singularity of B and of $K_2(B,r)$ is $(r-\nu')+(n-2r+\nu') = n-r$.

Formula (374) may be transformed, with the use of (62) for $r=2$ and $r=n$, into the direct trigonometric spectrum of the eigen oblique projector $\overleftarrow{B}$, it is called the *trigonometric spectrum of a null-prime matrix* B:

$$\overleftarrow{B} = \frac{K_2(B,r)}{k(B,r)} = \sum_{i=1}^{r-\nu'} \boxplus \frac{B_i^{2\times 2}}{tr\ B_i^{2\times 2}} \boxplus I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}. \tag{376}$$

Similar algebraic representation of the coefficient $K_2(BB',r)$ of the highest order and the eigen orthoprojector $\overleftarrow{BB'}$, as the *trigonometric spectrum of a multiplicative matrix* BB', are derived, according to the principle of binarity (see sect. 5.7):

$$K_2(BB',r) = \sum_{i=1}^{r-\nu'} \overrightarrow{S_i} \cdot K_2(BB',r) \cdot \overrightarrow{S_i} + \overrightarrow{S_m} \cdot K_2(BB',r) \cdot \overrightarrow{S_m}, \tag{377}$$

$$K_2(BB',r) = \sum_{i=1}^{r-\nu'} \boxplus\ B_i^{2\times 2}(B')_i^{2\times 2} \boxplus det^2\ B_m^{\nu'\times\nu'} \cdot I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{378}$$

$$\overleftarrow{BB'} = \frac{K_2(BB',r)}{k(BB',r)} = \sum_{i=1}^{r-\nu'} \boxplus \frac{B_i^{2\times 2}(B')_i^{2\times 2}}{tr\ [B_i^{2\times 2}(B')_i^{2\times 2}]} \boxplus I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}. \tag{379}$$

Note, that for a null-prime matrix B'. we use similar algebraic representations of the coefficient $K_2(B'B,r)$ and the eigen orthoprojector $\overleftarrow{B'B}$.

From direct spectra (374), (376) and (378), (379) we infer multiplicative formulae for the highest scalar coefficients for matrices B (or B') and BB' (or $B'B$):

$$k(B,r) = \prod_{i=1}^{r-\nu'} tr\ B_i^{2\times 2} det\ B_m^{\nu'\times\nu'} = \prod_{i=1}^{r-\nu'} tr\ (B')_i^{2\times 2} det\ (B')_m^{\nu'\times\nu'} = k(B',r), \tag{380}$$

$$k(BB',r) = \prod_{i=1}^{r-\nu'} tr\ [B_i^{2\times 2} \cdot (B')_i^{2\times 2}] det^2\ B_m^{\nu'\times\nu'} = k(B'B,r). \tag{381}$$

8.2 The general cosine inequality

For null-prime matrices $rank\{\cos\tilde{\Phi}_B\} = n$ $(\nu'' = 0)$, and due to (186), (194) we have

$$\overleftarrow{BB'} = \overleftarrow{B} \cdot \overleftarrow{B'} \cdot \cos^2\tilde{\Phi}_B = (\overleftarrow{B} \cdot \cos\tilde{\Phi}_B) \cdot (\overleftarrow{B} \cdot \cos\tilde{\Phi}_B)'. \qquad (382)$$

In $\overleftarrow{B} \cdot \overleftarrow{B'} \cdot \cos^2\tilde{\Phi}_B$, represent all the matrices as direct spectra, obtain the following inequalities for each trigonometric cell with the use of the principle of binarity:

$$0 \leq \cos^2\varphi_i = \frac{tr^2\ B_i^{2\times 2}}{tr\ [B_i^{2\times 2} \cdot (B')_i^{2\times 2}]} \leq 1. \qquad (383)$$

From (380), (381), and (383) the *general cosine inequality in the normalized form* for a square matrix (where $\varphi_i \in (0; \pi/2]$), i. e., in variant (138), follows:

$$0 \leq \prod_{i=1}^{r-\nu'} \cos^2\varphi_i = |\{B\}|^2_{\cos} = |det\ \cos\tilde{\Phi}_B| = \frac{k^2(B, r)}{k(BB', r)} \leq 1. \qquad (384)$$

Here $|\{B\}|_{\cos}$ defines the *cosine norm* of $\tilde{\Phi}_B$ and Φ_B. Its extremal special cases are: $|\{B\}|_{\cos} = 0$ if B is a null-defected matrix, $|\{B\}|_{\cos} = 1$ if B is a null-normal matrix. In terms of the dianal and the minorant of B (see Ch. 3) the general cosine inequality and the cosine norm of $\tilde{\Phi}_B$ and Φ_B (or the cosine ratio for B) are expressed as

$$0 \leq \frac{|\mathcal{D}l(r)B|}{\mathcal{M}t(r)B} = |\{B\}|_{\cos} = \frac{|\mathcal{D}l(r)B|}{\sqrt{\mathcal{D}l(r)BB'}} \leq 1.$$

Consider $(\overleftarrow{B} \cdot \cos\tilde{\Phi}_B) \cdot (\overleftarrow{B} \cdot \cos\tilde{\Phi}_B)'$ in (382) and obtain similar cosine inequalities in *the sign form* (where $\varphi_i \in (0; \pi]$):

$$-1 \leq \cos\varphi_i = \frac{tr\ B_i^{2\times 2}}{\sqrt{tr\ \{B_i^{2\times 2} \cdot (B')_i^{2\times 2}\}}} \leq +1. \qquad (385)$$

The cosine ratio $|\{B\}|_{\cos}$ is supplemented by the *signed cosine ratio* as in variant (137):

$$-1 \leq \prod_{i=1}^{r-\nu'} \cos\varphi_i = \{B\}_{\cos} = \frac{k(B, r)}{\sqrt{k(BB', r)}} = \frac{\mathcal{D}l(r)B}{\mathcal{M}t(r)B} = \frac{\mathcal{D}l(r)B}{\sqrt{\mathcal{D}l(r)BB'}} \leq +1. \quad (386)$$

The extremal cases (± 1) correspond to null-normal matrix B with positive or negative dianals – see before (138) in Ch. 3. Note, that (386) supplements independently the Inequality of H. Weyl for the eigen and singular numbers of $n \times n$-matrix B [4].

The *cosine distinct ranges* of the angles is similar to that for the angle between two undirected vectors and the angle between two directed vectors (or straight lines). (But the *sine distinct ranges* of the angles give algebraically $\varphi_i \in [-\pi/2; +\pi/2]$ – Ch. 3.)

Corollary. *For spherical functions of tensor angles* $\tilde{\Phi}_B$ *and* Φ_B, *their eigen angles* φ_i *have the following trigonometric sense: they are the scalar angles between planars or lineors, given by matrices* $B_i^{2\times 2}$ *and* $B'^{2\times 2}_i$ *of rank* 1 *in the trigonometric spectra of the eigen projectors* $\overleftarrow{B}$ *and* $\overleftarrow{B'}$ (see (186)–(189), (190)-(193) and Figure 1).

$|\{B\}|_{\cos}$ is the cosine ratio for the *planars* $\langle im\ B\rangle$, $\langle im\ B'\rangle$ as well as the planars $\langle ker\ B\rangle$, $\langle ker\ B'\rangle$; but $\{B\}_{\cos}$ is the cosine ratio for *lineors* determined by B and B'.

If a binary tensor angle $\tilde{\Phi}_{12}$ is determined by equirank lineors A_1, A_2 or planars $\langle im\ A_1\rangle$, $\langle im\ A_2\rangle$, then scalar angles φ_i in cells have the similar sense. Suppose that $B = A_1A'_2$, condition (224) holds and consequently bijection (226) between eigen orthoprojectors takes place. The trigonometric spectra for external multiplications are

$$K_2(AA',r) = \sum_{i=1}^{r-\nu'} \overrightarrow{S_i} \cdot K_2(AA',r) \cdot \overrightarrow{S_i} + \overrightarrow{S_m} \cdot K_2(AA',r) \cdot \overrightarrow{S_m} \equiv$$

$$\equiv \sum_{i=1}^{r-\nu'} \boxplus\ (AA')_i^{2\times 2} \boxplus det\ (AA')^{\nu'\times\nu'} \cdot I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{387}$$

$$K_2(A_1A'_2,r) = \sum_{i=1}^{r-\nu'} \overrightarrow{S_i} \cdot K_2(A_1A'_2,r) \cdot \overrightarrow{S_i} + \overrightarrow{S_m} \cdot K_2(A_1A'_2,r) \cdot \overrightarrow{S_m} \equiv$$

$$\equiv \sum_{i=1}^{r-\nu'} \boxplus\ (A_1A'_2)_i^{2\times 2} \boxplus det\ (A_1A'_2)^{\nu'\times\nu'} \cdot I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{388}$$

$$\overleftarrow{AA'} = \frac{K_2(AA',r)}{k(AA',r)} = \sum_{i=1}^{r-\nu'} \boxplus\ \frac{(AA')_i^{2\times 2}}{tr\ ((AA')_i^{2\times 2}} \boxplus I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{389}$$

$$\overleftarrow{A_1A'_2} = \frac{K_2(A_1A'_2,r)}{k(A_1A'_2,r)} = \sum_{i=1}^{r-\nu'} \boxplus\ \frac{(A_1A'_2)_i^{2\times 2}}{tr\ ((A_1A'_2)_i^{2\times 2}} \boxplus I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{390}$$

$$k(AA',r) = \prod_{i=1}^{r-\nu'} tr\ (AA')_i^{2\times 2} det\ (AA')^{\nu'\times\nu'} = det\ (A'A), \tag{391}$$

$$k(A_1A'_2,r) = \prod_{i=1}^{r-\nu'} tr\ (A_1A'_2)_i^{2\times 2} det\ (A_1A'_2)^{\nu'\times\nu'} = det\ (A'_1A_2). \tag{392}$$

According to (132) there holds $det^2\ (A_1A'_2)_i^{\nu'\times\nu'} = det\ (A_1A'_1)_i^{\nu'\times\nu'} \cdot det\ (A_2A'_2)_i^{\nu'\times\nu'}$. Further, from (186), (187), (196), and (226) we obtain

$$\overleftarrow{A_1A'_1} \cdot \overleftarrow{A_2A'_2} = \overleftarrow{A_1A'_2} \cdot \cos^2\tilde{\Phi}_{12} = (\overleftarrow{A_1A'_2}\cos\tilde{\Phi}_{12}) \cdot (\overleftarrow{A_2A'_1} \cdot \cos\tilde{\Phi}_{12}). \tag{393}$$

In addition, intermediately, by (68) in its special case for $n = 2$, and the obvious relation $[A'_2A_1]_i = [A'_1A_2]_i$, for the i-th 2×2-cells of rank 1 there holds

$$(A_1A'_2)_i^{2\times 2} \cdot (A_1A'_2)_i^{2\times 2} = tr\ (A_1A'_2)_i^{2\times 2} \cdot (A_1A'_2)_i^{2\times 2} = (A_1A'_1)_i^{2\times 2} \cdot (A_2A'_2)_i^{2\times 2}. \tag{394}$$

Represent the matrices in (393) as direct spectra and apply (394) in all the i-th cells, obtain the i-th elementary cosine inequalities

$$0 \le \cos^2 \varphi_i = \frac{tr^2\ (A_1A_2')_i^{2\times 2}}{tr\ (A_1A_1')_i^{2\times 2} tr\ (A_2A_2')_i^{2\times 2}} \le 1, \qquad (395)$$

and the general cosine inequality for equirank lineors A_1, A_2 in *the normalized form*:

$$0 \le \prod_{i=1}^{r-\nu'} \cos^2 \varphi_i = |\{A_1A_2'\}|^2_{\cos} = |det\ \cos \tilde{\Phi}_{12}| = \frac{\mathcal{D}l^2(r)(A_1A_2')}{\mathcal{M}t^2(r)A_1 \cdot \mathcal{M}t^2(r)A_2} \le 1, \quad (396)$$

where φ_i are the scalar angles between the planars $\langle im\ (A_1A_1')_i^{2\times 2}\rangle \equiv \langle im\ (A_1A_2')_i^{2\times 2}\rangle$ and $\langle im\ (A_2A_2')_i^{2\times 2}\rangle \equiv \langle im\ (A_2A_1')_i^{2\times 2}\rangle$. Under condition (224) there holds (sect. 3.3):

$$0 \le \prod_{i=1}^{r-\nu'} \cos^2 \varphi_i = |\{A_1A_2'\}|^2_{\cos} = |det\ \cos \tilde{\Phi}_{12}| = \frac{det^2\ (A_1'A_2)}{det\ (A_1'A_1) \cdot det\ (A_2'A_2)} \le 1,$$

(for non-orthogonal lineors: $rank\{\cos \tilde{\Phi}_{12}\} = n\ (\nu'' = 0)$). The extremal cases are
$|\{A_1A_2'\}|_{\cos} = 1$ if the lineors are entirely parallel, $\{A_1A_2'\}$ is null-normal;
$|\{A_1A_2'\}|_{\cos} = 0$ if the lineors are orthogonal, may be partially, $\{A_1A_2'\}$ is null-defected.
The general cosine inequality is a direct product of particular Cauchy Inequalities. It is inferred through the external or internal multiplications of cosine type of two lineors.

The signed forms of these inequalities and the cosine ratio are

$$-1 \le \cos \varphi_i = \frac{tr\ (A_1A_2')_i^{2\times 2}}{\sqrt{tr\ (A_1A_1')_i^{2\times 2} \cdot tr\ (A_2A_2')_i^{2\times 2}}} \le +1, \qquad (397)$$

$$-1 \le \prod_{i=1}^{r-\nu'} \cos \varphi_i = \{A_1A_2'\}_{\cos} = \frac{\mathcal{D}l(r)(A_1A_2')}{\mathcal{M}t(r)A_1 \cdot \mathcal{M}t(r)A_2} \le +1. \qquad (398)$$

The numerators and denominators in (384) and (396) under condition (224) are the same in accordance with (132). (If $r_1 \ne r_2$, then the cosine ratio formally is 0.)

In general cosine inequality (396), the value $|\{A_1A_2'\}|_{\cos}$ determines the *cosine norm* of $\tilde{\Phi}_{12}$ and Φ_{12}. In the special case $r = 1$, formula (396) is *the module form* of the geometric Cauchy inequality for two vectors. The Cauchy inequality is used in analytical geometry for normalizing the angle between two vectors in $[0; \pi/2]$. *The sign form* of the inequality similar to (141) determines the signed cosine of the angle between two directed vectors in $[0; \pi]$. It is the same special case of (398). Initially, the Cauchy inequality had the pure algebraic character. General inequalities (384), (386), and (396), (398) may be considered from the algebraic point of view too if they are applied to scalar elements of matrices.

From (299), (230) the following *internal multiplication criterion for at least partial orthogonality of two equirank $n \times r$-lineors* is inferred:

$$det\ C_{12} = det\ (A_1'A_2) = 0\ \Leftrightarrow\ \{A_1A_2'\}_{\cos} = 0. \qquad (399)$$

8.3 Spectral-cell representations of tensor trigonometric functions

Now it is possible to consider in details the structures of tensor trigonometric functions at the level of elementary 2×2-cells. It was shown in Ch. 5 that the eigen trigonometric planes corresponding to 2×2-cells are the same for projective and motive tensor angles. That is why from the left side of (301) and spectral formula (389) we obtain the following rotational connection between two equirank planars

$$\begin{bmatrix} \cos\varphi_i & -\sin\varphi_i \\ +\sin\varphi_i & \cos\varphi_i \end{bmatrix} \cdot \frac{(A_1A_1')_i^{2\times 2}}{tr\ (A_1A_1')_i^{2\times 2}} \cdot \begin{bmatrix} \cos\varphi_i & +\sin\varphi_i \\ -\sin\varphi_i & \cos\varphi_i \end{bmatrix} = \frac{(A_2A_2')_i^{2\times 2}}{tr\ (A_2A_2')_i^{2\times 2}}.$$

Further, represent the 2×2-cell $[\overleftarrow{AA'}]_i^{2\times 2}$ of rank 1 for the eigen projector $\overleftarrow{AA'}$ as the following exterior multiplication of the unity 2×1-vector $\mathbf{e}_i$:

$$[\overleftarrow{AA'}]_i^{2\times 2} = \frac{(AA')_i^{2\times 2}}{tr\ (AA')_i^{2\times 2}} = \mathbf{e_i e_i'} = \overleftarrow{\mathbf{e_i e_i'}}.$$

Here the unity 2×1-vector $\mathbf{e}_i$ determines the i-th basic line of the planar $\langle im\ A\rangle$ in the i-th eigen plane of the binary tensor angle $\tilde{\Phi}_{12}$.

Respectively the two sides of this tensor angle between planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ of rank r at the level of elementary 2×2-cells may be represented as two unity eigenvectors (straight lines). They may be transformed into each other with rotation or reflection according to (301). Express the Cartesian coordinates of these vectors as

$$\mathbf{e}_1 = \begin{bmatrix} \cos\varphi_1 \\ \sin\varphi_1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} \cos\varphi_2 \\ \sin\varphi_2 \end{bmatrix}.$$

Then their rotational transformation is

$$\mathbf{e}_2 = \begin{bmatrix} \cos\varphi_{12} & -\sin\varphi_{12} \\ +\sin\varphi_{12} & \cos\varphi_{12} \end{bmatrix} \cdot \mathbf{e}_1, \quad \varphi_{12} = \varphi_2 - \varphi_1.$$

The vector $\mathbf{e}_1$ and each of two its orthoprojections are rotated at the same angle. According to definition (171), the tensor cosine at the level of elementary 2×2-cells is

$$[\cos\tilde{\Phi}_{12}]^{2\times 2} = \overleftarrow{\mathbf{e_1 e_1'}} + \overleftarrow{\mathbf{e_2 e_2'}} - I^{2\times 2} = \mathbf{e_1 e_1'} + \mathbf{e_2 e_2'} - I^{2\times 2}.$$

This initial trigonometric definition gives final result

$$[\cos\tilde{\Phi}_{12}]^{2\times 2} = \cos\varphi_{12} \cdot \begin{bmatrix} +\cos(\varphi_1+\varphi_2) & \sin(\varphi_1+\varphi_2) \\ \sin(\varphi_1+\varphi_2) & -\cos(\varphi_1+\varphi_2) \end{bmatrix}. \tag{400}$$

Here

$$\cos(\varphi_2-\varphi_1)\cdot\cos(\varphi_2+\varphi_1) = \cos^2\varphi_2 + \cos^2\varphi_1 - 1,$$
$$\cos(\varphi_2-\varphi_1)\cdot\sin(\varphi_2+\varphi_1) = \cos\varphi_2\sin\varphi_2 + \cos\varphi_1\sin\varphi_1.$$

Consider a 2×2-cell of the tensor sine. According to definition (163) it is

$$[\sin \tilde{\Phi}_{12}]^{2\times 2} = \overleftarrow{\mathbf{e_2 e_2'}} - \overleftarrow{\mathbf{e_1 e_1'}} = \mathbf{e_2 e_2'} - \mathbf{e_1 e_1'}.$$

This initial trigonometric definition gives final result

$$[\sin \tilde{\Phi}_{12}]^{2\times 2} = \sin \varphi_{12} \cdot \begin{bmatrix} -\sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) \\ \cos(\varphi_1+\varphi_2) & +\sin(\varphi_1+\varphi_2) \end{bmatrix}. \tag{401}$$

Here

$$\sin(\varphi_2-\varphi_1)\cdot\sin(\varphi_2+\varphi_1) = \sin^2\varphi_2 - \sin^2\varphi_1,$$
$$\sin(\varphi_2-\varphi_1)\cdot\cos(\varphi_2+\varphi_1) = \cos\varphi_2\sin\varphi_2 - \cos\varphi_1\sin\varphi_1.$$

Condition $\varphi_1+\varphi_2 = 0$ and its tensor form $\tilde{\Phi}_1 + \tilde{\Phi}_2 = \tilde{Z}$ determines the Cartesian base of the diagonal cosine, i. e., the trigonometric base for angles $\tilde{\Phi}$ and Φ. Under this condition all tensor angles and their trigonometric functions as well as all their eigen reflectors have canonical forms determined in Ch. 5. Secants and tangents of tensor angles have similar representations. The mirror of the mid-reflector (253) is the mid-subspase of a tensor angle, it is clearly seen in the 2×2-cells considered above.

8.4 The general sine inequality

The sine ratio (135) defines the *sine trigonometric norm of a tensor angle.* It is nonzero if the two lineors are completely linearly independent. From (227), (228) the following *internal multiplication criterion for at least partial parallelism or linear dependence of two lineors of sizes $n \times r_1$ and $n \times r_2$* or planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ is derived:

$$det\ G_{1,2} = det\ [(A_1|A_2)'(A_1|A_2)] = 0 \ \Leftrightarrow\ |\{A_1A_2'\}|_{\sin} = 0. \tag{402}$$

Similar to the cosine ratio, the sine ratio may be represented as direct product of sine ratio (124) in each eigen planes according to the lineors sine trigonometric spectrum. If lineors A_1 and A_2 are linearly independent, the superposition matrix $(A_1|A_2)$ has rank $r_1 + r_2 \le n$. Its external homomultiplication $B_{1,2} = [(A_1|A_2)(A_1|A_2)']$ is a symmetric positive (semi-)definite $n \times n$-matrix. Due to (120) and (402) we have

$$k(B_{1,2}, r_1+r_2) = det\ G_{1,2} \ge 0. \tag{403}$$

Then, by the analogy with (135), through the *external multiplication* $\{B_{1,2}, r_1+r_2\}$ *(or internal multiplication* $\{G_{1,2}\}$*) of these two lineors of sine type*, we obtain

$$|\{A_1|A_2\}|^2_{\sin} = \frac{\mathcal{M}t^2(r_1+r_2)\{A_1|A_2\}}{\mathcal{M}t^2(r_1)A_1\cdot\mathcal{M}t^2(r_2)A_2} = \frac{k(B_{1,2}, r_1+r_2)}{k(A_1A_1', r_1)k(A_2A_2', r_2)}. \tag{404}$$

In addition, due to (62), (159), and (163), *for two completely linearly independent lineors* A_1 and A_2 ($\nu' = 0$), in the subspace of non-zero values of $\sin\tilde{\Phi}_{12}$, there holds

$$\overleftarrow{\sin\tilde{\Phi}_{12}} = \overleftarrow{B_{12}} = \frac{K_2(B_{1,2}, r_1+r_2)}{k(B_{1,2}, r_1+r_2)}, \quad (\nu' = 0 \to rank\{\sin\tilde{\Phi}_{12}\} = r_1+r_2 \le n). \tag{405}$$

Consider the trigonometric spectrum of the coefficient $K_2(B_{1,2}, r_1+r_2)$ and express it as the following algebraic sum with the use of the principle of binarity:

$$K_2(B_{1,2}, r_1+r_2) = \sum_{i=1}^{r_1-\nu''} \overrightarrow{S_i} \cdot K_2(B_{1,2}, r_1+r_2) \cdot \overrightarrow{S_i} + \overrightarrow{S_d} \cdot K_2(B_{1,2}, r_1+r_2) \cdot \overrightarrow{S_d}. \quad (406)$$

Here $\overrightarrow{S_d}$ is the orthogonal projector into the defect subspace of intersections $\langle \mathcal{P}_d \rangle \equiv \langle \langle im\ A_2 \cap ker\ A_1' \rangle \cup \langle im\ A_1 \cap ker\ A_2' \rangle \rangle$ of dimension $(r_2 - r_1 + 2\nu'')$.

This coefficient may be represented also as the direct orthogonal sum

$$K_2(B_{1,2}, r_1+r_2) = \sum_{j=1}^{r_1-\nu''} \boxplus\ det\ [(A_1|A_2)(A_1|A_2)']_j^{2\times 2} \cdot I_j^{2\times 2} \boxplus det\ (A_1A_1')^{\nu''\times\nu''} \cdot\ I^{\nu''\times\nu''} \boxplus$$

$$\boxplus\ det\ (A_2A_2')^{(r_2 - r_1 + \nu'') \times (r_2 - r_1 + \nu'')} \cdot I^{(r_2 - r_1 + \nu'') \times (r_2 - r_1 + \nu'')} \boxplus$$

$$\boxplus\ Z^{(n - r_1 - r_2) \times (n - r_1 - r_2)}, \quad (407)$$

where (as the illustration see Figure 2):
$[(A_1|A_2)(A_1|A_2)']_j^{2\times 2}$ is the nonsingular 2×2-matrix of rank 2, it corresponds to j-th trigonometric cell, its highest matrix coefficient is evaluated by (29), and the highest scalar coefficient is its determinant (their summary dimension here is $2(r_1 - \nu'')$;
$(A_1A_1')^{\nu''\times\nu''}$ and $(A_2A_2')^{\nu''\times\nu''}$ are the nonsingular matrices in the spectrum corresponding to the subspaces $\langle im\ A_1 \cap ker\ A_2' \rangle$ and $\langle im\ A_2 \cap ker\ A_1' \rangle$, their highest coefficients also are specified as determinants;
$Z^{(n - r_1 - r_2) \times (n - r_1 - r_2)}$ is the zero block; if $\nu' \neq 0$, the dimension rises by $2\nu'$.

In the direct sum, the orthoprojector onto the image of homomultiplication $B_{1,2}$ is

$$\overleftarrow{B_{1,2}} = \sum_{j=1}^{r_1-\nu''} \boxplus\ I_j^{2\times 2} \boxplus$$

$$\boxplus\ I^{(r_2 - r_1 + 2\nu'') \times (r_2 - r_1 + 2\nu'')} \boxplus Z^{(n - r_1 - r_2) \times (n - r_1 - r_2)}. \quad (408)$$

With the use of the principle of binarity, from (407), (408) and (378), (379) we may infer relations between higher scalar coefficients and direct products over the trigonometric subspaces as in sect. 8.1. But the two latter for lineors A_1 and A_2 transform into analogous formulae (387) and (389). Suppose in the sequel $r_2 \geq r_1$ (see Figure 2). If lineors are completely linearly independent, then $r_1 + r_2 \leq n$ and $\nu' = 0$. For the i-th trigonometric cell, due to (124) there holds

$$0 \leq \sin^2 \varphi_i = \frac{det\ [(A_1|A_2)(A_1|A_2)']_i^{2\times 2}}{tr\ (A_1A_1')_i^{2\times 2} tr\ (A_2A_2')_i^{2\times 2}} \leq 1, \quad (409)$$

where φ_i is the eigen angle between the planars $\langle im\ (A_1A_1')_i^{2\times 2} \rangle$ and $\langle im\ (A_2A_2')_i^{2\times 2} \rangle$ of rank 1 (similar to one in cosine variant (395)). Further, evaluate the highest scalar coefficient of matrix $B_{1,2}$ with the use of (407)–(409).

$$k(B_{1,2}, r_1 + r_2) =$$

$$= \prod_{i=1}^{r_1-\nu''} det\ [(A_1|A_2)(A_1|A_2)']_i^{2\times 2} \cdot det\ (A_1A_1')_i^{\nu''\times\nu''} \cdot det\ (A_2A_2')_i^{(r_2-r_1+\nu'')\times(r_2-r_1+\nu'')} =$$

$$= \prod_{i=1}^{r_1-\nu''} \{\sin^2\varphi_i \cdot tr(A_1A_1')_i^{2\times 2} \cdot tr(A_2A_2')_i^{2\times 2}\} det(A_1A_1')^{\nu''\times\nu''} det(A_2A_2')^{(r_2-r_1+\nu'')\times(r_2-r_1+\nu'')} =$$

$$= \prod_{i=1}^{r_1-\nu''} \sin^2\varphi_i \cdot k(A_1A_!', r_1) \cdot k(A_2A_2', r_2) \qquad (410)$$

(here ν'' values of $\sin^2\varphi_i = 1$, for $i > r_1 - \nu''$, are omitted.

Finally, the *general sine inequality in the normalized form* for lineors A_1 and A_2 of size $n \times r_1$ and $n \times r_2$ follows from (404) and (410) (where $\varphi_i \in (0; \pi/2]$):

$$0 \le \prod_{i=1}^{r_1-\nu''} \sin^2\varphi_i = |\{A_1|A_2\}|^2_{\sin} = \frac{\mathcal{M}t^2(r_1+r_2)\{A_1|A_2'\}}{\mathcal{M}t^2(r_1)A_1\mathcal{M}t^2(r_2)A_2} =$$

$$= |\mathcal{D}l(r_1+r_2)\sin\tilde{\Phi}_{12}| \le 1. \qquad (411)$$

If $n > 2$, the inequality has only the normalized form. The extremal special cases are:
$|\{A_1|A_2\}| = 0$ if the lineors are at least partially parallel,
$|\{A_1|A_2\}| = 1$ if the lineors are completely orthogonal.
If lineors A_1, A_2 are equirank, then general inequalities (396) and (411) may be united:

$$0 \le \sqrt[r]{|\{A_1A_2\}|^2_{\cos}} + \sqrt[r]{|\{A_1A_2\}|^2_{\sin}} \le 1. \qquad (412)$$

This is derived with applying the algebraic Cauchy inequalities for the arithmetic and geometric means to squared eigenvalues of the cosine and sine, and further summating both the results. The right equality in (412) holds iff $|\varphi_i| = \text{const}$, $i = 1, \dots r$.

If two planars have the same rank 1 (straight lines) or $n-1$ (hyperplanes), then the tensor angle between these planars has exactly one trigonometric cell, it corresponds to the unique trigonometric eigen plane. Then inequalities (412) are transformed into usual identity $\cos^2\varphi + \sin^2\varphi = 1$.

Consider a $n \times r$-matrix A of rank r and its arbitrary partition into j column blocks $A = \{A_1|A_2|\dots|A_j\}$. This form of the matrix corresponds to the polyhedral tensor angle, the sides of the angle are determined by the lineors $A_1, \dots, A_j$. If each block consists of exactly one column, then the polyhedral tensor angle is r-edges. Apply the general sine inequality j times sequentially to this block-matrix A, obtain

$$\mathcal{M}t(r)A \le \mathcal{M}t(r)A_1 \cdot \mathcal{M}t(r)A_2 \cdots \mathcal{M}t(r)A_j. \qquad (413)$$

Equality holds iff the lineors (the vectors) are mutually orthogonal. Inequality (413) is the most complete generalization of the Hadamard Inequality [28] of sine nature.

Chapter 9

Geometric norms of matrix objects

9.1 Quadratic norms of matrix objects in Euclidean spaces

Norms for matrices and matrix objects have as usually positive or non-negative values. The geometric norms must be invariant under admissible geometric transformations in the space containing the objects, including parallel translations. For example, homogeneous transformations in $\langle \mathcal{Q}^{n+q} \rangle$ are determined by a reflector tensor: they are trigonometrically compatible with pure rotations and reflections. In $\langle \mathcal{E}^n \rangle$ the reflector tensor is an unity matrix. As both these basis spaces have the same Euclidean metric (see in sect. 5.7), the geometric norms, defined in $\langle \mathcal{E}^n \rangle$, may be used in $\langle \mathcal{Q}^{n+q} \rangle$ too.

For objects of rank 1 (vectors) in arithmetic space $\langle \mathcal{E}^n \rangle$, the Euclidean norm of length is naturally used. However, for objects of rank r greater than 1, the Frobenius norm (i. e., a norm of the same order 1 similarly to Euclidean one) is only the first special norm from the *set of geometric norms of orders t (*$1 \leq t \leq r$). That is why defining geometric norms of higher orders (up to r) for objects of rank r is the problem of great interest. In principle, there are two ways for defining a geometric norm of a $r \times n$-lineor A as the geometric object (or a $r \times n$-matrix A as the algebraic transformation).

Way 1. At first, an intermediate norm of homomultiplication $A'A$ is evaluated, it depends on eigenvalues $\sigma_i^2 > 0$ of this matrix. Then the norm of the original matrix A may be obtained as the positive square root of the intermediate norm for $A'A$.

Way 2. A norm is defined in terms of positive eigenvalues σ_i of the *arithmetic square root* $\sqrt{A'A}$. But evaluating this square root is a long and complicated process.

(If $A = S$ is a symmetric matrix, then the results of both ways are equivalent.)

Thus, in the book, we use only way 1. Norms constructed with this method are called *quadratic*, as they are based on the set of eigenvalues σ_i^2. For example, symmetric matrix functions $\cos \tilde{\Phi}$, $\sin \tilde{\Phi}$, $\tan \tilde{\Phi}$, $\sec \tilde{\Phi}$ are sign-indefinite. Their nonzero quadratic norms depend on squared eigenvalues of $\cos^2 \varphi_i$, $\sin^2 \varphi_i$, $\tan^2 \varphi_i$, $\sec^2 \varphi_i$. Consequently, they are the same for trigonometric functions of motive and projective tensor angles. (For tensor angles, the general cosine and sine norms were defined in previous chapter.)

Correct definition of *general and particular quadratic norms* will be given with the use of geometric analogies similar to (126), (127) in sect. 3.1 and of the general inequality of means, more precisely, its chain (11) for algebraic means expressed in terms of positive Viéte coefficients (sect. 1.2). Our analysis of (126), (127) in section 3.1 gave clear interpretation of the positive Viéte coefficients for matrices homomultiplication. Remember, that algebraic means (and other ones), inferred from the positive Viéte coefficients, form a hierarchical sequence. (As before, we use a bar to denote means.)

Let A be a $r \times n$-matrix A and $rankA = r$. Define its *parametric* and *hierarchical* geometric norms of order t and degree h as

$$||A||_t^h = [\sqrt[2t]{k(A'A,t)}]^h > 0, \tag{414}$$

$$\overline{||A||_t^h} = [\sqrt[2t]{k(A'A,t)/C_r^t}]^h > 0. \tag{415}$$

Formally all these norms are regarded to be zero if $t > r$ and unity if $t = 0$.

Parametric norm (414) with $h = t$ may be consider geometrically as t-dimensional *volume parameter* for the lineor A, and with $t = 1$ as its *length parameter* – see formula (127) in sect. 3.1. Hierarchical norms (415) may be consider as hierarchical medians of order $t = 1, \ldots, r$ and degree h, according to original chain (11) for scalar coefficients of the matrix $B = AA'$ (see the general inequality of means in sect. 1.2). In particular,

$$||A||_r^r = \sqrt{det\ (A'A)} = \mathcal{M}t(r)A = \overline{||A||_r^r}.$$

For quadratic nonsingular and singular matrices B, there hold:

$$||B||_n^n = \sqrt{det\ (B'B)} = \sqrt{det\ (BB')} = |det\ B| = \overline{||B||_n^n},$$

$$||B||_r^r = \sqrt{k(B'B,r)} = \sqrt{k(BB',r)} = \mathcal{M}t(r)B = \overline{||B||_r^r}.$$

By the definition, *any general norms for a matrix* have maximal order t equal to its rank r. If in (414), (415) $h = r$, then the general norm of a matrix is its minorant.

In that number, this definition belongs to general norms for the tensor cosine and the tensor sine (projective and motive). For example, *general quadratic trigonometric norms of degree* $h = 1$ are defined similarly with maximal order, according their ranks:

$$0 \le ||\cos\Phi_{12}||_n^1 = \sqrt[2n]{det\ \cos^2\Phi_{12}} = \sqrt[n]{\prod_{i=1}^{r-\nu'} \cos^2\varphi_i} = \sqrt[n]{(A_1|A_2)|_{\cos}^2} \le 1, \tag{416}$$

$$0 \le ||\sin\Phi_{12}||_{r_1+r_2}^1 = \sqrt[2(r_1+r_2)]{\mathcal{D}_l(r_1+r_2)\sin^2\Phi_{12}} =$$

$$= \sqrt[r_1+r_2]{\prod_{i=1}^{r_1-\nu''} \sin^2\varphi_i} = \sqrt[r_1+r_2]{|(A_1|A_2)|_{\sin}^2} \le 1. \tag{417}$$

These norms characterize binary tensor angles $\tilde{\Phi}_{12}$ and Φ_{12} between the lineors A_1 and A_2 or between the planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ (the planars $\langle ker\ A_1'\rangle$ and $\langle ker\ A_2'\rangle$).

In its turn, the scalar characteristic

$$0 \le ||\cos\Phi_B||_n^1 = \sqrt[n]{|\{B\}|_{\cos}^2} \le 1 \tag{418}$$

is the general trigonometric norm of degree 1 for the cosine of binary tensor angles $\tilde{\Phi}_B$ and Φ_B between the planars $\langle im\ B\rangle$ and $\langle im\ B'\rangle$ (the planars $\langle ker\ B\rangle$ and $\langle ker\ B'\rangle$).

According to the Le Verrier-Waring direct recurrent formula or the Newton system of equations (see in sect. 1.1), there exist only r independent geometric norms of each type. Just norms (414) and (415) completely determine scalar properties of a linear matrix object of rank r by these two set of its geometric invariants. The quadratic geometric norm of degree 1 and order 1 is the Frobenius norm, for example:

$$||A||_1^1 = \sqrt{tr\ (A'A)} = \sqrt{\sum_{k=1}^{n}\sum_{j=1}^{r} a_{jk}^2} = \sqrt{\sum_{i=1}^{r} \sigma_i^2} = ||A||_F > 0, \tag{419}$$

where a_{jk} – elements of A, σ_i^2 – eigenvalues of $A'A$. The Euclidean norm $||\mathbf{a}||_E$ is similar. Note, that a power manner for norms defining (in terms of eigenvalues of $\sqrt{A'A}$) give the Euclidean and Frobenius norms as *degree norms of order* θ with $\theta = 2$:

$$\sqrt[\theta]{S_\theta(\sigma_i)} = \sqrt[\theta]{\sum_{i=1}^{r} \sigma_i^\theta}, \quad \theta = 1, \ldots, r.$$

On the other hand, the two ways of norms defining (see above) are equivalent only for norms of the greatest order, i. e., general ones:

$$||\sqrt{A'A}||_r^1 = \sqrt[2r]{s_r(\sigma_i)} = \sqrt[2r]{det\ (A'A)} = \sqrt[r]{\mathcal{M}t(r)A} = \sqrt[r]{s_r(\sigma_i)} = \sqrt[r]{det\sqrt{A'A}}.$$

(In particular, this holds, if $r = 1$.) In general, way 2 of norms defining demands computing a matrix arithmetic square root through eigenvalues of $A'A$. Way 1 defines norms propositionally in terms of scalar characteristic coefficients of the same internal homomultiplication $A'A$ (i. e., not directly in terms of eigenvalues of $A'A$). This is the essential difference between the two ways and the reason for choosing the first one.

The Frobenius norm of order 1 and degree 1 is the invariant of length. The general norm of order r and degree r (the minorant), is the invariant of r-dimensional volume. The characteristic $||A||_r^1 = \overline{||A||_r^1}$ is the invariant of degree 1 of this volume (the general hierarchical norm). The geometric norms $\overline{||A||_t^1}$ (the small medians) form the hierarchy in order of t values ($1 \leq t \leq r$) corresponding to inequality chain (11) – see sect. 1.1.

The *hierarchical quadratic trigonometric norms of order* $t = 1$ are defined similarly:

$$\overline{||\cos\Phi||_1^1} = \sqrt{\frac{tr\ \cos^2\Phi}{n}}, \qquad \overline{||\sin\Phi||_1^1} = \sqrt{\frac{tr\ \sin^2\Phi}{n}}.$$

Taking into account (182) and (264), we obtain also with $t = 1$ the simplest invariant:

$$\overline{||\cos\Phi||_1^2} + \overline{||\sin\Phi||_1^2} = 1. \tag{420}$$

Quadratic trigonometric norms of the greatest order are defined as (416) and (417). If chain (11) consists of mean invariants of a tensor trigonometric function, then (12) contains mean invariants of the inverse function (with respect to multiplication). The hierarchical invariants of the spherical cosine and sine range in $[0; 1]$, that of the spherical secant and the tangent range in $[1; \infty)$ and $[0; \infty)$.

9.2 Absolute and relative norms

Consider definitions and properties of various geometric norms for matrix objects. Let A be a complex-valued $n \times m$-matrix of rank r. It represents algebraically a certain geometric object such as either as an 1-valent tensor in $\langle \mathcal{A}^n \rangle$, $m < n$, or as a 2-valent tensor in $\langle \mathcal{A}^{n \times n} \rangle$, $m = n$.

For a complex-valued $n \times m$-matrix A of rank r, its *absolute geometric norm of order* t*,* $0 \le t \le r$*, and degree* h is the scalar characteristic $||A||_t^h$ with the following defining conditions:

(a) $||A||_t^h = [||A||_t^1]^h > 0 \ \text{ if } 1 \le t \le r$,

(a') $||A||_0^h = 1 \ \text{ if } t = 0$,

(a") $||A||_t^h = 0 \ \text{ if } t > r$,

(b) $||c \cdot A||_t^h = |c|^h \cdot ||A||_t^h$,

(c) $||U_1 \cdot A \cdot U_2||_t^h = ||A||_t^h$,

(d) $||A^*||_t^h = ||A||_t^h$.

For example, (414)–(419) are *definite* absolute geometric norms. If the symbol " $>$" in defining condition (a) is replaced by " $\ge$ " , then such norms are called *semi-definite* absolute geometric norms of order t and degree h. They are used only for square matrices B representing 2-valent tensors and denoted as $|\{B\}|_t^h$. Their examples are

$$|\{B\}|_t^t = |k(B,t)| \ge 0, \quad |\{B\}|_r^r = |k(B,r)| \ge 0, \quad |\{B\}|_1^1 = |tr\ B| \ge 0. \qquad (421)$$

A *relative norm of order* t *and degree* h is the ratio of a semi-definite absolute norm and definite one. They are always dimensionless and have here *trigonometric nature.* Examples of relative norms of order $t = r$ are the cosine and sine ratios introduced in Ch. 3. These geometric norms are called *general* if $t = r$ and *particular* if $t < r$. General norms were interpreted before. Reveal the geometrical sense of particular ones.

9.3 Geometric interpretation of particular quadratic norms

Consider particular norms, using as clear model, the *particular cosine ratio* (i. e., under condition $t < r$). The general cosine inequalities (396), (398) and the cosine ratios corresponding to these inequalities may be further developed and their quasi-analogs for orders $t < r$ may be inferred.

Let A_1 and A_2 be $n \times r$-lineors. For each j-th subset of t columns, $j = 1, \ldots, C_r^t$, choose the pair of $n \times t$-submatrices $\{A_1\}_j$ and $\{A_2\}_j$ with the same subset of columns.

Write down all the submatrices $\{A_1\}_j$ one under another and do the same with $\{A_2\}_j$. This operation transforms A_1 and A_2 into the pair of *ranged* $nC_r^t \times t$*-lineors* of rank t.

For each pair $\{A_1\}_j$ and $\{A_2\}_j$, the cosine inequalities similar to (396), (398) hold:

$$-1 \leq det\ \{A_1'A_2\}_j / \left(\sqrt{det\ (A_1'A_1)_j} \cdot \sqrt{det\ (A_2'A_2)_j}\right) \leq +1.$$

The numerator of the fraction is the j-th principal minor of order t of $\{A_1'A_2\}$, as the internal multiplication of $\{A_1\}_j$ and $\{A_2\}_j$. Summate separately j numerators and j denominators of these inequalities, we obtain from two sums a united inequality (that is generally a *Rule of summing homogeneous fraction inequalities, i. e., with constant left and right constraints and positive denominators, in a united fraction inequality*):

$$-1 \leq \frac{\sum_{j=1}^{C_r^t} det\ \{A_1'A_2\}_j}{\sum_{j=1}^{C_r^t} \sqrt{det\ \{A_1'A_1\}_j} \cdot \sqrt{det\ \{A_2'A_2\}_j}} \leq +1.$$

Further, apply to the denominator the geometric (cosine) Cauchy Inequality for a paired set of positive numbers, obtain the following intermediate inequality:

$$-1 \leq \frac{\sum_{j=1}^{C_r^t} det\ \{A_1'A_2\}_j}{\sqrt{\sum_{j=1}^{C_r^t} det\ \{A_1'A_1\}_j \cdot \sum_{j=1}^{C_r^t} det\ \{A_2'A_2\}_j}} \leq +1.$$

Using (120) and (121), obtain the *particular quasi-cosine inequalities* in the sign form:

$$-1 \leq \frac{k(A_1'A_2, t)}{\sqrt{k(A_1'A_1, t)} \cdot \sqrt{k(A_2'A_2, t)}} = \frac{k(A_1A_2', t)}{\sqrt{k(A_1A_1', t)} \cdot \sqrt{k(A_2A_2', t)}} \leq +1. \tag{422}$$

The quasi-cosine inequalities in the signless form define the particular relative norms:

$$0 \leq \frac{|\{A_1A_2'\}|_t^1}{||A_1||_t^1 \cdot ||A_2||_t^1} \leq 1. \quad (1 \leq t < r) \tag{423}$$

Trigonometric sense of the quasi-cosine ratio as a norm of order $t < r$ is explained with its inference, it is connected with ranged lineors. If $t = 1$, then

$$-1 \leq \frac{tr\ (A_1'A_2)}{\sqrt{tr\ (A_1'A_1)} \cdot \sqrt{tr\ (A_2'A_2)}} = \frac{tr\ (A_1A_2')}{\sqrt{tr\ (A_1A_1')} \cdot \sqrt{tr\ (A_2A_2')}} \leq +1, \tag{424}$$

$$0 \leq \frac{|\{A_1A_2'\}|_1^1}{||A_1||_1^1 \cdot ||A_2||_1^1} \leq 1. \tag{425}$$

From these inequalities the classical triangle and parallelogram inequalities for the Frobenius norms ($t = 1$) of the original $n \times r$-lineors follow:

$$||A_1 + A_2||_1^1 \leq ||A_1||_1^1 + ||A_2||_1^1. \tag{426}$$

$$\big|\ ||A_1||_1^1 - ||A_2||_1^1\ \big| \leq ||A_1 \pm A_2||_1^1 \leq ||A_1||_1^1 + ||A_2||_1^1. \tag{427}$$

These particular inequalities are of linear nature. They define the Frobenius norm of lineors as an invariant of extent (or length for vectors). However, particular inequalities (422), (424) and (426), (427) characterize the lineors A_1 and A_2 if $r > 1$ not directly, but in terms of ranged $nC_r^t \times t$-lineors $\{A_1\}$ and $\{A_2\}$. For illustrations, get Frobenius norms: they describe these lineors in terms of ranged $nr \times 1$-vectors $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$||A_1||_1^1 = ||\mathbf{a}_1||_E, \quad ||A_2||_1^1 = ||\mathbf{a}_2||_E, \quad ||A_1 \pm A_2||_1^1 = ||\mathbf{a}_1 \pm \mathbf{a}_2||_E;$$

$$tr\ (A_1' \cdot A_2) = tr\ (A_1 \cdot A_2') = \mathbf{a_1'a_2}.$$

Consequently, the *Pythagorean Theorem for the Frobenius norms of the lineors* A_1 *and* A_2 *holds iff ranged vectors* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *are orthogonal*:

$$\mathbf{a_1'a_2} = 0 = tr\ (A_1'A_2) \ \leftrightarrow\ ||A_1 \pm A_2||_1^2 = ||A_1||_1^2 + ||A_2||_1^2. \qquad (428)$$

Similarly, from the trigonometric point of view, particular quasi-cosine ratios (423) and (425) as relative norms characterize also tensor angles $\tilde{\Phi}_{12}$ and Φ_{12} between the lineors A_1 and A_2 not directly, but only in terms of ranged lineors $\{A_1\}$ and $\{A_2\}$.

9.4 Lineors of special kinds and simplest figures formed by lineors

In the *lineor Euclidean space* $\langle\mathcal{E}^n\rangle$, according to (130) in sect. 3.1, an $n \times r$-lineor $(of\ rank\ A = r)$ may be represented in the *unambiguous quasi-polar decomposition*:

$$A = \{A \cdot (\sqrt{A'A})^{-1}\} \cdot \sqrt{A'A} = Rq \cdot |A|,$$

where $|A| = \sqrt{A'A}$ is the $r \times r$-*matrix module* of the original $n \times r$-lineor A, and $Rq = \{A \cdot (\sqrt{A'A})^{-1}\}$ is its *own quasi-orthogonal lineor*. This decomposition is similar to one for a vector: $\mathbf{a} = \mathbf{e} \cdot |\mathbf{a}|$, where $|\mathbf{a}| = \sqrt{\mathbf{a'a}} = ||\mathbf{a}||_E$. The $r \times r$-matrix module of the lineor is similar to the scalar module of a vector, but with respect to the set of r basis unity vectors $\{\mathbf{e}_i\} = Rq$ in $\langle\mathcal{E}^n\rangle$. These vectors determine independent directional axes in $\langle\mathcal{E}^n\rangle$ of the given $n \times r$-lineor A. Consequently, there hold

$$\overleftarrow{AA'} = \overleftarrow{Rq \cdot Rq'} = Rq \cdot Rq', \quad Rq' \cdot Rq = Rq^+ \cdot Rq = I_{r\times r}, \quad (Rq' = Rq^+).$$

Each lineor formally belongs to its basis planar: $A \in \langle im\ A\rangle$ (as $\mathbf{a} \in \langle im\ \mathbf{a}\rangle$). The condition (154) determines the set of *coplanar lineors* with respect to the basis planar $\langle im\ A_1\rangle$ (for the vector $\mathbf{a}_2$ this condition is $\mathbf{a}_2 \in \langle im\ A_1\rangle$).

Equirank lineors ($r_A = r =$ const) with the same basis planar $\langle im\ A\rangle$ form the complete set of *colplanar lineors* with respect to the basis planar $\langle im\ A\rangle$. If $r = 1$, they are *collinear vectors*. Two equirank lineors are colplanar iff they satisfy (153). The complete set of colplanar $n \times r$-lineors $\langle AC\rangle$ with respect to the basis planar $\langle im\ A\rangle$ is parametrically determined with a free nonsingular $r \times r$-matrix C by relation

$$\overleftarrow{AA'} = \overleftarrow{(AC)(AC)'} = \text{Const}. \qquad (429)$$

Colplanar lineors A_k are defined by the following invariant relations:

$$\overleftarrow{A_k A_k'} = \overleftarrow{Rq_k Rq_k'} = Rq_k \cdot Rq_k' = \overleftarrow{AA'} = \mathrm{Const}_{n\times n}, Rq_k' \cdot Rq_k = I_{r\times r} = \mathrm{Const}_{r\times r}. \quad (430)$$

Further, in the set of colplanar lineors $\langle A\rangle$, separate the subset of *coaxial lineors*. They are defined stronger with additional condition $Rq_k = Rq = \mathrm{Const} = \{\mathbf{e}_i\}$. Such lineors differ only by their matrix moduli $|A_k|$. If A_1, A_2 are coaxial lineors, then

$$|A_1 \pm A_2|^2 = \Big|\, |A_1| \pm |A_2| \,\Big|^2 = (|A_1| \pm |A_2|)^2, \;\; A_1' A_2 = |A_1| \cdot |A_2|, \;\; A_2' \cdot A_1 = |A_2| \cdot |A_1|.$$

Let A_1 and A_2 be equirank lineors, may be linearly entirely independent or not, but under the same conditions (224) and (230), and lying in their own basis planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$. Then the oblique projector $\overleftarrow{A_1 A_2'}$ exists. Using formulae (186) and (187) from sect. 5.2, and (226) from sect. 5.4, we obtain:

$$\overleftarrow{A_1 A_2'} = \overleftarrow{Rq_1 Rq_2'} = Rq_1\ Rq_1' \cdot \sec \tilde{\Phi}_{12} = \sec \tilde{\Phi}_{12} \cdot Rq_2\ Rq_2'. \quad (431)$$

Expressions (430) and (431) may be useful in *QR-factorizations* of lineors with similar conditions. They can be illustrated easily and visually on the simplest unit lineors $\mathbf{e}_1$ and $\mathbf{e}_2$, as we have done earlier too.

In conclusion, define also *rotationally congruent lineors*:

$$A_2 = Rot\ \Phi_{12} \cdot A_1 \;\Rightarrow\; \{Rq_2 = Rot\ \Phi_{12} \cdot Rq_1,\ |A_1| = |A_2| = |A|\}. \quad (432)$$

Such lineors differ only by their quasi-orthogonal lineors Rq_k.

For these lineors A_1 and A_2 we have these symmetric matrix module expressions:

$$\left.\begin{array}{c} |A_1 \pm A_2|^2 = 4 \cdot |A|^2 \cdot [(I \pm \cos\Phi_{12})/2], \\ |A_1 + A_2|^2 = 4 \cdot |A|^2 \cdot \sin^2(\Phi_{12}/2), \\ |A_1 - A_2|^2 = 4 \cdot |A|^2 \cdot \cos^2(\Phi_{12}/2). \end{array}\right\} \quad (433)$$

With the use of parallel translations, rotationally congruent lineors A_1 and A_2 form a $2r$-dimensional rhombus. In particular, centered equimodule vectors are rotationally congruent. If $\Phi_{12} = \pi/2$, then these lineors form the following $2r$-dimensional square:

$$|A_1 + A_2| = \sqrt{2} \cdot |A|.$$

One may construct from such lineors corresponding triangles, parallelograms and so on. Thus lineors, as well as vectors, can form, but more complex, geometric figures with various geometric properties. Euclidean and quasi-Euclidean spaces of lineors (*lineor spaces*) have, as well as vector spaces, valency 1.

Chapter 10
Complexification of tensor trigonometry

10.1 Adequate complexification

Complex-valued projective and motive *spherical* angles are expressed adequately in terms of real-valued spherical and hyperbolic tensor angles in the following forms

$$\tilde{\Psi} = \tilde{\Phi} + i\tilde{\Gamma},\ (\tilde{\Psi}' = \tilde{\Phi} - i\tilde{\Gamma});\quad \Psi = \Phi + i\Gamma,\ (\Psi' = -\Phi + i\Gamma);\quad \psi_j = \varphi_j + i\gamma_j,\ (434)$$

where $\tilde{\Phi}' = \tilde{\Phi}$, $\tilde{\Gamma}' = -\tilde{\Gamma}$; $\Phi' = -\Phi$, $\Gamma' = \Gamma$ (see the angles in Chs. 5 and 6).

In a *complex n-dimensional Euclidean space*, tensor trigonometry is realized with the use of adequate complexification (sect. 4.2). Complex tensor angles have their transposed forms indicated above. All geometric notions and formulae except norms and inequalities stay valid and do not change. In particular, complex minorants and complex matrix modules are evaluated with the use of transposing.

Complex numbers $+c$ and $-c$ have the analogous adequate complex module, it is evaluated also in terms of c^2 by Moivre formula:

$$\pm c = \pm\rho(\cos\alpha + i\sin\alpha),\quad 0 \le \alpha < \pi,$$
$$(\pm c)^2 = c^2 = \rho^2(\cos 2\alpha + i\sin 2\alpha) = \rho^2(\cos\beta + i\sin\beta),\ 0 \le \beta < 2\pi,$$
$$|\pm c| = |c| = \rho(\cos(\beta/2) + i\sin(\beta/2)) = \rho(\cos\alpha + i\sin\alpha). \qquad (435)$$

It is seen that $|c^2| = c^2$.

The adequate matrix Euclidean module $|A| = \sqrt{A'A}$ of a matrix A (sect. 9.4) is evaluated with intermediate diagonalization of its interior multiplication and complex orthogonal modal transformation:

$$R' \cdot A'A \cdot R = D\{A'A\} = \{\sigma_j^2\},\quad \sigma_j^2 = \rho_j^2(cos\beta_j + i\sin\beta_j) = |\sigma_j|^2,\quad 0 \le \beta_j < 2\pi.$$

From this, by Moivre formula, we obtain

$$|\sigma_j| = \rho_j[\cos(\beta_j/2) + i\sin(\beta_j/2)],\qquad |A| = R\cdot\{|\sigma_j|\}\cdot R',\qquad |A|^2 = A'A.$$

In the adequate complexification variant, all geometric characteristics, in particular angles and their trigonometric functions, are decomposed into real and imaginary parts, though each whole characteristic may be represented in the most suitable form. The adequate variant in its simplest form is used in complex-valued Euclidean plane geometry, in particular, in scalar complex Euclidean trigonometry. In general case, complex squared identity (142), in that number in variant of the sine-cosine Lagrange Identity for two vectors, does not change. The scalar sine and cosine ratios in (124) and (141) may be used for evaluating of the complex angles between two vectors and their trigonometric functions. The general scalar ratios (135) and (140) have also their adequate complex-valued forms.

10.2 Hermitean complexification

In the *Hermitean space*, Hermitean complexification of real-valued Euclidean geometry with tensor trigonometry is used (sect. 4.3). A projective *spherical* tensor angle is an Hermitean matrix $\tilde{\mathcal{H}} = \tilde{\Phi} + i\tilde{\Gamma} = \tilde{\mathcal{H}}^*$, where $\tilde{\Phi}^* = \tilde{\Phi}$, $\tilde{\Gamma}^* = -\tilde{\Gamma}$. Its eigenvalues are real spherical angles $\pm\eta_j$ and zero. A motive *spherical* tensor angle is a skew-Hermitean matrix $\mathcal{K} = \Phi + i\Gamma = -\mathcal{K}^*$, where $\Phi^* = -\Phi$, $\Gamma^* = \Gamma$ (Chs. 5, 6). Its eigenvalues are imaginary pseudohyperbolic angles $\pm i\eta_j$ and zero. Hermitean modules of linear objects are positive definite. Normalized general inequalities (Ch. 8), geometric and trigonometric norms (Ch. 9) preserve their real positive forms in Hermitean variants.

The principle of binarity also stays valid in complex adequate and Hermitean variants of tensor trigonometry, as all its preliminaries do hold.

Hermitean analogs of cell formulae (399) and (400) in sect. 8.3 are inferred with analogous complex-valued unity vectors. Here the two sides of the tensor angle $\tilde{\mathcal{H}}_{12}$ between planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ of rank r are represented at the level of elementary trigonometric 2×2-cells as unity eigenvectors:

$$\mathbf{u}_1 = \begin{vmatrix} \cos\alpha_1 \\ \sin\alpha_1 \end{vmatrix}, \qquad \mathbf{u}_2 = \begin{vmatrix} \cos\alpha_2 \\ \sin\alpha_2 \end{vmatrix},$$

where

$$\cos\alpha \cdot \overline{\cos\alpha} + \sin\alpha \cdot \overline{\sin\alpha} = 1,$$

$$\cos\alpha = \cos\eta \cdot \exp i\beta_c, \qquad \sin\alpha = \sin\eta \cdot \exp i\beta_s,$$

$$\cos\alpha \cdot \overline{\cos\alpha} = \cos^2\eta, \qquad \sin\alpha \cdot \overline{\sin\alpha} = \sin^2\eta;$$

$$[\cos\tilde{\mathcal{H}}_{12}]^{2\times 2} = \overleftarrow{\mathbf{u_1} \cdot \mathbf{u_1^*}} + \overleftarrow{\mathbf{u_2} \cdot \mathbf{u_2^*}} - I^{2\times 2} = \mathbf{u_1} \cdot \mathbf{u_1^*} + \mathbf{u_2} \cdot \mathbf{u_2^*} - I^{2\times 2} =$$

$$= \left[\begin{array}{l|l} \cos\alpha_1 \cdot \overline{\cos\alpha_1} + \cos\alpha_2 \cdot \overline{\cos\alpha_2} - 1 & \cos\alpha_1 \cdot \overline{\sin\alpha_1} + \cos\alpha_2 \cdot \overline{\sin\alpha_2} \\ \overline{\cos\alpha_1} \cdot \sin\alpha_1 + \overline{\cos\alpha_2} \cdot \sin\alpha_2 & \overline{\sin\alpha_1} \cdot \sin\alpha_1 + \overline{\sin\alpha_2} \cdot \sin\alpha_2 - 1 \end{array}\right] =$$

$$= \begin{bmatrix} +|c_1| & \overline{s_1} \\ s_1 & -|c_1| \end{bmatrix};$$

$$-det\ [\cos\tilde{\mathcal{H}}_{12}]^{2\times 2} = |c_1|^2 + s_1 \cdot \overline{s_1} = \cos^2(\eta_2 - \eta_1) - \Delta = \cos^2\eta_{12},$$

$$\Delta = (1/2) \cdot \sin(2\eta_1) \cdot \sin(2\eta_2) \cdot [1 - \cos(\beta_{c_1})\cos(\beta_{c_2})\cos(\beta_{s_1})\cos(\beta_{s_2}];$$

$$[\sin\tilde{\mathcal{H}}_{12}]^{2\times 2} = \overleftarrow{\mathbf{u_2} \cdot \mathbf{u_2^*}} - \overleftarrow{\mathbf{u_1} \cdot \mathbf{u_1^*}} = \mathbf{u_2} \cdot \mathbf{u_2^*} - \mathbf{u_1} \cdot \mathbf{u_1^*} =$$

$$= \left[\begin{array}{l|l} \cos\alpha_2 \cdot \overline{\cos\alpha_2} - \cos\alpha_1 \cdot \overline{\cos\alpha_1} & \cos\alpha_2 \cdot \overline{\sin\alpha_2} - \cos\alpha_1 \cdot \overline{\sin\alpha_1} \\ \overline{\cos\alpha_2} \cdot \sin\alpha_2 - \overline{\cos\alpha_1} \cdot \sin\alpha_1 & \overline{\sin\alpha_2} \cdot \sin\alpha_2 - \overline{\sin\alpha_1} \cdot \sin\alpha_1 \end{array}\right] = \begin{bmatrix} -|s_2| & \overline{c_2} \\ c_2 & +|s_2| \end{bmatrix};$$

$$-det\ [\sin\tilde{\mathcal{H}}_{12}]^{2\times 2} = |s_2|^2 + c_2 \cdot \overline{c_2} = \sin^2(\eta_2 - \eta_1) + \Delta = \sin^2\eta_{12}.$$

For the residue Δ we have $\Delta = 0 \ \Leftrightarrow\ \cos(\beta_{c_1})\cos(\beta_{c_2})\cos(\beta_{s_1})\cos(\beta_{s_2}) = 1 = |\cos\beta_k|$;

$$\Delta = 0 \ \Leftrightarrow\ \eta_{12} = \eta_2 - \eta_1, \quad \Delta \neq 0 \ \Leftrightarrow\ \eta_{12} \neq \eta_2 - \eta_1. \tag{436}$$

The cell forms with respect to the *trigonometric base* (see sect. 5.5) are

$$[\cos\tilde{\mathcal{H}}_{12}]^{2\times 2} = \cos\eta_{12}\cdot\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}, \quad [\sin\tilde{\mathcal{H}}_{12}]^{2\times 2} = \sin\eta_{12}\cdot\begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix}. \quad (437, 438)$$

In the Hermitean variant, all canonical W-forms of tensor trigonometric functions are real-valued and do not change. They are constructed with complex unitary modal matrices U_W. In an Hermitean plane and with respect to the trigonometric base (of the diagonal cosine), Hermitean shift of paired functions (cosine-sine, secant-tangent) at a phase angle β may take place – see respectively (179) and (259):

$$Exp\ (-i\beta/2)\cdot Ref\ \{B^*B\}_r\cdot Exp\ (+i\beta/2) = Ref\ \{B^*B\}_c, \quad (439)$$

$$Exp\ (-i\beta/2)\cdot Rot\ \{\mathcal{H}\}_r\cdot Exp\ (+i\beta/2) = Rot\ \{\varepsilon\}_c, \quad (440)$$

i. e.,

$$\begin{bmatrix} \ddots & & & \\ & \exp\left(\frac{-i\beta}{2}\right) & 0 & \\ & 0 & \exp\left(\frac{+i\beta}{2}\right) & \\ & & & \ddots \end{bmatrix}\cdot\begin{bmatrix} \ddots & & & \\ & +\cos\eta & \sin\eta & \\ & \sin\eta & -\cos\eta & \\ & & & \ddots \end{bmatrix}\cdot\begin{bmatrix} \ddots & & & \\ & \exp\left(\frac{+i\beta}{2}\right) & 0 & \\ & 0 & \exp\left(\frac{-i\beta}{2}\right) & \\ & & & \ddots \end{bmatrix} =$$

$$= \begin{bmatrix} \ddots & & & \\ & +\cos\eta & \sin\eta\cdot\exp(-i\beta) & \\ & \sin\eta\cdot\exp(+i\beta) & -\cos\eta & \\ & & & \ddots \end{bmatrix},$$

$$\begin{bmatrix} \ddots & & & \\ & \exp\left(\frac{-i\beta}{2}\right) & 0 & \\ & 0 & \exp\left(\frac{+i\beta}{2}\right) & \\ & & & \ddots \end{bmatrix}\cdot\begin{bmatrix} \ddots & & & \\ & \cos\eta & -\sin\eta & \\ & +\sin\eta & \cos\eta & \\ & & & \ddots \end{bmatrix}\cdot\begin{bmatrix} \ddots & & & \\ & \exp\left(\frac{+i\beta}{2}\right) & 0 & \\ & 0 & \exp\left(\frac{-i\beta}{2}\right) & \\ & & & \ddots \end{bmatrix} =$$

$$= \begin{bmatrix} \ddots & & & \\ & \cos\eta & -\sin\eta\cdot\exp(-i\beta) & \\ & +\sin\eta\cdot\exp(+i\beta) & \cos\eta & \\ & & & \ddots \end{bmatrix}.$$

That is why the Hermitean trigonometric base should lead to the diagonal cosine as before and also to real-valued W-forms. In each eigen Hermitean plane (at the level of each 2×2-cells), Hermitean shift at a phase angle β may be eliminated with the special unitary rotational modal transformation $Exp\ (i\beta/2)$, and as final result with reducing in real-valued canonical forms of tensor trigonometric functions.

Hermitean analogs of Cauchy and Hadamard Inequalities (of cosine and sine types) with complex trigonometric identity (142) for coordinates of 2 vectors or in general of 2 lineors are inferred with the use of Hermitean transposing in their internal products. Hermitean spherical angle is a composite function of the linear objects coordinates. But in its trigonometric base, the tensor angle have the real-valued canonical form.

10.3 Pseudoization in binary complex spaces

Consider pseudoization as the important special case of adequate complexification of real-valued algebraic and geometric notions (see Ch. 4). Fix a binary complex affine space $\langle \mathcal{A}^{n+q} \rangle_c$ of index q. In any admissible binary affine base, this space may be considered as linear one. In particular, with respect to a certain affine pseudounity base $\tilde{E}_0$, the space $\langle \mathcal{A}^{n+q} \rangle_c$ is the direct sum of the real and imaginary affine subspaces:

$$\langle \mathcal{A}^{n+q} \rangle_c \equiv \langle \mathcal{A}^n \rangle \oplus \langle i\mathcal{A}^q \rangle \equiv \text{CONST}. \tag{441}$$

Here the sum space and dimensions of summand subspaces are constant. In $\langle \mathcal{A}^{n+q} \rangle$, we admit linear transformations V preserving the binary structure:

$$\overset{V}{\begin{bmatrix} V_{11} & iV_{12} \\ iV_{21} & V_{22} \end{bmatrix}} \cdot \overset{\tilde{E}_0}{\begin{bmatrix} I^{n\times n} & Z^{n\times q} \\ Z^{q\times n} & \pm iI^{q\times q} \end{bmatrix}} = \overset{\tilde{E}}{\begin{bmatrix} V_{11} & \pm V_{12} \\ iV_{21} & \pm iV_{22} \end{bmatrix}}, \; det\ V_{jk} \neq 0. \tag{442}$$

First n columns of the base matrices generate $\langle \mathcal{A}^n \rangle$, other q columns generate $\langle i\mathcal{A}^q \rangle$. The modal matrix V^{-1} has the same structure, this matrix transforms an arbitrary binary base $\tilde{E}$ into the simplest, i. e., diagonal (pseudounity) base $\tilde{E}_0$ and performs a *passive* modal transformation of a linear element: $\mathbf{z}\{\tilde{E}\} = V \cdot \mathbf{z}\{\tilde{E}_0\}$.

The *binary local complex trigonometric bases* are expressed in the *left and right mutual forms* connected with the local real-valued trigonometric base $\tilde{E}_1 = \{I\}$ by pseudounity modal matrices:

$$\tilde{E}_{01} = \begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & +i & \\ & & & \ddots \end{bmatrix} \cdot \tilde{E}_1 = (\sqrt{I^{\pm}})_D \cdot \{I\} = R_{c1} \cdot \{I\} = \{R_{c1}\}, \tag{443}$$

$$\tilde{E}_{02} = \begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & -i & \\ & & & \ddots \end{bmatrix} \cdot \tilde{E}_1 = (\sqrt{I^{\pm}})_D^{-1} \cdot \{I\} = R_{c2} \cdot \{I\} = \{R_{c2}\}. \tag{444}$$

With respect to an admissible binary complex base $\tilde{E}$, a linear element and the whole space are direct sums of their real and imaginary affine projections:

$$\mathbf{z} = \mathbf{x} \oplus i\mathbf{y} = \begin{bmatrix} \mathbf{x} \\ i\mathbf{y} \end{bmatrix}. \tag{445}$$

The space $\langle \mathcal{A}^{n+q} \rangle_c$ is affine, and hence the translations in it at linear elements (445) are admissible, and hence the space is homogeneous.

Right local base (443) is identical to one in (271) and used in canonical forms of *pseudohyperbolic* trigonometric matrices with angle eigenvalues $-i\varphi_j = \varphi_j/(+i)$ (see sect. 5.9). The sign "minus" at angles is due to the multiplier $+i$ at ordinates.

Left local base (444) represents canonical forms of trigonometric matrices in the *pseudospherical* variant of tensor trigonometry with angle eigenvalues $+i\gamma_j = \gamma_j/(-i)$. This base is identical to inverse (271), i. e., with the multiplier $-i$ at ordinates.

The modal transformation translates into base $\tilde{E}_{01}$ (443) similarly (322):

$$\overset{(\sqrt{I^{\pm}})^{-1}}{\begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & -i & \\ & & & \ddots \end{bmatrix}} \cdot \begin{bmatrix} \ddots & & & \\ & \cos\varphi_j & -\sin\varphi_j & \\ & +\sin\varphi_j & \cos\varphi_j & \\ & & & \ddots \end{bmatrix} \cdot \overset{\sqrt{I^{\pm}}}{\begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & +i & \\ & & & \ddots \end{bmatrix}} =$$

$$= \begin{bmatrix} \ddots & & & \\ & \cos\varphi_j & -i\sin\varphi_j & \\ & -i\sin\varphi_j & \cos\varphi_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & \cosh(-i\varphi_j) & \sinh(-i\varphi_j) & \\ & \sinh(-i\varphi_j) & \cosh(-i\varphi_j) & \\ & & & \ddots \end{bmatrix}.$$

And the modal transformation translates into base $\tilde{E}_{02}$ (444) similarly to (323):

$$\overset{\sqrt{I^{\pm}}}{\begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & +i & \\ & & & \ddots \end{bmatrix}} \cdot \begin{bmatrix} \ddots & & & \\ & \cosh\gamma_j & \sinh\gamma_j & \\ & \sinh\gamma_j & \cosh\gamma_j & \\ & & & \ddots \end{bmatrix} \cdot \overset{(\sqrt{I^{\pm}})^{-1}}{\begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & -i & \\ & & & \ddots \end{bmatrix}} =$$

$$= \begin{bmatrix} \ddots & & & \\ & \cosh\gamma_j & -i\sinh\gamma_j & \\ & +i\sinh\gamma_j & \cosh\gamma_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & \cos(i\gamma_j) & -\sin(i\gamma_j) & \\ & +\sin(i\gamma_j) & \cos(i\gamma_j) & \\ & & & \ddots \end{bmatrix}.$$

Accordingly, in $\tilde{E}_{01}$ and $\tilde{E}_{02}$ of $\langle \mathcal{Q}^{n+q} \rangle_c$, we have the mixed pseudoized angles in two forms: $\gamma_k \boxplus (-i\varphi_j)$ and $\varphi_j \boxplus i\gamma_k$ (at the counter-clockwise angle φ) – see sect. 6.1 too.

Express coordinates of linear elements (445) in $\langle \mathcal{A}^{n+q} \rangle_c$ with respect to base (444). Define in $\tilde{E}_{02}$ the scalar product for elements $\mathbf{z}$ (445) in $\langle \mathcal{Q}^{n+q} \rangle_c$ as in Euclidean space:

$$\mathbf{z_1'}\{I^+\}\mathbf{z_2} = \mathbf{z_1'}\mathbf{z_2} = \mathbf{x_1'} \cdot \mathbf{x_2} + \mathbf{iy_1'} \cdot \mathbf{iy_2} = \mathbf{x_1'}\mathbf{x_2} - \mathbf{y_1'}\mathbf{y_2}.$$

This is valid in the special *complex quasi-Euclidean space* with index q, its metric is as if *Euclidean-like*; but it is either real-valued or zero or imaginary-valued. First this construction (with $n = q = 1$) was made by H. Poincaré in his *group variant* of the relativistic theory [47]. The space is binary, it is the direct spherically orthogonal sum of the real-valued Euclidean subspace and the imaginary-valued *anti-Euclidean* one:

$$\langle \mathcal{Q}^{n+q} \rangle_c \equiv \langle \mathcal{E}^n \rangle \boxplus \langle i\mathcal{E}^q \rangle \equiv \text{CONST} \Leftrightarrow \langle \mathcal{P}^{n+q} \rangle \equiv \langle \mathcal{E}^n \rangle \boxtimes \langle \mathcal{E}^q \rangle \equiv \text{CONST}. \quad (446)$$

Here $\boxplus$ and $\boxtimes$ stand for direct *spherically and hyperbolic orthogonal-like summation.* Admissible transformations in these space are determined by the reflector-tensor $\{I^{\pm}\}$.

Chapter 11

Tensor trigonometry of general pseudo-Euclidean spaces

11.1 Realification of complex quasi-Euclidean spaces

Return to *binary complex quasi-Euclidean space* (446). It is defined by the *unity metric tensor* $\{I^+\}$ and the reflector tensor $\{I^\pm\}$; its trigonometric base is $\tilde{E}_{02}$. Further, apply to the complex-valued space $\langle \mathcal{Q}^{n+q}\rangle_c$ realifying here modal transformation (443), i. e., $R_c = (\sqrt{I^\pm})_D$ (the transformation has contrary to (271) sense):

$$\tilde{E}_{02} = \{(\sqrt{I^\pm})_D^{-1}\} \;\rightarrow\; (\sqrt{I^\pm})_D \cdot \tilde{E}_{02} = \tilde{E}_1 = \{I\}, \;\; (\sqrt{I^\pm})_D^{-1} \cdot \mathbf{z}_{02} = \mathbf{u}. \tag{447}$$

The modal matrix is not admissible as $\sqrt{I^\pm}' \cdot \sqrt{I^\pm} \neq \{I^+\}$; it transfers into a *realificated pseudo-Euclidean* space $\langle \mathcal{P}^{n+q}\rangle$ of index q with the *metric and reflector tensor* $\{I^\pm\}$. Its quadratic metric is *pseudo-Euclidean.* The scalar product for the same element is

$$\mathbf{z}'_{02} \cdot \mathbf{z}_{02} = [(\sqrt{I^\pm})_D \cdot \mathbf{u}]' \cdot [(\sqrt{I^\pm})_D \cdot \mathbf{u}] = \mathbf{u}' \cdot \{I^\pm\} \cdot \mathbf{u} = \text{const}. \tag{448}$$

So, the spaces $\langle \mathcal{Q}^{n+q}\rangle_c$ and $\langle \mathcal{P}^{n+q}\rangle$ are isometric and expressed only in different forms! Now the same element is expressed in the base $\tilde{E}_1$, and it is denoted as $\mathbf{u}$. The new metric tensor $\{I^\pm\}$ in this *coaxially oriented* space $\langle \mathcal{P}^{n+q}\rangle$ is also its *reflector tensor*! Realification $\langle \mathcal{Q}^{n+q}\rangle_c \rightarrow \langle \mathcal{P}^{n+q}\rangle_r$ with introducing the metric tensor $\{I^\pm\}$ at $q = 1$ was suggested by H. Minkowski in 1909 [49], at the beginning into space-time $\langle x, ct\rangle$.

Further, realize next and also isometric modal transformation in the similar binary space, but with an affine base $\tilde{E}$, connected with $\tilde{E}_1 = \{I\}$ by the constant binary real-valued modal matrix V. Matrix V is not compatible again with the former metric tensor $\{I^\pm\}$. We have the sequential transformations of the original base and element

$$\tilde{E} = V \cdot \tilde{E}_1, \quad \tilde{E}_1 = (\sqrt{I^\pm})_D \cdot \tilde{E}_{02}, \tag{449}$$

$$\mathbf{w} = V^{-1} \cdot (\sqrt{I^\pm})_D^{-1} \cdot \mathbf{z}_{02} = [(\sqrt{I^\pm})_D \cdot V]^{-1} \cdot \mathbf{z}_{02}. \tag{450}$$

Now the original element $\mathbf{z}_{02}$ is expressed in the affine base $\tilde{E}$, it is denoted as $\mathbf{w}$. The inverse modal matrices of the passive transformations are written *in direct order* for sequential ones. The scalar product of this element, as its immanent characteristic at isometric space transformations, does not change with respect to the new and now *affine base* $\tilde{E}$, the metric reflector tensor (with the same reflector tensor) $\{I^\pm\}$ does change into the new certain symmetric metric reflector tensor:

$$\mathbf{z}'_{02} \cdot \mathbf{z}_{02} = [(\sqrt{I^\pm})_D \cdot V \cdot \mathbf{w}]' \cdot [\sqrt{I^\pm})_D \cdot V \cdot \mathbf{w}] = \mathbf{w}' \cdot \{V' \cdot I^\pm \cdot V\} \cdot \mathbf{w} = \text{const}. \tag{451}$$

What is important, in fact, the binary *basis space* $\langle \mathcal{P}^{n+q}\rangle)$ is preserved again, because we introduced in it only other (affine) base with one-valued linear transformation V.

Let $V \neq$ Const and respectively to its changing the metric reflector tensor does change too, because it is subjected to the permanent general congruent transformation

$$\{G^{\pm}\} = \{V' \cdot I^{\pm} \cdot V\} = \{V' \cdot I^{\pm} \cdot V\}' = \{G^{\pm}\}', \quad (V \leftrightarrow \{G^{\pm}\}). \tag{452}$$

Then the new metric tensor operates in *Special curvilinear coordinates* in the binary space with *Riemannian local metric* due to function $\{G^{\pm}\}(\mathbf{w})$. Its *mutual tensor* is

$$\{\hat{G^{\pm}}\} = \{G^{\pm}\}^{-1} = \{V^{-1} \cdot I^{\pm} \cdot V'^{-1}\}. \tag{453}$$

The binary space with this variable local metric and with zero Riemannian–Christoffelian curvature is isometric and topologically equivalent to $\langle \mathcal{P}^{n+q} \rangle$, where latter is the basis space by the definition. Curvilinear and pseudo-Cartesian coordinates act in such flat space. However, if the curvature is non-zero, we have the pseudo-Riemannian space. These binary spaces will be used in Chapter 9A. The geometry, if $V =$ Const, may be considered as linear mapping of a pseudo-Euclidean one in admissible affine bases:

$$\langle \tilde{E}_{af} \rangle \equiv \langle T_{af} \rangle \cdot \tilde{E} \tag{454}$$

with the constant metric reflector tensor $G^{\pm}$. There holds

$$T'_{af} \cdot \{V' \cdot I^{\pm} \cdot V\} \cdot T_{af} = T'_{af} \cdot \{G^{\pm}\} \cdot T_{af} = \{G^{\pm}\}. \tag{455}$$

Equalities $det\ T_{af} = \pm 1$ follow from (455). We define the group of affine *continuous* trigonometric transformations $\langle T_{af} \rangle$ with respect to $G^{\pm}$ by more exact conditions:

$$T'_{af} \cdot \{G^{\pm}\} \cdot T_{af} = \{G^{\pm}\} = \text{Const}, \quad det\ T_{af} = +1. \tag{456}$$

Due to (448), the metric tensor $\{I^{\pm}\}$ is identical to its *mutual* analog. This condition, generally, is $G = G' = G^{-1} \rightarrow \{G^{\pm}\} = \{\sqrt{I}\}_S$. Hence, in any metric space $\langle \mathcal{P}^{n+q} \rangle$ and only in them, contravariant and covariant coordinates are identical, in particular, if $q = 0$ or $n = 0$. That is why pseudo-Cartesian bases are uniquely applicable in $\langle \mathcal{P}^{n+q} \rangle$. The metric reflector tensor $\{\sqrt{I}\}_S$ is the general variant of ones in a pseudo-Euclidean space, its metric has no linear distortions.

11.2 The general Lorentzian group of pseudo-Euclidean rotations

In (452) put $V = R$, this spherically orthogonal transformation is not compatible with the simplest metric reflector tensor $\{I^{\pm}\}$ too (sect. 6.3). Then we obtain the following metric reflector tensor in the general form, what is identical to its mutual analog:

$$\{R' \cdot I^{\pm} \cdot R\} = \{\sqrt{I}\}_S = \{\sqrt{I}\}'_S = \{\sqrt{I}\}^{-1}_S. \tag{457}$$

Here $\{\sqrt{I}\}_S$ is a symmetric certain square root of I. Formula (457) describes a metric reflector tensor of the *non-coaxially oriented* pseudo-Euclidean space $\langle \mathcal{P}^{n+q} \rangle$ as well as a reflector tensor of the similar quasi-Euclidean space (see in sect. 6.3).

According to (253), (330) and (449), the orthogonal matrix R in (457) is contrary it its sense to the modal matrix R_W, i. e., here may put $R' = R_W$. Then the *group* $\langle T \rangle$ *of rotational trigonometric transformations* in $\langle \mathcal{P}^{n+q} \rangle$ is determined by conditions (456) with the new metric reflector tensor $\{\sqrt{I}\}_S$ as follows:

$$T' \cdot \{\sqrt{I}\}_S \cdot T = \{\sqrt{I}\}_S = T \cdot \{\sqrt{I}\}_S \cdot T' = \text{Const}, \quad det\ T = +1. \tag{458}$$

In $\langle \mathcal{P}^{n+q} \rangle$, admissible transformations may be defined in terms of internal or external products, what is equivalent to the identity of contravariant and covariant coordinates:

$$\left.\begin{array}{c} T' \cdot \{\sqrt{I}\}_S \cdot T = \{\sqrt{I}\}_S \ \leftrightarrow T' \cdot \{\sqrt{I}\}_S \cdot T \cdot \{\sqrt{I}\}_S = I \ \leftrightarrow \\ \leftrightarrow T' \cdot \{\sqrt{I}\}_S \cdot T \cdot \{\sqrt{I}\}_S \cdot T' = T' \ \leftrightarrow \\ \leftrightarrow \{\sqrt{I}\}_S \cdot T \cdot \{\sqrt{I}\}_S \cdot T' = I \ \leftrightarrow T \cdot \{\sqrt{I}\}_S \cdot T' = \{\sqrt{I}\}_S . \end{array}\right\} \tag{459}$$

The relation $T' \cdot \{\sqrt{I}\}_S \cdot T = T \cdot \{\sqrt{I}\}_S \cdot T' = \{\sqrt{I}\}_S$ is pseudo-analog of Euclidean one $R' \cdot R = R \cdot R' = I$. But, if $V = I$ in (452), we have again the coaxially oriented space with the metric reflector tensor $\{I^{\pm}\}$ and admissible trigonometric transformations:

$$T' \cdot \{I^{\pm}\} \cdot T = \{I^{\pm}\} = T \cdot \{I^{\pm}\} \cdot T' = \text{Const}, \quad det\ T = +1. \tag{460}$$

In (458)–(460) the set $\langle T \rangle$ is called the *Lorentz continuous group of homogeneous transformations* in $\langle \mathcal{P}^{n+q} \rangle$. (Its complex analog exists for the binary complex pseudo-Euclidean space $\langle \mathcal{P}^{n+q} \rangle_c$!) The groups $\langle T \rangle$ and $\langle T_{af} \rangle$ are isomorphic and homothetic:

$$(V^{-1} \cdot T \cdot V)' \cdot \{V' \cdot I^{\pm} \cdot V\} \cdot (V^{-1} \cdot T \cdot V) = \{V' \cdot I^{\pm} \cdot V\}, \quad \langle T_{af} \rangle = V^{-1} \cdot \langle T \rangle \cdot V. \tag{461}$$

An *absolute* pseudo-Euclidean space with respect to its metric reflector tensor $\{\sqrt{I}\}_S$ may be represented in any its pseudo-Cartesian base $\tilde{E}_k$ by the *hyperbolically orthogonal direct sum* of the two real-valued *relative* Euclidean subspaces:

$$\langle \mathcal{P}^{n+q} \rangle \equiv \langle \mathcal{E}^n \rangle^{(k)} \boxtimes \langle \mathcal{E}^q \rangle^{(k)} \equiv \text{CONST}. \tag{462}$$

Moreover, the real-valued subspace $\langle \mathcal{E}^q \rangle$ is obtained as result of realification (447) from the imaginary anti-Euclidean subspace $\langle i\mathcal{E}^q \rangle$. In original complex variant, the *absolute* quasi-Euclidean space $\langle \mathcal{Q}^{n+q} \rangle_c$ is represented in any its quasi-Cartesian base $\{\tilde{E}_k\}_c$ as a *spherically orthogonal direct sum* of the Euclidean and anti-Euclidean subspaces:

$$\langle \mathcal{Q}^{n+q} \rangle_c \equiv \langle \mathcal{E}^n \rangle^{(k)} \boxplus \langle i\mathcal{E}^q \rangle^{(k)} \equiv \text{CONST}. \tag{463}$$

Here and in the sequel, $\boxplus$ and $\boxtimes$ stand for spherically and hyperbolically orthogonal direct summation with respect to a metric reflector tensor. In the both absolute spaces decompositions, these paired summands as the orthogonal complements of each other (in admissible bases $\tilde{E}_k$) are connected one-to-one functionally, as $\langle \mathcal{E}^q \rangle \equiv Y \langle \mathcal{E}^n \rangle$ and $\langle \mathcal{E}^n \rangle \equiv Y^{-1} \langle \mathcal{E}^q \rangle$. These subspaces are *relative*, but the whole space is *absolute* ! Here $Y(X)$ is the matrix function, connected one-to-one these two spaces. So, for example, we have $y(x) = a - x \leftrightarrow \ x(y) = a - y$, where a is an absolute.

For 1-valent tensor objects, internal and external multiplications (in the certain base $\tilde{E} = T \cdot \tilde{E}_1$) are determined as follow:

$$\left.\begin{array}{l} \mathbf{a}_1' \cdot \{I^{\pm}\} \cdot \mathbf{a}_2 = c_{12}, \qquad A_1' \cdot \{I^{\pm}\} \cdot A_2 = C_{12}; \\ \\ \sqrt{I^{\pm}} \cdot T \cdot (\mathbf{a_1 a_2}') \cdot T'\sqrt{I^{\pm}} = B_{12}, \quad \sqrt{I^{\pm}} \cdot T \cdot \{A_1 A_2'\} \cdot T'\sqrt{I^{\pm}} = B_{12}. \end{array}\right\} \quad (464)$$

Note, that $\mathbf{a}$, A, T and $\{I^{\pm}\}$ must have here an unified compatible binary structure. These multiplications are translated into original complex quasi-Euclidean space (463). Thus they may be used in Euclidean geometry including its tensor trigonometry!
Hence, a *metric* reflector tensor in the space $\langle \mathcal{P}^{n+q} \rangle$ executes the following operations:

- it defines the space binary structure,
- it determines the admissible transformations,
- it translates internal and external products into the original space $\langle \mathcal{P}^{n+q} \rangle_c$.

In particular, by this way the following analogs of (120) and (121) are inferred:

$$\left.\begin{array}{l} c_{12} = tr\ B_{12}, \qquad k(C_{12}, t) = k(B_{12}, t); \\ \mathbf{a}' \cdot \{I^{\pm}\} \cdot \mathbf{a} = tr\ (\sqrt{I^{\pm}} \cdot T \cdot \mathbf{aa}' \cdot T' \cdot \sqrt{I^{\pm}}); \\ k[(A' \cdot \{I^{\pm}\} \cdot A), t] = k[(\sqrt{I^{\pm}} \cdot T \cdot AA' \cdot T' \cdot \sqrt{I^{\pm}}), t]. \end{array}\right\} \quad (465)$$

These scalar characteristics of admitted vector and lineor objects in a pseudo-Euclidean space are their *real-valued pseudonorms*, in addition to semi-definite norms of sect. 9.2.

For $t = r$, define the *pseudominorant* and the *pseudodianal*:

$$\left.\begin{array}{l} \mathcal{M}p^2(r)A = k[(\sqrt{I^{\pm}} \cdot T \cdot AA' \cdot T' \cdot \sqrt{I^{\pm}}), r] = det\ (A' \cdot \{I^{\pm}\} \cdot A), \\ \\ \mathcal{D}l(r)B_{12} = k(B_{12}, r) = det\ C_{12}. \end{array}\right\} \quad (466)$$

Rotational matrices and reflectors compatible with a metric tensor do not change internal multiplications (464) and scalar angles in W-forms of projective trigonometric functions of tensor angles between linear objects (vectors, lineors, planars). Note, that in $\langle \mathcal{P}^{n+q} \rangle$, reflectors as well as projectors may be also spherically, hyperbolically, or, generally, pseudo-Euclidean orthogonal. The same relates to geometric objects too.

1-valent tensor objects are *pseudo-orthogonal* if $C_{12} = Z$, this is similar to (155); and they are *at least partially pseudo-orthogonal* if $detC_{12} = 0$, this is similar to (229). If two objects are spherically orthogonal, then they both are either in $\langle \mathcal{E}^n \rangle$, or in $\langle \mathcal{E}^q \rangle$. If two objects are hyperbolically orthogonal, then one of them is in $\langle \mathcal{E}^n \rangle$ and another one is in $\langle \mathcal{E}^q \rangle$, and this is true for decompositions of $\langle \mathcal{P}^{n+q} \rangle$ into its relative subspaces.

Also hyperbolic and spherical analogs of eigenprojectors considered in Ch. 2 operate in this space as shown, for example, in sect. 6.3.

The set of *universal bases* is identical to the set of orthospherical rotational matrices compatible with $I^{\pm}$ with respect to the trigonometric base $\tilde{E}_1 = \{I\}$ – see (352):

$$\left.\begin{array}{l} \langle \tilde{E}_{1u} \rangle \equiv \langle Rot\ \Theta \rangle \cdot \{I\} \equiv \langle \{Rot\ \Theta\} \rangle, \\ Rot'\ \Theta \cdot \{I^{\pm}\} \cdot Rot\ \Theta = \{I^{\pm}\} = Rot\ \Theta \cdot \{I^{\pm}\} \cdot Rot'\ \Theta. \end{array}\right\} (det\ Rot\ \Theta = +1) \quad (467)$$

The scalar angles in trigonometric rotations (460) and invariant scalar angles between linear objects (in W-forms) are real-valued numbers, they may be spherical (θ_k) or hyperbolic (γ_j) compatible separately with the constant-sign or alternating-sign parts of the metric reflector tensor $I^{\pm}$. In their W-forms, these structures correspond to exactly pure rotational trigonometric types considered in Chs. 5 and 6:

$$T = \{Rot\ (\pm\Theta)\}_{can} \qquad\qquad \{I^{\pm}\}$$

$$\begin{bmatrix} \ddots & & & \\ & \cos\theta_k & \mp\sin\theta_k & \\ & \pm\sin\theta_k & \cos\theta_k & \\ & & & \ddots \end{bmatrix} \Leftrightarrow \begin{bmatrix} \ddots & & & \\ & \pm 1 & 0 & \\ & 0 & \pm 1 & \\ & & & \ddots \end{bmatrix}, \tag{468}$$

$$T = \{Roth(\pm\Gamma)\}_{can}$$

$$\begin{bmatrix} \ddots & & & \\ & \cosh\gamma_j & \pm\sinh\gamma_j & \\ & \pm\sinh\gamma_j & \cosh\gamma_j & \\ & & & \ddots \end{bmatrix} \Leftrightarrow \begin{bmatrix} \ddots & & & \\ & \pm 1 & 0 & \\ & 0 & \mp 1 & \\ & & & \ddots \end{bmatrix}. \tag{469}$$

These structures generate with not admissible modal transformation R'_W two *pure types* of general rotational matrices determined with respect to reflector tensor (457) as in (458) and a certain new base. These types are orthospherical and hyperbolic:

$$\left.\begin{array}{l} R_W \cdot \{Rot\ \Theta\}_{can} \cdot R'_W = Rot\ \Theta = T_{(1)}, \quad (T'_{(1)} \cdot T_{(1)} = T_{(1)} \cdot T'_{(1)} = I), \\ T'_{(1)} \cdot \{I^{\pm}\} \cdot T_{(1)} = \{I^{\pm}\} = T_{(1)} \cdot \{I^{\pm}\} \cdot T'_{(1)}, \quad det\ T_{(1)} = +1; \end{array}\right\} \tag{470}$$

$$\left.\begin{array}{l} R_W \cdot \{Roth\ \Gamma\}_{can} \cdot R'_W = Roth\ \Gamma = T_{(2)}, \quad (T_{(2)} = T'_{(2)}), \\ T_{(2)} \cdot \{I^{\pm}\} \cdot T_{(2)} = \{I^{\pm}\} = T_{(2)} \cdot \{I^{\pm}\} \cdot T_{(2)}, \quad det\ T_{(2)} = +1. \end{array}\right\} \tag{471}$$

Modal matrices R'_W not compatible with $\{I^{\pm}\}$ change it as in (457) and condition (460) into (458). Thus the group $\langle T\rangle$ contains as pure types $Rot\ \Theta$ and $Roth\ \Gamma$ (Ch. 6).

Generally, an arbitrary transformation T may be a composition of them with respect to certain unity base $\tilde{E}_1$ of their definition:

$$T = \cdots Rot\ \Theta_{(t-1)t} \cdot Roth\ \Gamma_{(t-1)t} \cdots. \tag{472}$$

All hyperbolic rotations in their trigonometric cells, by (469), must correspond to two different blocks from the positive and negative unity parts of the reflector tensor. If $q = 1$, the elementary hyperbolic rotations with frame axes are (363) and (364).

All orthospherical rotations must be compatible with the positive and negative unity parts of the reflector tensor as indicated below:

$$Rot\ \Theta \qquad\qquad I^{\pm}$$

$$\begin{bmatrix} Rot\ \Theta^{n\times n} & Z^{n\times q} \\ Z^{q\times n} & Rot\ \Theta^{q\times q} \end{bmatrix}, \qquad \begin{bmatrix} +I^{n\times n} & Z^{n\times q} \\ Z^{q\times n} & -I^{q\times q} \end{bmatrix}. \tag{473}$$

According to (462), the pseudo-Euclidean space has binary structures determined generally by the reflector metric tensor $\{\sqrt{I}\}_S$ and pseudo-Cartesian bases $\tilde{E}_k$. In this space, an 1-valent tensor is decomposed in the two hyperbolically orthogonal projections into $\langle\mathcal{E}^n\rangle^{(k)}$ and $\langle\mathcal{E}^q\rangle^{(k)}$; a 2-valent tensor is decomposed in the homogeneous $n \times n$-biprojection into $\langle\mathcal{E}^n\rangle^{(k)}$ and $q \times q$-biprojection into $\langle\mathcal{E}^q\rangle^{(k)}$), and the mixed $n \times q$ and $q \times n$ projections into $\langle\mathcal{E}^n\rangle^{(k)}$ and $\langle\mathcal{E}^q\rangle^{(k)}$ transposed to each other.

A metric of the pseudo-Euclidean space, centralized with respect to any admissible base, partitions the space into three subspaces. For metric tensor $\{I^{\pm}\}$, the middle of them is the following dividing conic hypersurface of the 2-nd order:

$$\rho^2(\mathbf{u}) = \sum_{s=1}^{n} x_s^2 - \sum_{t=1}^{q} y_t^2 = \rho^2(\mathbf{x}) - \rho^2(\mathbf{y}) = 0, \quad \textit{or} \quad \rho^2(\mathbf{u}) = \mathbf{u}' \cdot \{I^{\pm}\} \cdot \mathbf{u} = 0.$$

The hypersurface is invariant with respect to Lorentz bases transformations (460). According to this equation, the metric $\rho(\mathbf{u})$ is zero over all of the dividing conic hypersurface. Its generating lines are central middle straight rays. This hypersurface divides $\langle\mathcal{P}^{n+q}\rangle$ into its invariant conic internal and external cavities (if $n > q$) called the *internal and external isotropic cones.* The vertex of these isotropic cones with this hypersurface is the origin of all the centralized admissible pseudo-Cartesian bases.

For visuality and determinacy, we choose an universal base $\tilde{E}_1$ for trigonometric descriptions with the use of this dividing hypersurface and these two cones (at $n > q$). The external isotropic cone ($\rho^2(\mathbf{u}) > 0$) is the open region outside the dividing conic hypersurface, it is also the union of the subspaces $\langle\mathcal{E}^n\rangle^{(k)}$ in decompositions (462). The internal isotropic cone ($\rho^2(\mathbf{u}) < 0$) is the open region inside the dividing conic hypersurface, it is also the union of the subspaces $\langle\mathcal{E}^q\rangle^{(k)}$ in decompositions (462).

The set of admissible rotations in the space $\langle\mathcal{P}^{n+q}\rangle$ with respect to any centralized pseudo-Cartesian base consists of the two connected subsets of Lorentz continuous homogeneous transformations inside and outside the dividing conic hypersurface, what stipulates *isotropy* of these internal and external cones. In general, these motions of any tensor objects have hyperbolically orthogonal homogeneous and mixed projections into instantaneous $\langle\mathcal{E}^n\rangle^{(k)}$ and $\langle\mathcal{E}^q\rangle^{(k)}$ (see above), i. e., these motions realized always in these two instantaneous isotropic cones. Hence, $\langle\mathcal{P}^{n+q}\rangle$ *on the whole is isotropic* too for any admissible motions. On the other hand, the parallel translations into its any point are admissible too, and stipulates *homogeneity* of the space $\langle\mathcal{P}^{n+q}\rangle$.

If $q = 1$, then $\langle\mathcal{P}^{n+1}\rangle$ is the Minkowski space (see in Ch. 12) with its *internal double isotropic cone* ($\rho^2(\mathbf{u}) < 0$) and *external circle isotropic cone* ($\rho^2(\mathbf{u}) > 0$). In special theory of relativity (STR), the double internal isotropic cone, where $\mathbf{u}$ is *time-like*, is formed by the upper and lower conic parts so called the *cone of the future* and the *cone of the past*, i. e., in accordance with the positive and negative directions of the ordinate $\overrightarrow{y}^{(k)}$-axis. These parts are situated inside the dividing conic hypersurface, in STR called the *light cone.* They are the union of the ordinate $\overrightarrow{y}^{(k)}$-axes. The external circle isotropic cone, where $\mathbf{u}$ is *space-like*, is the union of the spaces $\langle\mathcal{E}^n\rangle^{(k)}$.

11.3 Polar representation of general pseudo-Euclidean rotations

Any composite continuous transformation (460), for example (472), of some geometric objects in the internal and external cavities of an isotropic cone, with respect to an universal base $\tilde{E}_1$, may be reduced to the non-commutative product of a hyperbolic rotational matrix and orthospherical one in the following *two polar forms*:

$$T = Roth\ \Gamma \cdot Rot\ \Theta = Rot\ \Theta \cdot Roth\ \overset{\angle}{\Gamma}, \qquad (474), (475)$$

where $Roth\ \Gamma = \{\sqrt{TT'}\}_{S^+} = \sqrt{Roth\ 2\Gamma} = Roth'\ \Gamma = Roth^{-1}\ (-\Gamma)$,

$$Roth\ \overset{\angle}{\Gamma} = \{\sqrt{T'T}\}_{S^+} = \sqrt{Roth\ 2\ \overset{\angle}{\Gamma}}$$

are one-valued symmetric arithmetic (and trigonometric) square roots (sect. 5.7, 6.2);

$$Rot\ \Theta = \sqrt{TT'}^{-1} \cdot T = Roth\ (-\Gamma) \cdot T = T \cdot \sqrt{T'T}^{-1} = T \cdot Roth\ (-\ \overset{\angle}{\Gamma}) = Rot'\ (-\Theta).$$

Note (!): the polar representations strictly correspond to definition (351) of $\langle \mathcal{P}^{n+q} \rangle$.

From (474), (475) the simple connection between these two principal rotations as well as their two motive hyperbolic tensor angles follows:

$$Roth\ \overset{\angle}{\Gamma} = Rot'\ \Theta \cdot Roth\ \Gamma \cdot Rot\ \Theta = Rot\ (-\Theta) \cdot Roth\ \Gamma \cdot Rot\ \Theta. \qquad (476)$$

Polar representation can be inferred with the use of arithmetic roots by the two ways:

1) $T = S^+ \cdot R \ \Rightarrow\ TT' = S^2,\ \ T'T = R' \cdot S^2 \cdot R \Rightarrow\ T'T = R' \cdot TT' \cdot R\ \Rightarrow$

$\Rightarrow\ \sqrt{T'T} = R' \cdot \sqrt{TT'} \cdot R\ \Rightarrow\ T = \sqrt{TT'} \cdot R = R \cdot \sqrt{T'T};\ \ det\ T = +1\ \Rightarrow\ R = Rot\ \Theta;$

2) $(460), (267), (325)\ \Rightarrow\ (TT') \cdot I^{\pm} \cdot (TT') = I^{\pm} = (T'T) \cdot I^{\pm} \cdot (T'T)\ \Rightarrow\ (471)\ \Rightarrow$

$$\Rightarrow \begin{cases} TT' = Roth\ 2\Gamma, & \sqrt{TT'} = Roth\ \Gamma\ \Rightarrow\ (474), \\ T'T = Roth\ 2\ \overset{\angle}{\Gamma}, & \sqrt{T'T} = Roth\ \overset{\angle}{\Gamma}\ \Rightarrow\ (475); \end{cases} \quad det\ T = +1\ \Rightarrow (476).$$

By (476), Γ and $\overset{\angle}{\Gamma}$ *have the same angles eigenvalues spectrum* $\langle \gamma_j \rangle$.

We use widely such polar representations of a general rotational transformation for simple description of multistep hyperbolic or spherical principal rotations, for example, of relativistic motions in STR, motions in spherical and hyperbolic geometries.

Further consider the polar representation of trigonometric modal transformations:

$$\left.\begin{aligned} T = \sqrt{TT'} \cdot R = S_1 \cdot R = (S_1 \cdot R \cdot S_1^{-1}) \cdot S_1 = \\ = R \cdot \sqrt{T'T} = R \cdot S_2 = (R \cdot S_2 \cdot R') \cdot R. \end{aligned}\right\} (R = Rot\ \Theta) \qquad (477, 478)$$

The symmetric matrices of principal rotations $S_1 = Roth\ \Gamma$ and $S_2 = Roth\ \overset{\angle}{\Gamma}$ are expressed in (474), (475) in canonical form (324) in the unity base $\tilde{E}_1 = \{I\}$. But the latter acts in the base $\tilde{E}_{1u} = Rot\ \Theta \cdot \tilde{E}_1$ and then is transformed in it by the rotation R.

The orthospherical rotation $Rot\ \Theta$ is expressed initially in $\tilde{E}_1 = \{I\}$ too. But $Rot\ \Theta$ acts really in the base $\tilde{E}_{1h} = Roth\ \Gamma \cdot \tilde{E}_1$ and then is transformed in it by rotation S_1.

According to (477) the matrix S_1 acts in the base $\tilde{E}_1$ and realizes the base rotation at the angle Γ, and then the orthospherical matrix R acts in this hyperbolically rotated base $\tilde{E}_{1h}$ and realizes the base rotation at the angle Θ. According to (478) the matrix R acts in the base $\tilde{E}_1$ and realizes the base rotation at the angle Θ, and then the matrix S_2 acts in this spherically rotated base $\tilde{E}_{1u}$ and realizes the base rotation at the angle Γ. Both these modal transformations of the base $\tilde{E}_1$ are *formally* equivalent.

Similar sense of these two variants of multiplications S and R appears in a *passive transformation* of an element $\mathbf{u}^{(1)}$ coordinates:

$$\mathbf{u}^{(2)} = (S_1 \cdot R)^{-1} \cdot \mathbf{u}^{(1)} = R^{-1} \cdot S_1^{-1} \cdot \mathbf{u}^{(1)} = \{R' \cdot S_1 \cdot R\}^{-1} \cdot R^{-1} \cdot \mathbf{u}^{(1)} =$$
$$= (R \cdot S_2)^{-1} \cdot \mathbf{u}^{(1)} = S_2^{-1} \cdot R^{-1} \cdot \mathbf{u}^{(1)} = \{S_2^{-1} \cdot R \cdot S_2\}^{-1} \cdot S_2^{-1} \cdot \mathbf{u}^{(1)}. \quad (479)$$

In a linear pseudo-Euclidean space, separate the full set of right pseudo-Cartesian bases $\langle T \cdot \tilde{E}_1 \rangle$. All these bases are rotationally connected as $det\ T = +1$. Transition from $\tilde{E}_1$ to a new base $\tilde{E}$ may be represented, by (474) and (475), in the following two polar forms – straight and inverse:

$$\tilde{E} = T \cdot \tilde{E}_1 = Roth\ \Gamma \cdot Rot\ \Theta \cdot \tilde{E}_1 = (Roth\ \Gamma \cdot Rot\ \Theta \cdot Roth^{-1}\ \Gamma) \cdot Roth\ \Gamma \cdot \tilde{E}_1, \quad (480)$$

$$\tilde{E} = T \cdot \tilde{E}_1 = Rot\ \Theta \cdot Roth\ \overset{\angle}{\Gamma} \cdot \tilde{E}_1 = (Rot\ \Theta \cdot Roth\ \overset{\angle}{\Gamma} \cdot Rot'\ \Theta) \cdot Rot\ \Theta \cdot \tilde{E}_1. \quad (481)$$

These two forms give the two possible sequences of these hyperbolic and orthospherical rotations execution. For both these variants: in the left multiplications these matrices are expressed in the base $\{I\}$ of their definitions; in the right multiplications these matrices are expressed in the bases of their actions! Hence, these two polar forms realize the principal hyperbolic rotation in different bases: straight polar form (480) in the base $\tilde{E}_1$ and inverse polar form (481) in the other universal base $\tilde{E}_{1u} = Rot\ \Theta \cdot \tilde{E}_1$.

For any pseudo-Cartesian base $\tilde{E}_k$, first n columns of its matrix determine the subspace $\langle \mathcal{E}^n \rangle^{(k)}$, other q columns determine $\langle \mathcal{E}^q \rangle^{(k)}$ in hyperbolically orthogonal sum (462). The matrix $Rot\ \Theta$ has structure (473), that is why only hyperbolic rotations of any pseudo-Cartesian base $\tilde{E}_k$ give new subspaces $\langle \mathcal{E}^n \rangle^{(j)}$ and $\langle \mathcal{E}^q \rangle^{(j)}$ determined by the columns of the new base $\tilde{E}_j$ matrix. If the new base $\tilde{E}$ connected with $\tilde{E}_1 = \{I\}$ by a modal matrix T or $Roth\ \Gamma$, then in the base we have the following identities:

$$\left.\begin{array}{l} \langle \mathcal{E}^n \rangle \equiv im\ [\tilde{E}]^{(n+q)\times n} \equiv im\ [T]^{(n+q)\times n} \equiv im\ [Roth\ \Gamma]^{(n+q)\times n}, \\ \langle \mathcal{E}^q \rangle \equiv im\ [\tilde{E}]^{(n+q)\times q} \equiv im\ [T]^{(n+q)\times q} \equiv im\ [Roth\ \Gamma]^{(n+q)\times q}. \end{array}\right\} \quad (482)$$

This means that all trigonometric rotations (460) applied to the Euclidean subspaces $\langle \mathcal{E}^n \rangle$ and $\langle \mathcal{E}^q \rangle$ *on the whole* as sets of point elements are reduced to their pure hyperbolic rotation from (474). In particular, for a Minkowski space $\langle \mathcal{P}^{n+1} \rangle$, the n and 1 columns of the matrices $\tilde{E}$, T, $roth\ \Gamma$ determine the space $\langle \mathcal{E}^n \rangle$ and the axis $\overrightarrow{y}$ as the relative subspaces in the base $\tilde{E}$ after the base $\tilde{E}_1$ rotation by the matrix T or $roth\ \Gamma$.

Hence, the polar formula (474) reduces any admissible transformation T of the two relative subspaces in the whole from the original base $\tilde{E}_1 = \{I\}$ into any admissible pseudo-Cartesian base $\tilde{E}$ till their pure hyperbolic rotation $Roth\ \Gamma = \sqrt{TT'}$.

The polar representation of a general trigonometric transformation of the relative subspaces in the whole as hyperbolic rotation does not hold for subsets of these subspaces, in particular, the base coordinate axes. This can be seen in (481): the coordinate axes are subjected to orthospherical rotation and then hyperbolic rotation.

The matrix of a transformation T, due to (460), is a bivalent pseudo-Euclidean quasibiorthogonal tensor. This is true for the matrix of the base $\tilde{E} = T \cdot \{I\}$ too. The tensor is splitted projectively into the pair of symmetric homogeneous ($n \times n$ and $q \times q$) and the pair of mutually transposed mixed ($n \times q$ and $q \times n$) tensor projections:
$[\tilde{E}]^{n\times n}$ is orthoprojection of space-like unity basis vectors into the subspace $\langle \mathcal{E}^n \rangle^{(1)}$;
$[\tilde{E}]^{q\times q}$ is orthoprojection of time-like unity base vectors into the subspace $\langle \mathcal{E}^q \rangle^{(1)}$;
$[\tilde{E}]^{n\times q}$ and $[\tilde{E}]^{q\times n}$ are mutually transposed oblique projections into $\langle \mathcal{E}^n \rangle^{(1)}$ and $\langle \mathcal{E}^q \rangle^{(1)}$.
If the base matrix is transposed, then these projections are reflected with respect to the matrix diagonal. This takes place, in particular, under changing the direction of a multistep hyperbolic rotation sequence (see in next sect.).

If $q = 1$, then the matrix $Rot\ \Theta^{q\times q}$ in (473) degenerates into I. In a space $\langle \mathcal{P}^{n+1} \rangle$, an 1-valent tensor is decomposed in two hyperbolically orthogonal projections into $\langle \mathcal{E}^n \rangle^{(k)}$ and onto $\overrightarrow{y}^{(k)}$; a 2-valent tensor is decomposed in an homogeneous projection $n \times n$-tensor into $\langle \mathcal{E}^n \rangle^{(k)}$, an invariant scalar onto $\overrightarrow{y}^{(k)}$-axis, and two mixed projections – $n \times 1$-vectors into $\langle \mathcal{E}^n \rangle^{(k)}$ and onto $\overrightarrow{y}^{(k)}$. World events in STR are described here from the view-point of a relatively immobile Observer with respect to an universal base. Among them, $\tilde{E}_1 = \{I\}$ is the simplest original one. Any concrete spherical-hyperbolic analogy (from sect. 6.2) is realized with respect to this base!

In this Minkowski space, Lorentz transformation (460) of a point element on the $\overrightarrow{y}^{(1)}$-axis is reduced by polar representation up to either it hyperbolic rotation together with the ordinate axis (under passive transformation), or it hyperbolic rotation off the ordinate axis in the direction given by the orthospherical tensor angle (under active transformation). Consider two examples with elementary matrices useful in STR.

Example 1.

$$\mathbf{u}^{(j)} = \{rot'\ \Theta \cdot roth\ \Gamma \cdot rot\ \Theta\}^{-1} \cdot rot'\ \Theta \cdot \mathbf{u}^{(1)} = \{rot'\ \Theta \cdot roth\ \Gamma \cdot rot\ \Theta\}^{-1} \cdot \mathbf{u}^{(1)}, \quad (483)$$

where $\mathbf{u}^{(1)} \in \langle \overrightarrow{y}^{(1)} \rangle$ is a point object with respect to $\tilde{E}_1$, and $\mathbf{u}^{(j)}$ is the same object with respect to $\tilde{E}_j = T_{1j} \cdot \tilde{E}_1$. However, its pure hyperbolic passive transformation (in brackets) was realized here from the base $\tilde{E}_{1u} = rot\ \Theta \cdot \tilde{E}_1$ into the final base $\tilde{E}_j$!

Example 2.

$$\mathbf{u}_j = T_{1j} \cdot \mathbf{u}_1 = \{rot\ \Theta \cdot roth\ \overset{\angle}{\Gamma} \cdot rot'\ \Theta\} \cdot rot\ \Theta \cdot \mathbf{u}_1 = \{rot\ \Theta \cdot roth\ \overset{\angle}{\Gamma} \cdot rot'\ \Theta\} \cdot \mathbf{u}_1, \quad (484)$$

where $\mathbf{u}_1 \in \langle \overrightarrow{y}^{(1)} \rangle$ is a point element, it generated in $\tilde{E}_1$ the element $\mathbf{u}_j = T_{1j} \cdot \mathbf{u}_1$. Here the pure hyperbolic active rotation was realized off $\overrightarrow{y}^{(1)}$ under the angle Θ!

11.4 Multistep hyperbolic rotations

The summarized multistep hyperbolic rotation are pure hyperbolic if its particular rotations are trigonometrically compatible with each other (see Ch. 6), i. e., they may be reduced to form (324) in common bases. In particular, vectors of directional cosines for elementary hyperbolic rotations in (363) are equal to each other up to coefficients ± 1. If particular rotational matrices are not trigonometrically compatible (though each of them is compatible with the set metric reflector tensor), then a composite formula of non-symmetric (in general) multistep hyperbolic rotation may be always reduced till polar forms (474), (475).

Specify the sequence of particular hyperbolic rotations (realizing geodesic motions on hyperboloid hypersurfaces (i. e., at $\rho^2(\mathbf{u}) = \text{const}$) in the original base $\tilde{E}_1 = \{I\}$:

$$Roth\ \Gamma_{12},\ Roth\ \Gamma_{23}, \ldots,\ Roth\ \Gamma_{(t-1)t}.$$

For descriptive analysis in $\tilde{E}_1 = \{I\}$, these matrices in the own *unity* bases $\tilde{E}_k = \{I\}$ had canonical form (324) in $\langle \mathcal{P}^{n+q} \rangle$, either elementary one (363) in $\langle \mathcal{P}^{n+1} \rangle$, and both these forms correspond to a metric reflector tensor $\{I^{\pm}\}$. The following matrices realize hyperbolic rotations in other bases. As result, in the base $\tilde{E}_1 = \{I\}$, they have now the forms corresponding to this original base! The bases are transformed as follows:

$$\tilde{E}_1 = \{I\}, \tilde{E}_2 = \{Roth\ \Gamma_{12}\}_{(\tilde{E}_1)} \cdot \tilde{E}_1, \ldots, \tilde{E}_t = \{Roth\ \Gamma_{(t-1)t}\}_{(\tilde{E}_{t-1})} \cdot \tilde{E}_{t-1}.$$

Translate the matrix $\tilde{E}_t$ from the base of its action into the original base $\tilde{E}_1 = \{I\}$ for rotations analysis, obtain the dual formula for resulting multistep transformation:

$$\tilde{E}_t = T_{1t} \cdot \tilde{E}_1 = \{Roth\ \Gamma_{(t-1)t}\}_{(\tilde{E}_{t-1})} \cdots \{Roth\ \Gamma_{23}\}_{(\tilde{E}_2)} \cdot \{Roth\ \Gamma_{12}\}_{(\tilde{E}_1)} \cdot \tilde{E}_1 =$$

$$= T_{1t} \cdot \tilde{E}_1 = Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} \cdots Roth\ \Gamma_{(t-1)t} \cdot \tilde{E}_1. \tag{485}$$

This is **Rule** of executing multistep transformations (proved by induction on $t \geq 3$).

$$\tilde{E}_3 = \{Roth\ \Gamma_{23}\}_{(\tilde{E}_2)} \cdot \tilde{E}_2 = \{Roth\ \Gamma_{23}\}_{(\tilde{E}_2)} \cdot \{Roth\ \Gamma_{12}\}_{(\tilde{E}_1)} \cdot \tilde{E}_1 =$$

$$= \{Roth\, \Gamma_{12} \cdot Roth\, \Gamma_{23} \cdot Roth^{-1}\, \Gamma_{12}\} \cdot \{Roth\, \Gamma_{12}\} \cdot \tilde{E}_1 = Roth\, \Gamma_{12} \cdot Roth\, \Gamma_{23} \cdot \tilde{E}_1. \tag{486}$$

Here the sequence of the particular matrices is inversed (see, for example, [16, p. 428]).

Coordinates of linear objects are transformed *passively*, but the sequence of inverse rotational matrices in their canonical form is direct:

$$\mathbf{u}^{(t)} = Roth\ (-\Gamma_{(t-1)t}) \cdots Roth\ (-\Gamma_{23}) \cdot Roth\ (-\Gamma_{12}) \cdot \mathbf{u}^{(1)} =$$

$$= \{Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} \cdots Roth\ \Gamma_{(t-1)t}\}^{-1} \cdot \mathbf{u}^{(1)}, \tag{487}$$

$$\mathbf{u}^{(3)} = Roth\ (-\Gamma_{23}) \cdot \mathbf{u}^{(2)} = Roth\ (-\Gamma_{23}) \cdot Roth\ (-\Gamma_{12}) \cdot \mathbf{u}^{(1)} =$$

$$= \{Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23}\}^{-1} \cdot \mathbf{u}^{(1)}. \tag{488}$$

Active multistep hyperbolic rotational transformations of a generating element $\mathbf{u}$, for example, in $\tilde{E}_1 = \{I\}$, are realized similarly to analogous multistep hyperbolic transformations of the base, when particular rotational matrices are ordered inversely (as in (485), because they are determined and act sequentially with respect to $\tilde{E}_1$:

$$\mathbf{u}_t = Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} \cdots Roth\ \Gamma_{(t-1)t} \cdot \mathbf{u}_1, \tag{489}$$

$$\mathbf{u}_3 = Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} \cdot \mathbf{u}_1 = \{Roth\ \Gamma_{23}\}_{\tilde{E}_2} \cdot \mathbf{u}_2. \tag{490}$$

Formulae (485)–(490) are the special cases of the *General rule of multistep linear transformations*. Other special cases of the rule relate to similar sequences of principal spherical rotations – motions in a quasi-Euclidean binary space $\langle \mathcal{Q}^{n+q} \rangle$ (Ch. 8A).

In pseudo-Euclidean geometry, matrices of pure hyperbolic (principal) rotations may be or not be symmetric, but they are always prime. This depends on the bases of their definition and action. A matrix is symmetric in canonical forms (324), (362), (363) with respect to any unity base of its definition. The matrix $T \cdot Roth\ \Gamma \cdot T^{-1}$ represents the hyperbolic rotation with respect to the universal base $\tilde{E}_1$ and acting in the pseudo-Cartesian base $\tilde{E} = T \cdot \tilde{E}_1$. Prime matrices of hyperbolic rotations also belong to the Lorentz group with the metric tensor $I^{\pm}$. A prime hyperbolic matrix may be represented in $\tilde{E}_1$ in polar forms (474), (475) for its analysis. The analogous statements hold for orthospherical rotations $Rot\ \Theta$ and $T \cdot Rot\ \Theta \cdot T^{-1}$ too. They may be expressed with respect to either the original base $\tilde{E}_1$, or the base $\tilde{E} = T \cdot \tilde{E}_1$ of their action. *All pure orthospherical rotations form their complete continuous subgroup* of the Lorentz continuous group of homogeneous transformations.

For a generating or transforming element $\mathbf{u}$*, its continuous Lorentz transformations do not change the value of the invariant* $\rho^2(\mathbf{u}) = [T \cdot \mathbf{u}]' \cdot \{I^{\pm}\} \cdot [T \cdot \mathbf{u}] = \mathbf{u}' \cdot \{I^{\pm}\} \cdot \mathbf{u}$ *similar to continuous motions on the hyperboloid surface with invariant* $\rho^2(\mathbf{u}) = const$!

Further, in order to analyze and reduce expressions for two-step and multistep hyperbolic rotations, we use again polar representations (474), (475). There hold:

$$\tilde{E}_t = T_{1t} \cdot \tilde{E}_1 = Roth\ \Gamma_{1t} \cdot Rot\ \Theta_{1t} \cdot \tilde{E}_1 = Rot\ \Theta_{1t} \cdot Roth\ \overset{\angle}{\Gamma}_{1t} \cdot \tilde{E}_1, \tag{491}$$

$$\left.\begin{array}{l} Roth\ \Gamma_{13} = \sqrt{TT'} = \sqrt{Roth\ \Gamma_{12} \cdot Roth\ (2\Gamma_{23}) \cdot Roth\ \Gamma_{12}} = \\ = \sqrt{Roth\ (2\Gamma_{13})}, \\ Rot\ \Theta_{13} = Roth\ \Gamma_{31} \cdot Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} = Rot'\ (-\Theta_{13}) = \\ = Rot^{-1}\ (-\Theta_{13}) = Rot'\ \Theta_{31} = Rot\ (-\Theta_{31}), \end{array}\right\} (t = 3) \tag{492}$$

$$\left.\begin{array}{l} \mathbf{u}^{(t)} = (Rot'\ \Theta_{1t} \cdot Roth\ \Gamma_{1t} \cdot Rot\ \Theta_{1t})^{-1} \cdot Rot'\ \Theta_{1t} \cdot \mathbf{u}^{(1)} = \\ = \{Roth\ \Gamma_{1t}\}^{-1}_{\tilde{E}_{1u}} \cdot \mathbf{u}^{(1u)}, \\ A^{(j)} = (Rot'\ \Theta_{1t} \cdot Roth\ \Gamma_{1t} \cdot Rot\ \Theta_{1t})^{-1} \cdot Rot'\ \Theta_{1t} \cdot A^{(1)} = \\ = \{Roth\ \Gamma_{1t}\}^{-1}_{\tilde{E}_{1u}} \cdot A^{(1u)}. \end{array}\right\} (t \geq 3) \tag{493}$$

Here the rotation $Rot\ \Theta_{13}$ is executed in the bases of particular rotations actions in the sequence 31, 12, 23 along of legs of the orthospherical triangle 123 – see Rule (485)!

Multistep *hyperbolic* geodesic motions of a point element, when $\rho^2(\mathbf{u}) = \rho^2 = \text{const}$, sequentially produce apices of a certain geometric figure, for examples, a triangle or a polygon. A necessary condition for such entire construction be a geometric figure is that the sequential hyperbolic rotations form a closed circuit with summarized hyperbolic angle annihilation: $\prod_{k\geq 3} Roth\ \Gamma_{(k)}\mathbf{u}_1 = Rot\ \Theta_{1t} \cdot \mathbf{u}_1$.

Geometry of the figures from geodesic segments is realized in *invariant hyperboloid surfaces* of maximal dimension for a given quadratic metric invariant $\rho^2(\mathbf{u}) = \text{const}$:

$$\mathbf{u}' \cdot \{I^{\pm}\} \cdot \mathbf{u} = \sum_{s=1}^{n} x_s^2 - \sum_{t=1}^{q} y_t^2 = \rho^2(\mathbf{x}) - \rho^2(\mathbf{y}) = \rho^2(\mathbf{u}) = \pm R^2, \quad (R = \text{const}). \quad (494)$$

If $R = 0$, then, in any admissible to $\{I^{\pm}$ pseudo-Cartesian bases with the same origin, we have a centralized invariant conic surface dividing the pseudo-Euclidean space into its internal and external cavities – sect. 11.2. For such geometric figures, their segments are continuous, that is why, a constructed figure is contained in exactly one cavity of this conic surface: either inside the internal cone with $\rho^2(\mathbf{u}) = -R^2$ $(\rho^2(\mathbf{y}) > \rho^2(\mathbf{x}))$, or inside the external cone with $\rho^2(\mathbf{u}) = +R^2$ $(\rho^2(\mathbf{x}) > \rho^2(\mathbf{y}))$.

However from (494) we may get else, as trivial cases, real-valued n- and q-dimensional spheres with their equations: $\sum_{s=1}^{n} x_s^2 = \rho^2(\mathbf{x}) = +R^2$ if we put $y_t = 0 \to \rho^2(\mathbf{y}) = 0$ and $-\sum_{t=1}^{q} y_t^2 = -\rho^2(\mathbf{y}) = -R^2$ if we put $x_t = 0 \to \rho^2(\mathbf{x}) = 0$. They have the usual spherical geometry for a sphere in Euclidean space. Here the geometry may have place on the spheres with the radius R in two Euclidean subspaces $\langle\mathcal{E}^n\rangle_{(x)}$ and $\langle\mathcal{E}^q\rangle_{(y)}$ in any admissible bases of the pseudo-Euclidean space $\langle\mathcal{P}^{n+1}\rangle$.

The active continuous homogeneous Lorentz transformations perform motions of a generating point element $\mathbf{u} = T \cdot \mathbf{u}_1$ on all this hyperboloid hypersurface with the given metric invariant $\rho^2(\mathbf{u}_1) = \rho^2(\mathbf{u}) = const$. If this circuit of hyperbolic motions is complete and closed at $t = 3$ or $t \geq 3$ in (485), i. e., the principal hyperbolic rotations form a closed geometric figure (a hyperbolic triangle or a hyperbolic polygon) with the constant quadratic metric invariant $\rho^2(\mathbf{u})$, then as result is the appearance of the concrete orthospherical precession $Rot\ \Theta$. In Appendix we shall prove that its orthospherical angle θ formally is equal in the case to the figure spherical angular deviation of Gauss–Bonnet in non-Euclidean geometries, and the precession is the deviation algebraic cause explained by tensor trigonometry!

In the Minkowskian space-time $\langle\mathcal{P}^{3+1}\rangle$ of STR, the orthospherical rotation in (491) is the result of summing motions (velocities) with different directions $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$. In STR, this is a *secondary rotation*. Its well-known case is a *Thomas precession*. The principal hyperbolic motion is called a *boost*. The feature of velocities summation law is explained by hyperbolic nature of principal motions in this space.

In conclusion of this chapter, note that a sum of motions is invariant under a choice of passive or active transformations of objects coordinates. We choose T for an original base transformation as a more universal variant, and we shall use this in Appendix.

Chapter 12

Tensor trigonometry of Minkowski pseudo-Euclidean space

12.1 Trigonometric models for two concomitant hyperbolic geometries

Now consider more in details a *coaxially oriented* pseudo-Euclidean space $\langle \mathcal{P}^{n+1} \rangle$, i. e., *Minkowskian space* and *Minkowskian space-time* in STR (at $n = 3$) [49]. Due to (462), it is expressed in any base $\tilde{E}_k$ as the following *hyperbolically orthogonal direct sum*

$$\langle \mathcal{P}^{n+1} \rangle \equiv \langle \mathcal{E}^n \rangle^{(k)} \boxtimes \overrightarrow{y}^{(k)} \equiv \mathrm{CONST}. \tag{495}$$

Tensor trigonometry in the pseudo-Euclidean space are realized with *elementary* tensor angles and trigonometric functions (as $q = 1$). It is rational to use these angles and functions in its special form (see in sect. 6.5) with the frame abscissa axis $\overrightarrow{y}$. Note, that in any pseudo- and quasi-Euclidean spaces, the tensor trigonometry in its different forms is realizable and applicable due to homogeneity and isotropy of the spaces! First homogeneity and isotropy to the space-time of events was given by H. Poincaré [47].

We use (494) for determination in the universal base $\tilde{E}_1$ of $\langle \mathcal{P}^{n+1} \rangle$ of two hyperboloid hypersurfaces, as the *invariant geometric objects*, with different signs of their metric invariant ρ^2 at $+R^2$ and $-R^2$ with $R = \mathrm{const}$, i. e., by two quadric equations:

$$\mathbf{v}' \cdot \{I^{\pm}\} \cdot \mathbf{v} = \sum_{s=1}^{n} x_s^2 - y^2 = \rho^2(\mathbf{x}) - y^2 = \rho^2(\mathbf{v}) = +R^2, \quad (||\mathbf{x}||_E > |y|_P), \tag{ I}$$

$$\mathbf{u}' \cdot \{I^{\pm}\} \cdot \mathbf{u} = \sum_{s=1}^{n} x_s^2 - y^2 = \rho^2(\mathbf{x}) - y^2 = \rho^2(\mathbf{u}) = -R^2, \quad (||\mathbf{x}||_E < |y|_P). \tag{II}$$

Here $\mathbf{u}$ and $\mathbf{v}$ are the radius-vectors of points on these hypersurfaces, $\mathbf{x}$ is its vectorial projection into $\langle \mathcal{E}^n \rangle$, y is its scalar projection onto $\overrightarrow{y}$ (and $||\mathbf{u}||_P = ||\mathbf{v}||_P = R$). If put $\rho^2 = 0$, the equations give an asymptotic conic and invariant hypersurface (isotropic or light cone) dividing the objects in its external and internal cavities (Figure 4). As *invariant geometric objects*, the hypersurfaces are concomitant Minkowskian hyperboloids I and II (they are trigonometric at $R = 1$). For their construction in the original base $\tilde{E}_1$, the two generating hyperbolae (see at Figure 3 in sect. 6.4) are rotated with respect to the abscissa axis $\overrightarrow{y}^{(1)}$ with $(n-1)$ degrees of freedom.

These hyperboloids, besides their internal n-dimensional hyperbolic geometries, have $(n-1)$-dimensional orthospherical geometry. Due to equation (I), for any value of the abscissa y it is possible on a hyperboloid I to realize real-valued spherical figures (till circles) of radius $r = +\sqrt{R^2 + y^2}$. And due to equation (II), for these values of the abscissa $|y| > R$ it is possible on a hyperboloid II to realize real-valued spherical figures (till circles) of radius $r = +\sqrt{y^2 - R^2}$. Their equations are $\sum_{s=1}^{k} x_s^2 = r, (k \leq n)$.

The *trigonometric* hyperboloids with $n = 2$, $R = 1$ in cut may be seen with their centered projections in $\tilde{E}_1$ at Figure 4. Their initial points in $\tilde{E}_1$ (the unique $\mathbf{u}_1$ for II and, for example, $\mathbf{v}_1$ for I) are rotated hyperbolically into other ones $\mathbf{u}_k$ and $\mathbf{v}_k$.

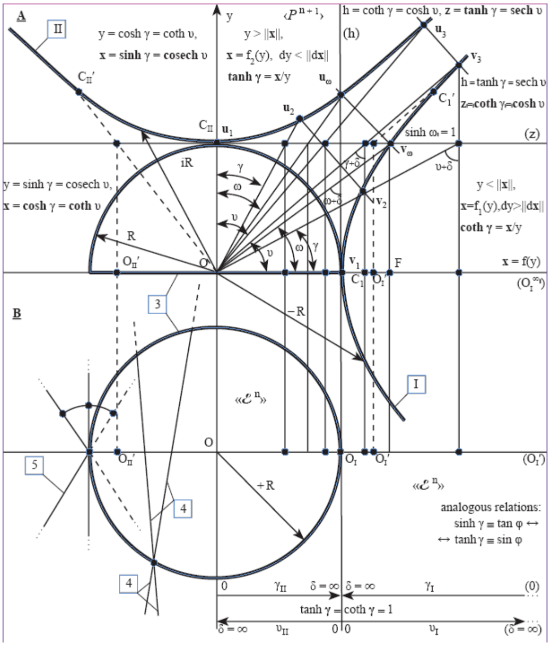

Figure 4. Trigonometric models of hyperboloids I and II (upper part) with hyperspheroid (upper parts).

A. Trigonometric correspondence between points of Minkowskian hyperboloids I and II ($R = 1$) cat by eigen pseudoplane in $\langle \mathcal{P}^{n+1} \rangle$. The following hyperbolic angles are present: principal γ, complementary υ, special ω (sinh $\omega = 1$), right (infinite) δ, obtuse ($\upsilon \uplus \delta$).
B. Trigonometric models in the universal trigonometric base $\tilde{E}_1 = \{I\}$, or projective models with respect to the Cayley absolute oval into the projective hyperplane $\langle\langle \mathcal{E}^n \rangle\rangle$.

(I) a one-sheet hyperboloid I of radii $\rho = \pm R$ with its cotangent ($\coth \gamma$) model ($\gamma \leftrightarrow \upsilon$) as coth $(\gamma, \upsilon) = \cosh(\upsilon, \gamma)$, or projective model outside the Cayley oval,

(II) a two-sheets hyperboloid II of radii $\rho = +iR$ (upper) and $\rho = -iR$ (lower) with its tangent ($\tanh \gamma$) model, or projective Klein model inside the Cayley oval,

(3) Klein disk with Cayley oval as tangent-cotangent projection of an isotropic cone,

(4) conjugate parallel straight lines inside and outside the Cayley oval,

(5) correspondences between straight lines inside and outside the Cayley oval.

Rotation of a time-like hyperbola generates a hyperboloid I of one sheet (seeming as an *hourglass*) inside the external cone. Rotation of space-like hyperbolae generates a hyperboloid II of two coupled sheets (seeming as two *symmetric cups*) inside the internal double cone. The one sheet of I has radii $-R$ and $+R$, the two sheets of II have radii $+iR$, $-iR$. This stipulates their negative constant Gaussian curvature.

Centered geodesic motions of generating elements $\mathbf{u}_1$ and $\mathbf{v}_1$ (Figure 4) on these hyperboloids are expressed by rotational matrix functions $roth\ \Gamma = F(\gamma)$ according to (362), (363). Noncentralized geodesic following motion of the element $\mathbf{u}_2 = T_{12}\mathbf{u}_1$ on a hyperboloid II is represented in the base $\tilde{E}_2$ as $T_{12}\{roth\ \Gamma_{23}\}_{\tilde{E}_1}T_{12}^{-1}$ (see sect. 11.4). The angle γ ranges in $[0; +\infty)$ if $dy > 0$ and in $[0; -\infty)$ if $dy < 0$.

The real-valued extent of a geodesic hyperbolic line's segment on hyperboloids I and II is a *pseudo-Euclidean length* in a certain pseudoplane in $\langle \mathcal{P}^{n+1} \rangle$, which includes this hyperbolic line. Here and further this extent is called the *pseudo-Euclidean natural measure* of a length for *mensurations along geodesic*. (The radius of a hyperboloid II in this measure is *realificated* too.) The extent of geodesic hyperbolic arcs-segments (either differentials) is $\lambda_\gamma = R\gamma$ (sect. 6.4), and of orthospherical arcs is $\lambda_\theta = R\theta$. In geometries of these hyperboloids as well as in real-valued hyperbolic non-Euclidean geometries, the *angular Lambert measure γ of a length* [33] is applied at calculations with the use of exponential and trigonometric formulae through the hyperbolic angle γ.

Both concomitant Minkowskian hyperboloids are conjugate and have isometric each to other external and internal geometries. They are simplest descriptive isometric maps in $\langle \mathcal{P}^{n+1} \rangle$ of two isomorphic to them real-valued hyperbolic n-dimensional spaces with the *Euclidean natural measure l* and the *angular measure* $\gamma = l/R$ of a length for *straight segments* as internal parameters. Thus symbols λ and l stand for these pseudo-Euclidean and natural Euclidean lengths. Hyperbolic trajectories on hyperboloids are identical to straight lines in these concomitant hyperbolic non-Euclidean spaces.

The hyperboloid II with space-like hyperbolae as main geodesics has both natural measures λ_γ and λ_θ. It is mapped by *tangent projecting* to its isomorphic finite tangent model onto the Klein disk (ball at $n > 2$), equivalent topologically to $\langle \mathcal{E}^n \rangle$, inside the Cayley oval of radius R (trigonometric circle at $R = 1$), when $\gamma \to \tanh\gamma$, $\cos\alpha_k = \mathrm{const}_k$, $k = 1, \ldots, n$ (Figure 4). Its external geometry is isometric on the whole and hence homeomorphic to internal Lobachevsky–Bolyai geometry [37–39], with the natural measures and the same parameters n and R [31]. Indeed, the upper and lower parts of a hyperboloid II are reduced by tangent projecting to the same projection, which is the Klein model of the hyperbolic hyperspace onto a part of the projective hyperplane $\langle\langle \mathcal{E}^n \rangle\rangle$ inside the Cayley oval. In the disk, the hyperbolic lines are mapped as straight lines [42; 9, part II, p. 178–193]. For a hyperboloid II, the Klein model is its *projective map* onto the same projective n-dimensional disk of radius R without its border. (This first projective model for the Lobachevsky–Bolyai plane was anticipated by E. Beltrami in 1868 [41].) Note, that tangent projections of two *limit circumferences* from the upper and lower parts of a hyperboloid II are asymptotes inside to the Cayley oval. The geometries on both sheets of a hyperboloid II are different only in the signs of the hyperbolic angle and its directional vector in trigonometric matrices for mirror-symmetric motions with respect to $\langle \mathcal{E} \rangle^n$. Latter statements are also true for the *two antipodal parts of the really two-sheet Lobachevsky-Bolyai space.* If $R = c$, then it is the hyperboloid of velocities $\mathbf{v}^*$ and $\mathbf{v}$ (see in Chs. 5A, 7A).

In the Klein model, the natural measure λ and the angular measure γ of a length (for both geometries) are transformed into the projective *tangent measure* $R\tanh\lambda/R$, identifiable in the projective hyperplane $\langle\langle\mathcal{E}^n\rangle\rangle$ with the Euclidean natural measure inside the Cayley oval. This projective measure is bounded by the radius R of the oval. If $R\to\infty$, then the disk of the Klein model together with the hyperboloid II and the Lobachevsky–Bolyai space are transforming into the infinite Euclidean space $\langle\mathcal{E}^n\rangle$.

Centered hyperbolae passing through the hyperboloid center C_{II} are mapped in the model into the diametrical straight lines passing through the center O with the same angle; non-centered hyperbolae are mapped into the chords (Figure 4). Express connection of the natural length λ_{23} of a segment between two points on some centered geodesic line and the Euclidean tangent projective lengths $R\tanh\gamma = R\tanh(\lambda/R)$ in the Klein model. The segment is formed as follows (see above and Rule 2 in sect. 6.2):

$$\mathbf{u}_2 = \{roth\,\Gamma_{12}\}\cdot\mathbf{u}_1,\ \mathbf{u}_3 = \{roth\,\Gamma_{12}\}\cdot\{roth\,\Gamma_{23}\}\cdot\mathbf{u}_1 \to\ \mathbf{u}_{23} = \mathbf{u}_3 - \mathbf{u}_2,\ \Gamma_{23} = \Gamma_{13} - \Gamma_{12}.$$

Diametrical lines inside the Cayley oval has the center O, which is the center of projecting in $\tilde{E}_1$ and the origin for counting the tangent function. Then the Euclidean length by the same centered tangent measure inside the absolute oval of the line segment is $R(\tanh\gamma_{13} - \tanh\gamma_{12})$ for the base $\tilde{E}_1$. Its non-Euclidean length by the natural measure is $\lambda_{23} = R\gamma_{23}$. Hence, there holds:

$$\lambda_{23} = \lambda_{13} - \lambda_{12} = R\cdot(\gamma_{13} - \gamma_{12}) = R\,\gamma_{23} = R\cdot[\operatorname{artanh}(\tanh\gamma_{13}) - \operatorname{artanh}(\tanh\gamma_{12})] =$$

$$= \frac{1}{2}\cdot R\cdot\left[\ln\frac{1+\tanh\gamma_{13}}{1-\tanh\gamma_{13}} - \ln\frac{1+\tanh\gamma_{12}}{1-\tanh\gamma_{12}}\right] = R\cdot\ln\sqrt{\frac{(R+R\tanh\gamma_{13})(R-R\tanh\gamma_{12})}{(R-R\tanh\gamma_{13})(R+R\tanh\gamma_{12})}}.$$

The formula corresponds to the Rule of summing collinear hyperbolic motions (Ch. 5A). If *straight lines are non-centered*, then the values for distances along them and angles between them are modified, the general formulae are given in Ch. 7A of Appendix. Put here $\gamma_{12} = 0, \gamma_{13} = \gamma$, then $\lambda_{23} = R\gamma$. If $\gamma = \lambda/R \to 0$, then either $R\to\infty$ (see above) or $\lambda\to 0$. In both variants $R\tanh\lambda/R \to \lambda$, and the natural measure of length $\lambda = R\gamma$ became equivalent to the Euclidean projective measure in $\langle\langle\mathcal{E}^n\rangle\rangle$: $\lambda \equiv l$!

Corollary 1. *The general n-dimensional geometry of a hyperboloid II is hyperbolic with additional spherical rotations, it is Lobachevsky-Bolyai and Riemann geometry of a constant negative curvature with Euclidean topology and infinitesimally Euclidean.*

The hyperboloid I with time-like hyperbolae is mapped by *cotangent projecting* to its isomorphic infinite cotangent model onto the *projective n-dimensional double ring of internal radii R* without its internal borders outside the double Cayley oval, when $\gamma \to \coth\,\gamma$, $\cos\alpha_k = \mathrm{const}_k$, $k = 1,\ldots,n$ (Figure 4). It has measures λ_γ and λ_θ too.

It associates with the hyperboloid II tangent model, as here hyperbolae are mapped into straight lines too. Indeed, conjugate hyperbolic lines on concomitant hyperboloids I and II may be interpreted as a quadrohyperbola in a pseudoplane (see it at Figure 3). The pseudoplane is determined by two coupled eigenvectors (along isotropic diagonals) of a hyperbolic rotation matrix as its trigonometric subspace with the axes x and $\overrightarrow{y}^{(1)}$.

In result of projecting, the pseudoplane with the included quadrohyperbola cats the *projective two-sided hyperplane* $[\langle\langle\mathcal{E}^n\rangle\rangle]$ along these four straight lines as maps in the tangent-cotangent projective models inside and outside the double Cayley oval.

For realization of opportunities of this cotangent model (see at Figure 4), let choose on the hyperboloid I any certain point M. Its cotangent projection on the open projective ring is the point M'. This is an isomorphic mapping of the hyperboloid I into the projective ring outside the double Cayley oval. Next, draw from the point M two tangent rays (a cone at $n > 2$) to the oval on both sides. They form an internal angular sector, where we may see the direct cotangent projections of the set of geodesic hyperbolae passing through the point M on the given hyperboloid. These rays are the same projections of horocycles on the hyperboloid, also passing through the point M. Outside this angular sector, there are ellipsoidal closed cotangent projections passing through the point M' of the set of geodesic ellipsoidal closed curves passing through point M on the given hyperboloid. The further point M' is from the oval, the smaller this angular sector. If point M' is located on an oval, then there are two rays as the same projections of horocycles on the hyperboloid, also passing through the point M. In this figure, we also show intuitive correspondences of geodesics on both hyperboloids. We will explain these correspondences more clearly below on the cylindrical model.

For a hyperboloid I the pseudo-Euclidean measure of length $\lambda = R\gamma$ of a hyperbolic arc is transformed into the *projective measure* $R \coth \lambda/R$ of a segment, identifiable in the projective two-sided hyperplane with the Euclidean measure. Further we have:

$$\mathbf{v}_2 = \{roth\,\Gamma_{12}\}\cdot\mathbf{v}_1,\ \mathbf{v}_3 = \{roth\,\Gamma_{12}\}\cdot\{roth\,\Gamma_{23}\}\cdot\mathbf{v}_1 \rightarrow\ \mathbf{v}_{23} = \mathbf{v}_3 - \mathbf{v}_2,\ \Gamma_{23} = \Gamma_{13} - \Gamma_{12}.$$

Express connection of the natural length λ_{23} of a segment between two points on some centered geodesic line and the Euclidean cotangent projective lengths $R \coth\gamma = R\coth(\lambda/R)$, i. e., with $[-R(\coth\gamma_{13} - \coth\gamma_{12})]$ in the base $\tilde{E}_1$ in the plane cotangent model. The segment is formed as follows (see also Rule 2 in sect. 6.2):

$$\lambda_{23} = \lambda_{13} - \lambda_{12} = R\cdot(\gamma_{13} - \gamma_{12}) = R\,\gamma_{23} = R\cdot[\operatorname{arcoth}(\coth\gamma_{13}) - \operatorname{arcoth}(\coth\gamma_{12})] =$$

$$\lambda_{23} = R \ln\sqrt{\frac{(\coth\ \gamma_{13}+1)(\coth\ \gamma_{12}-1)}{(\coth\ \gamma_{13}-1)(\coth\ \gamma_{12}+1}} \equiv R\cdot\ln\sqrt{\frac{(1+\tanh\gamma_{13})(1-\tanh\gamma_{12})}{(1-\tanh\gamma_{13})(1+\tanh\gamma_{12})}}.$$

These identical formulae for a distance, expressed with respect to the Cayley oval for *collinear motions*, correspond together to the tangent and cotangent projective models.

A hyperboloid I and corresponding to it the some real-valued hyperbolic space have topology of an open region as the double ring outside the two Cayley ovals (without them) in the closed whole projective hyperspace $[\langle\langle\mathcal{E}^n\rangle\rangle]$. On the whole, the region is equivalent topologically to cylindrical space $\langle\mathcal{C}^n\rangle$. The double ring is produced in $[\langle\langle\mathcal{E}^n\rangle\rangle]$ as the united map continuously through the conventional infinitely far border between two sides of this projective hyperplane (in its upper and lower halves). (This conventional infinite border is also the same cotangent map of the *equator* of the hyperboloid I.) Topology of a double ring and a cylinder is identical. If $R \rightarrow \infty$, the hyperboloid I is transforming into the infinite cylindrical pseudo-Euclidean space (but its cotangent projection is transforming into the infinite Euclidean double ring). If $R = c$, then it is the hyperboloid of supervelocities $\mathbf{s}^*$ and $\mathbf{s}$ (see in Chs. 4A, 6A).

Corollary 2. *The general n-dimensional geometry of a hyperboloid I is hyperbolic with additional spherical rotations, it is pseudo-Riemannian geometry of a constant negative curvature with cylindrical topology and infinitesimally pseudo-Euclidean.*

A more descriptive isomorphic *finite tangent model* of **the hyperboloid I** is realized as its tangent projection onto a central part of the *Closed infinite projective cylindrical pseudo-Euclidean hypersurface* $[\langle\langle \mathcal{C}^n \rangle\rangle]$ *with the same radius* R, centered in $\tilde{E}_1$ along the axis $\overrightarrow{y}^{(1)}$. This lateral cylindrical segment with radius R and height $\pm R$ is bounded on the hypersurface from above and below by two Cayley absolute ovals without them. From the trigonometric point of view, this model is the *tangent map* (see Figure 4) at $\gamma \to \tanh \gamma$, $\cos \alpha_k = \mathrm{const}_k$, $k = 1, \ldots, n$, i. e., the finite tangent projection outside two *trigonometric circles* of the radius R onto the projective cylindrical hypersurface.

For a hyperboloid I, this map is the projective *Special cylindrical model*, realized on the lateral cylindrical pseudo-Euclidean hypersurface $[\langle\langle \mathcal{C}^n \rangle\rangle]$, which consists conventionally of two adjacent parts (upper and lower) with the heights $\pm R$, where hyperbolae are mapped in straight lines under inclination $\varphi_R > |\pi/4|$. The hyperbolic measures λ and γ of a length are transformed into the *tangent projective measure* $R \tanh a/R$, identifiable in the projective hypersurface with pseudo-Euclidean measure. This model is topologically identical to the open cylindrical region outside the two Cayley ovals without them. It includes also the conventional internal border between upper and lower sides of this model and this hyperboloid I as a spherical n-equator. *This cylindrical tangent model is ideal for projective summing geodesic time-like hyperbolic ("straight") and space-like (ellipsoidal) motions in a hyperbolic geometry with cylindrical topology.* If $R \to \infty$, the hyperboloid with its model are transforming into the infinite $\langle \mathcal{C}^n \rangle$.

Both flat and cylindrical models of a hyperboloid I are conventionally two-sided, as they are divided *not topologically* into halves, with positive and negative values of y. Passage from one side to another of the models as well as passage through the equator of the hyperboloid I are accomplished by free transition of this conventional border.

The united *Whole Cylinder-model* of the hyperboloids I and II consists of two parts: (1) the Special cylindrical model of I as the lateral segment of the cylinder of radius R, and (2) on the heights $\pm R$ of this cylinder, the two Kleinian disks of radius R of the flat tangent model of II as upper and lower bases of the same cylinder. Two parts of a such united big model are situated onto both these whole projective hypersurfaces $[\langle\langle \mathcal{C}^n \rangle\rangle]$ and $[\langle\langle \mathcal{E}^n \rangle\rangle]$. The Whole Cylinder-model has obviously the spherical topology. For both these hyperboloids and their trigonometric models, the dividing hypersurface and its tangent-cotangent projection as the double $(n-1)$-dimensional Cayley oval (or trigonometric circle at $R = 1$) are *automorphisms*. And in the universal base $\tilde{E}_1$, this oval is determined by the equation: $x_1^2 + \cdots + x_n^2 = R^2$.

Main Inference. *Tensor trigonometry of* $\langle \mathcal{P}^{n+1} \rangle$ *applied to the unity Minkowskian hyperboloids I and II as trigonometric objects, embedded into the space, with exactness up to scale parameter* R*, is equivalent to the united external and internal hyperbolic non-Euclidean geometry on these concomitant hyperboloids with the radii* $\pm R$*. Their geometries are isometric to the internal real-valued hyperbolic non-Euclidean geometries with affine and cylindrical topologies. And these hyperboloids are the simplest geometric objects for isometric interpretation in the large of these real geometries.*

Dissecting at $n = 2$ the combined projective tangent *Whole Cylinder-model* by a centered cutting plane under a certain angle $\varphi_R(\gamma)$ to $\langle \mathcal{E}^n \rangle$. If the angle is zero, we have an equivalent map of the real equator of the hyperboloid I. If the angle less $\pi/4$, we get on the cylindrical hypersurface one (at $n = 2$) closed ellipsoidal curve as a map of the geodesic space-like ellipsoidal curve on the hyperboloid I. If the angle is $\pi/4$, we get on the cylindrical hypersurface two isotropic straight segments as a map of two horocyrcles (a parabolic trajectory at $n = 2$) on the hyperboloid I with zero metric. If the angle more $\pi/4$, we get four one-to-one connected straight segments: two ones on the cylindrical hypersurface as a map of two time-like hyperbolae on the hyperboloid I and two ones on the two disk-bases as a map of two space-like hyperbolae on the hyperboloid II. On the model, they form an united closed projective quadrangle cycle of the four as if connected infinite parallel lines. Its four apexes lie on two Cayley ovals. The similar quadrangle may be realized in the united flat model, but less descriptively.

A geometric sum in $\langle \mathcal{P}^{n+1} \rangle$ of the two concomitant complex-valued n-dimensional hyperboloids with the isotropic hypersurface as well as a geometric sum of the two concomitant real-valued hyperbolic n-dimensional spaces (including the antipodal part of the really two-sheet Lobachevsky-Bolyai space – see above) with the dividing them hypersurface can be mapped entirely onto the whole two-sided projective n-dimensional hypersurface $[\langle\langle \mathcal{E}^n \rangle\rangle]$ with topology of n-sphere. (The tangent-cotangent projections of four *limit circumferences* from the upper and lower parts of these two hyperboloids are four asymptotes inside and outside to the double Cayley oval.) The same summands as a geometric sum in $\langle \mathcal{P}^{n+1} \rangle$ can be mapped entirely onto the Whole Cylinder-model of the hyperboloids I and II also with topology of n-sphere. That is, this geometric sum may be represented by different manners.

This completely closed construction is the *United hyperbolic hypersurface of 3 sheets* in $\langle \mathcal{P}^{n+1} \rangle$ with its finite projective tangent map (as "the world in a water drop"). On the hypersurface, the two concomitant hyperbolic geometries should be considered as *United hyperboloidal geometry* with the complete Lorentzian group of homogeneous motions – pure hyperbolic and pure orthospherical ones, that is, at $|R| = \text{const}$. In sect. 6.4 we considered analogously the hyperbolic trigonometry on a pseudoplane with solving interior and exterior right triangles, where imaginary and real hyperbolae were as prototypes of Minkowskian hyperboloids in $\langle \mathcal{P}^{n+1} \rangle$. If $R = c$, the flat tangent-cotangent model is a vector space of coordinate velocities $\mathbf{v}$ and supervelocities $\mathbf{s}$.

Note, that a hyperboloid I and a Beltrami pseudosphere are the geometric objects in one parameter R (see more in Ch. 6A). They are homeomorphic one to another. The pseudosphere was discovered by Ferdinand Minding in 1838 [40] as a surface of constant negative Gaussian curvature. These surfaces and their general n-dimensional geometries are isometric, and they have additional identical $(n-1)$-dimensional orthospherical geometry. In Ch. 6A we constructed a tractrix with a Beltrami pseudosphere in the especial quasi-Euclidean space with their pure trigonometric equations in one parameter R similarly to circles and spheres, and with applications in STR.

The idea about possibility of rigorous geometry, in which the famous Fifth Euclidean Postulate may be not hold, or the Hypothesis of the acute Saccheri angle [32) can be valid on a "certain imaginary sphere" was expressed first by J. Lambert in 1766 [33]. Later it became more precise: the first property is the feature of *geometry in the large*, the second one is the feature of geometry *in the small.* They are mutually connected in geometry with completely free motions of figures. C. Gauss made some drafts in the direction [34]. F. Schweikart introduced the factor parameter R of the geometry [35].

F. Taurinus (his nephew) suggested a model of such geometry on a *hypothetic sphere of imaginary radius* similar to geometry on a real sphere [36]. So, Taurinus proved internal consistency of its planimetry ($n = 2$). Intuitive Lambert–Taurinus geometry anticipated non-Euclidean geometry on a hyperboloid II and a Lobachevsky–Bolyai plane as its real-valued isometric analog [37-39]. E. Beltrami realized it [41] as geometry in the small on a real pseudosphere in $\langle \mathcal{E}^3 \rangle$ as a peculiar surface with constant negative curvature (it was discovered and analyzed earlier by F. Minding [40]). The Klein projective model [42] reduced the problem of its non-contradiction on the whole to that of Euclidean geometry. D. Hilbert proved that 2-dimensional Lobachevsky–Bolyai geometry can not be realized on the whole on any *non-peculiar* Riemannian surface embedded into the 3-dimensional Euclidean space, as the Gaussian interior geometry [42]. But it does not mean that this geometry can not be realized on a saddle Riemannian surface in a $(3+k)$-dimensional Euclidean space. Such surface must have constant negative curvature. If its embedding into an Euclidean space of minimal dimension is possible, then this should mean solvability of the *Beltrami problem.* The first results in this direction was obtained for $\langle \mathcal{E}^6 \rangle$ and more for $\langle \mathcal{E}^{6n-5} \rangle$ by D. Blanusha in 1955 [43]. Later other authors made their contributions, particularly, E. Rosendorn in 1960 for $\langle \mathcal{E}^5 \rangle$ [44]. But the Beltrami problem was solved peculiarly due to embeddability of n-dimensional hyperbolic non-Euclidean spaces into $\langle \mathcal{P}^{n+1} \rangle$!

Definition of a n-dimensional Riemannian surface and its geometry is not interrupted of an enveloping Euclidean superspace, but it is interrupted only of its dimension, which a priori may be in $[(n+1), \infty)$. A posteriori the dimension may be quite definite. However, dimension of a Riemannian surface is the same for all its homeomorphisms, it is equal to dimension of a tangent Euclidean space. The latter generalized an one-dimensional *tangent to a curve*, but dimension of its embedding may be in $[2, \infty)$. So, an infinite regular curve of *constant spherical curvature* can not be realized on a plane, however, it is realizable in the 3-dimensional Euclidean space as a screw line. On the contrary, a similar curve of *constant hyperbolic curvature* is realizable in a pseudoplane as a hyperbola. Isometric images of the non-Euclidean geometry in different surfaces (a hyperboloid II upper, a Lobachevsky–Bolyai space and a real-valued Riemannian surface of constant negative curvature) differ very much in their visuality and complexity. But the cylindrical hyperbolic geometry may be realized isometrically both in $\langle \mathcal{P}^{2+1} \rangle$ on the hyperboloid I and in the real-valued *Especial quasi-Euclidean space* $\langle \mathcal{Q}^{2+1} \rangle$ (Ch. 6A) on the Beltrami pseudosphere with the same R.

Besides, there is isomorphism of all admissible rotations in the enveloping pseudo- or quasi-Euclidean spaces and motions on the embedded into them hyperbolic or spherical surfaces with the fixed radius R in both these cases, what is inferred by the rotational and projective tensor trigonometry!

The point elements $\mathbf{v}$ and $\mathbf{u}$ on hyperboloids I and II in $\langle \mathcal{P}^{n+1} \rangle$ are determined by their pseudo-Cartesian coordinates (x_k, y), $k = 1, \ldots, n$, for example, in the base $\tilde{E}_1$ (Figure 4). Any element on a certain Minkowski hyperboloid with its radius-vector may be also uniquely determined by four special parameters in the base $\tilde{E}_1$ as follows: $\mathbf{u} = R\mathbf{i}$ for a hyperboloid II (with $\rho = +iR$) and $\mathbf{v} = R\mathbf{j}$ for a hyperboloid I (with $\rho = +R$), where $\mathbf{i} = (\sinh\gamma \cdot \mathbf{e}_\alpha,\ \cosh\gamma)$ and $\mathbf{j} = (\cosh\gamma \cdot \mathbf{e}_\alpha,\ \sinh\gamma)$ are unity time-like and space-like radius-vectors and $\mathbf{e}_\alpha = \langle \cos\alpha_k \rangle$ is their Euclidean vector of directional cosines $\cos\alpha_k$, $k = 1, \ldots, n$ (for vectorial sine on II or vectorial cosine on I). In brackets, the orthoprojections in $\tilde{E}_1$ of these radius-vectors are given. For the point $\mathbf{u}$ on the hyperboloid II (on its upper part), $\overrightarrow{y}$ is the frame axis for counting absolute value of the scalar hyperbolic motive angle γ formed with its radius-vector $\mathbf{i}$. Therefore, n independent coordinates are sufficient, because $\sum_{k=1}^{n} \cos^2\alpha_k = 1$. For the point $\mathbf{v}$ on the one-sheet hyperboloid II, its frame axis lies in $\langle \mathcal{E}^n \rangle$, and it is always symmetrical with respect to the axis $\overrightarrow{y}$ relatively to the dividing isotropic conic hypersurface. It forms the angle γ with $\mathbf{j}$. A pair γ and $\mathbf{e}_\alpha$ determines a movable base $\tilde{E}_m = \{\mathbf{j}, \mathbf{i}\}$.

Recall that in tensor trigonometry, in general, we use tensor, vector and scalar angles. The tensor angle with its functions, including the tensor of motion and the tensor of deformation, contain complete geometric information. Tensor functions in angles Γ and Υ can be reduced to their adopted canonical forms. We have in $\langle \mathcal{P}^{n+1} \rangle$ the decompositions from (31A) into functions in a tensor angle similarly to (324)-(327):

$$\left.\begin{array}{l} roth\ \Gamma = \cosh\Gamma + \sinh\Gamma = \coth\ (\pm\Upsilon) + \mathrm{csch}\ \Upsilon = \overline{roth}\ \Upsilon, \\ defh\ \Gamma = \mathrm{sech}\ \Gamma + \tanh\Gamma = \tanh(\pm\Upsilon) + \mathrm{sech}\ \Upsilon = \overline{defh}\ \Upsilon. \end{array}\right\} (\text{with} \{I^{\pm}\}) \quad (496)$$

For complementary angles, there holds: $\gamma + \upsilon < \delta = \infty$ – see (361) in sect. 6.4.

Corollary 3. *In a right triangle in* $\langle \mathcal{P}^{n+1} \rangle$ *with the acute angles* γ, υ *and right angle* δ *there hold*: $\gamma + \upsilon < \delta$; $\gamma = \upsilon \leftrightarrow \gamma = \omega \leftrightarrow \upsilon = \omega$, $\Gamma = \Upsilon \Leftrightarrow \Gamma = \Omega \Leftrightarrow \Upsilon = \Omega$.

From (331), (359)–(361), we obtain the following tensor analogs of spherical formulae (175), (202), (206), (182), (208), with respect to metric reflector tensor $\{I^{\pm}\}$ of $\langle \mathcal{P}^{n+1} \rangle$

$$\sinh(\Gamma, \Upsilon) = \mathrm{csch}\ (\Upsilon, \Gamma), \quad \cosh(\Gamma, \Upsilon) = \coth\ (\pm\Upsilon, \Gamma).$$

$$[\tanh(\pm\Gamma, \Upsilon) = \mathrm{sech}(\Upsilon, \Gamma).]$$

$$\cosh^2(\Gamma, \Upsilon) - \sinh^2(\Gamma, \Upsilon) = I = \coth^2(\Upsilon, \Gamma) - \mathrm{csch}^2(\Upsilon, \Gamma) - \text{invariants for } \{I^{\pm}\}.$$

$$\tanh^2(\Gamma, \Upsilon) + \mathrm{sech}^2(\Gamma, \Upsilon) \equiv \sinh^2(\Phi, \Xi) + \cosh^2(\Phi, \Xi) = I - \text{quasi-invariant for} \{I^{+}\}.$$

Invariants and quasi-invariants for vector functions in the same angles are similar to ones for scalar functions of angles as in (361), because their valency is equal to 1. Their modulus are bond by scalar relations (359)–(361).

Note: After the functions exchange in (496) or in (324) and further in matrix functions as operation $\Upsilon \leftrightarrow \Gamma$, the new cotangent-cosecant function in the angle Γ (compatible with tensor $\{I^{\pm}\}$ too) gives hyperbolic rotation at the angle $\Upsilon(\Gamma)$! (See the similar note in sect. 5.8 also for spherical rotations.) This is spread into $\langle \mathcal{P}^{n+1} \rangle$.

These connections of Γ and Υ in pseudo-Euclidean right triangles in $\langle \mathcal{P}^{n+1} \rangle$ rise to the fact that cross projecting in determination of cotangent and cosecant of these angles is equivalent to usual orthoprojecting in determination of cosine and sine of the complementary angles Υ and Γ, what was shown and used in Ch. 12 at Figure 4!

As we have seen, tensor trigonometry in its vector projective forms gives descriptive isomorphic models of both concomitant hyperbolic non-Euclidean geometries with unity trigonometric objects – Minkowskian hyperboloids of unity radius. Moreover, tangent and cotangent models display hyperbolic geodesic motions or segments into rectilinear mappings onto a projective plane or a projective cylinder. However, as far as the mapping of the first differentials of any geodesic motions is concerned, they are always rectilinear as their linear part. Therefore, for the mappings of the first differentials onto the projective tangent plane or the projective tangent cylinder, any three-dimensional (at $n = 3$) vector trigonometric functions paired with scalar functions can be used. They form the above invariants in the pairs sine-cosine, cosine-sine, cotangent-cosecant, cosecant-cotangent in hyperbolic geometries.

The vector nature of such linear mappings allows us in external geometries to impart the vector nature to two- and multistep metric forms of the 1st order (traditionally scalar) with the preservation of their scalar characteristics as the modular values of the vectors. But these vectors determine the directions of motions in these forms. Therefore, such vector-scalar metric forms of absolute multistep motions or segments in their geometries can be mapped either into the tangent Euclidean and pseudo-Euclidean plane, or into the tangent Euclidean and pseudo-Euclidean cylindrical surface.

Considered above unity hyperboloids I and II are ideal models for displaying the first metric forms of relativistic motions - the most varied! Even cylindrical enveloping surfaces for swirling motions are in fact also partial fragments of these models. Examples of such partial multistep and integral and simplest types motions are studied in Chs. 5A, 6A, 7A and more in details in 10A.

A similar classification and examples take place also in spherical geometry and on its hyperspheroid with the mapping of metric forms into the tangent Euclidean plane or into the tangent Euclidean cylinder with an axis of the hyperspheroid.

In process of non-collinear multi-step or integral hyperbolic motions in $\langle \mathcal{P}^{n+1} \rangle$ and, in partiular, on these hyperboloids we deal with the secondary orthospherical rotation of the local and final pseudo-Cartesian bases with the non-point objects in them. There is a deep distinction between matrix representations of *rot* Θ and *roth* Γ. For *roth* Γ, the angle γ is counted from the current time-like frame axis $\overrightarrow{y}$ and space-like frame axis x_k. Structure (362), (363) and the pseudoplane of rotation γ are determined by the directional cosines with respect to the Cartesian part of the base.

12.2 Rotations and deformations in elementary tensor forms

Representation of *rot* Θ is defined by its general structure (473). For example, in $\langle \mathcal{P}^{2+1} \rangle$, the rotation *rot* Θ includes the 2×2-block as its elementary spherical rotational cell. In $\langle \mathcal{P}^{3+1} \rangle$, its 3×3-block *rot* $\Theta_{3\times 3}$ represents a orthospherical rotation with *fixed normal axis* $\mathbf{r}_N$ [16, p. 447–448]. Structure of *rot* Θ and the plane of the rotation are determined by the directional cosines of the *normal axis of rotation* $\mathbf{r}_N \in \langle \mathcal{E}^3 \rangle$ with respect to the Cartesian part of the universal trigonometric base $\tilde{E}_1 = \{I\}$.:

$$rot\ \Theta$$

$$\begin{array}{|c|c|c|c|}\hline \cos\theta + \frac{r_1^2}{1+\cos\theta} & -r_3 + \frac{r_1 r_2}{1+\cos\theta} & +r_2 + \frac{r_1 r_3}{1+\cos\theta} & 0 \\ \hline +r_3 + \frac{r_1 r_2}{1+\cos\theta} & \cos\theta + \frac{r_2^2}{1+\cos\theta} & -r_1 + \frac{r_2 r_3}{1+\cos\theta} & 0 \\ \hline -r_2 + \frac{r_1 r_3}{1+\cos\theta} & +r_1 + \frac{r_2 r_3}{1+\cos\theta} & \cos\theta + \frac{r_3^2}{1+\cos\theta} & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array}\ . \qquad (497)$$

Consider the angles Γ and $\overset{\angle}{\Gamma}$ in (495) in Ch. 11 for the cases of direct and inverse orders of two-step hyperbolic motions γ_{12}, γ_{23} (γ_{23}, γ_{12}) with their tensor structure (363) and their directional cosines $\cos\sigma_k$, $\cos\overset{\angle}{\sigma}_k$, $k = 1, 2, 3$: $\mathbf{e}_\sigma = \{\cos\sigma_k\}$, $\mathbf{e}_{\overset{\angle}{\sigma}} = \{\cos\overset{\angle}{\sigma}_k\}$. Applying polar representations (474), (475), from formula (476) we obtain sequentially:

$$\left.\begin{array}{l} rot'\ \Theta_{3\times 3} \cdot \{\mathbf{e}_\sigma \cdot \mathbf{e}'_\sigma\} \cdot rot\ \Theta_{3\times 3} = \mathbf{e}_{\overset{\angle}{\sigma}} \cdot \mathbf{e}_{\overset{\angle}{\sigma}}{}', \\ \mathbf{e}_{\overset{\angle}{\sigma}} = rot'\ \Theta_{3\times 3} \cdot \mathbf{e}_\sigma = rot\ (-\Theta_{3\times 3}) \cdot \mathbf{e}_\sigma, \\ \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\angle}{\sigma}} = \mathbf{e}'_{\overset{\angle}{\sigma}} \cdot \mathbf{e}_\sigma = \cos\ \theta = (tr\ [rot\ \Theta]_{3\times 3} - 1)/2. \end{array}\right\} (\mathbf{e} \cdot \mathbf{e}' = \overleftarrow{\mathbf{e} \cdot \mathbf{e}'}) \qquad (498)$$

In $\langle \mathcal{P}^{3+1} \rangle$, the unity vectors $\mathbf{e}_\sigma$ and $\mathbf{e}_{\overset{\angle}{\sigma}}$, by (498), uniquely determine the vector of spherically normal axis of rotation *rot* $\Theta_{3\times 3}$ as the following vectorial (sine) product:

$$\overrightarrow{\mathbf{r}_N} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \mathbf{e}_{\overset{\angle}{\sigma}} \times \mathbf{e}_\sigma = \begin{bmatrix} \cos\overset{\angle}{\sigma}_2 \cos\sigma_3 - \cos\overset{\angle}{\sigma}_3 \cos\sigma_2 \\ \cos\overset{\angle}{\sigma}_3 \cos\sigma_1 - \cos\overset{\angle}{\sigma}_1 \cos\sigma_3 \\ \cos\overset{\angle}{\sigma}_1 \cos\sigma_2 - \cos\overset{\angle}{\sigma}_2 \cos\sigma_1 \end{bmatrix} = \sin\theta \cdot \overrightarrow{\mathbf{e}_N}, \qquad (499)$$

We have ($det\{\mathbf{e}_{\overset{\angle}{\sigma}}, \mathbf{e}_\sigma, \overrightarrow{\mathbf{r}_N}\} > 0 \to \theta > 0$), i. e., if the triple ($\mathbf{e}_{\overset{\angle}{\sigma}}, \mathbf{e}_\sigma, \overrightarrow{\mathbf{r}_N}$) is *right-handed*, then the angle θ in $\langle \mathcal{E}^2 \rangle$ is counter-clockwise, what corresponds to $\theta > 0$ in (497) and the sign at tensor angle Θ in (497) and (498), due to original defining angle Θ in (476). In $\langle \mathcal{P}^{3+1} \rangle$ and hyperbolic geometries this is **General Rule** *for the sign of* θ *in result of summing non-collinear principal (hyperbolic) rotations* γ_{12}, γ_{23}: $\boxed{sgn\ \theta_{13} = -sgn\ \varepsilon\ !}$, where ε is the angle between principal motions γ_{12} and γ_{23} in the Cartesian base part. (Scalar angles θ and ε act in the same plane perpendicular to $\mathbf{r}_N$ in $\langle \mathcal{E}^3 \rangle$; $\pm\theta \leftrightarrow \pm\varepsilon$.)

For sine θ module in (499): $|\sin\theta| = ||\mathbf{r}_N|| = \sqrt{r_1^2 + r_2^2 + r_3^2}$, $tr\ rot\ \theta = 2(\cos\theta + 1)$. (In $\langle \mathcal{Q}^{2+1} \rangle$ and in spherical geometry for principal rotations: $\boxed{sgn\ \theta_{13} = +sgn\ \varepsilon\ !}$.)

In pseudo-Euclidean Minkowskian spaces, not only hyperbolic rotations as principal ones may be used. Admitted hyperbolic deformations *with respect to a universal base* are of interest too. They have tangent-secant form (496) and canonical structure (364) in the universal base. Deformations are made in the pseudoplane at the same tensor angle Γ. With respect to the base of the diagonal cosine Γ, these matrices and the metric reflector tensor have the following *binary-cell structure* in the eigen pseudoplane:

$$\begin{matrix} \{defh\ \Gamma\}_{can} & & \{roth\ \Gamma\}_{can} & & I^{\pm} \quad (q=1) \\ \begin{bmatrix} \ddots & & & \\ & \operatorname{sech}\gamma & -\tanh\gamma & \\ & +\tanh\gamma & \operatorname{sech}\gamma & \\ & & & \ddots \end{bmatrix} & , & \begin{bmatrix} \ddots & & & \\ & \cosh\gamma & \sinh\gamma & \\ & \sinh\gamma & \cosh\gamma & \\ & & & \ddots \end{bmatrix} & , & \begin{bmatrix} \ddots & & & \\ & +1 & 0 & \\ & 0 & -1 & \\ & & & \ddots \end{bmatrix} . \end{matrix}$$

This structure generates, similarly to (471), the *pure* type of the elementary (as $q = 1$) hyperbolic deformational matrices, for example, in the original base $\tilde{E} = R'_W \cdot \tilde{E}_1$:

$$\left. \begin{aligned} & R_W \cdot \{defh\ \Gamma\}_{can} \cdot R'_W = defh\ \Gamma, \\ & defh'\ \Gamma \cdot defh\ \Gamma = I = defh\ \Gamma \cdot defh'\ \Gamma, \end{aligned} \right\} (det\ defh\ \Gamma = +1.) \qquad (500)$$

Modal matrices R'_W not compatible with $\{I^{\pm}\}$ change it as in (457). The deformation do not belong to the Lorentz group as they do not satisfy (458) or (460). Equalities (500) have the form of (470) for $Rot\ \Theta$, however concordance of the matrices $rot\ \Theta$ and $defh\ \Gamma$ with the metric reflector tensor $I^{\pm}$ are different. Rotations $rot\ \Theta$ are compatible only with respect to the unit block of $I^{\pm}$, deformations $defh\ \Gamma$ are concordant with respect to a certain 2×2-cell with alternating signs. In other words, matrices $rot\ \Theta$ act in planes, matrices $defh\ \Gamma$ act in pseudoplanes. Deformational matrices do not satisfy pseudo-Euclidean metric relation (460), but they satisfy (only one-step) the Euclidean metric relation, this follows from (500) due to analogy (341). For the deformation, its *quasi-Euclidean cross invariant* in the pseudoplane of hyperbolic rotation and deformation is defined as an important parameter (see in sect. 5.10). By spherical-hyperbolic analogy defined in the base $\tilde{E}_1$, we have the circle as in (341):

$$\begin{array}{ccccccc} defh\ \Gamma(\Phi) & \equiv & rot\ \Phi(\Gamma) & \Leftrightarrow & \tanh\Gamma(\Phi) & \equiv & \sin\Phi(\Gamma) \\ & \updownarrow & & & & \updownarrow & \\ roth\ \Gamma(\Phi) & \equiv & def\ \Phi(\Gamma) & \Leftrightarrow & \sinh\Gamma(\Phi) & \equiv & \tan\Phi(\Gamma). \end{array}$$

All the matrices compatible with the metric reflector tensor act here in the same planes and pseudoplanes in the universal base $\tilde{E}_1$. Thus, as initial conditions there hold:

$$defh\ \Gamma \cdot I^{\pm} \cdot defh\ \Gamma = rot\ \Phi \cdot I^{\pm} \cdot rot\ \Phi = I^{\pm} =$$

$$= roth\ \Gamma \cdot I^{\pm} \cdot roth\ \Gamma\ = def\ \Phi \cdot I^{\pm} \cdot def\ \Phi.$$

And four relations in the circle with respect to the universal base $\tilde{E}_1$ hold in hyperbolic as well as spherical geometry. That is why they are represented with angles Γ and Φ of rotation, and their middle reflector tensor is $I^{\pm} \equiv Ref\ \{\cos\tilde{\Phi}\}^{\ominus} \equiv Ref\ \{\cosh\tilde{\Gamma}\}^{\ominus}$.

In the pseudo-Euclidean trigonometry in $\langle \mathcal{P}^{n+1} \rangle$ and external hyperbolic geometry on hyperboloids, with respect to admissible pseudo-Cartesian bases, defining relations (348), (349) hold; in quasi-Euclidean trigonometry in $\langle \mathcal{Q}^{n+1} \rangle$ and external spherical geometry on hyperspheroid, with respect to admissible quasi-Cartesian bases, defining relations (257), (258) hold. Between them, the simple trigonometric relations act in the universal base with the use of functions $\varphi(\gamma)$ and $\gamma(\varphi)$ introduced in Ch. 6.

Recall also the following distinction of tensor deformations: Rule 2 for summation of trigonometrically compatible angles-arguments does not hold for deformations (though deformational matrices with compatible angles commute with each other). However these matrices are used for *cross* non-Cartesian *projecting* in $\langle \mathcal{P}^{n+1} \rangle$. Cross projecting in the space $\langle \mathcal{P}^{3+1} \rangle$ is the complete mathematical model for Lorentz contraction of a moving object extents coaxially to the direction of its physical motion in $\langle \mathcal{E}^3 \rangle$.

12.3 The special mathematical principle of relativity

All statements concerning ⟨Euclidean, quasi-Euclidean, pseudo-Euclidean⟩ geometry without its affine contents have covariant forms in any ⟨Cartesian, quasi-Cartesian, pseudo-Cartesian⟩ base of an ⟨Euclidean, quasi-Euclidean, pseudo-Euclidean⟩ space. So, any geometry with the simplest quadratic invariant as a set of its own theorems does not depend in part of these theorems on a choice of its admissible base. In other words, ⟨Euclidean, quasi-Euclidean, pseudo-Euclidean⟩ geometries conserve covariant forms under their admissible transformations as ⟨orthogonal, quasi-orthogonal, pseudo-orthogonal⟩ and translations.

The *special mathematical principle of relativity* takes place in any flat quadratic-type geometry – so, in the Minkowskian geometry. In STR, it is a mathematical source for the great Poincaré Principle of Relativity: all physical laws have covariant forms in any uniformly rectilinearly moving frames of reference up to nearly light velocity, i. e., under Lorentz transformations. *Physical-mathematical isomorphism* unites two principles. The Lorentzian transformations do not change the absolute Minkowskian space-time and its dividing asymptotic conic hypersurface called the *light cone*:

$$\langle \mathcal{P}^{3+1} \rangle \equiv \langle \mathcal{E}^3 \rangle^{(k)} \boxtimes \overrightarrow{ct}^{(k)} \equiv \text{CONST}, \quad (n = 3,\ q = 1);\ \Delta ct > 0\ !$$

Contrary, the subspaces $\langle \mathcal{E}^3 \rangle$ and $\overrightarrow{ct}$ are relative and are changed under the Lorentzian transformations of the base, although these subspaces and their coordinate axes stay in the same external or internal cavities of the cone. Here the space $\langle \mathcal{E}^3 \rangle$ and the time-arrow $\overrightarrow{ct}$ are relative, but mutually dependent as direct orthogonal complements in $\langle \mathcal{P}^{3+1} \rangle$. According to (495, 462), there exists an one-to-one correspondence between them. Therefore, for STR this formula is the mathematical expression of the Poincaré–Einstein Law about relativity, mutual dependence and unity of the space and the time! *Pay special attention to the fact that the space is just as relative as the time.*

This is the mutual cause, that in the universal base $\tilde{E}_1$ in the pseudo-plane $\langle\mathcal{P}^{1+1}\rangle$ of the rotation $roth\Gamma$, the space and time axes of the base $\tilde{E}_m$, connected with a moving point M, are dilated with the identical coefficient $\cosh^{-1}\gamma(v)$. This directly explains the relativistic Einsteinian dilation of time and Lorentzian contraction of extent, and, due to Lorentz, ensured invariance of the Maxwell electromagnetic wave equation [46].

In the 4-dimensional Lagrangian space-time $\langle\mathcal{L}^{3+1}\rangle \equiv \langle\mathcal{E}^3 \oplus \overrightarrow{t}\rangle, \Delta t > 0$, laws of the classical mechanics are invariant with respect to a choice of inertial frames of reference, or under Galilean transformations. This is the physical-mathematical form of the Galilean special principle of relativity. From the mathematical point of view, Lagrangian space-time is the special case ($n = 3$, $q = 1$) of a Euclidean-affine space $\langle\mathcal{E}^n \oplus \mathcal{A}^q\rangle$ with general Galilean transformations and the Euclidean-affine geometry. This total binary space is absolute too, i. e., invariant under these transformations. Homogeneous Galilean transformations do not change its Euclidean subspace $\langle\mathcal{E}^3\rangle$ and scalar time t: they are also absolute in Newton sense. However, the time-arrow $\overrightarrow{t}$ is not invariant as a directed world line in $\langle\mathcal{L}^{3+1}\rangle$. It is subjected to *mid-rotations*, i. e., rotations middle between spherical and hyperbolic ones with respect to $\langle\mathcal{E}^3\rangle$: rotations at angles φ with compensating dilations with coefficients sec φ, where $\varphi = \arctan$ v (it is defined only by a physical velocity). In addition, $\langle\mathcal{E}^3\rangle$ and $\overrightarrow{t}$ may be subjected to parallel translations. The Lagrangian space and time-arrow form an unity, as their sum is direct, but is not orthogonal; and, hence, they are not mutually dependent. So, theorems of the Euclidean-affine geometry do not depend on a choice of a binary Cartesian-affine base with choosed scale factors. The Euclidean-affine geometry of Lagrangian space-time of index 1 corresponds to the Galilean principle of relativity. Its affine space projection parallel to a time-arrow and divided by time-factor (with three homogeneous Klein's coordinates) is a vector Euclidean space of physical velocities.

From the other side, continuous transformations in Minkowskian space-time $\langle\mathcal{P}^{3+1}\rangle$ carry out relativistic elementary hyperbolic principal rotations and elementary ortho-spherical secondary ones in accordance with its reflector tensor $\{I^{\pm}\}$. Moreover, the space-time fixations of any geometric objects are subjected to relativistic hyperbolic deformations, which are described also completely in the cross base $\tilde{E}_{i,j}$ with immobile Observer. Relativistic nature of the Lorentz–Poincaré–Einstein transformations takes place according to hyperbolic nature of principal rotations and deformations. With Einsteinian physical approach [48], STR was created with the use of definition of simultaneity. This definition corresponds to the trigonometric theorem in $\langle\mathcal{P}^{3+1}\rangle$, that the median and height in the pseudo-Euclidean right triangle are identical, which motivated the quadratic metric in the space-time of STR.

What else is important, the abstract and concrete spherical–hyperbolic analogies, with respect to the universal base, connects spherical and hyperbolic non-Euclidean geometries of the constant radius R, and enables one to describe them in the enveloping spaces $\langle\mathcal{P}^{n+1}\rangle$ and $\langle\mathcal{Q}^{n+1}\rangle$ by the similar approaches based on the tensor trigonometry.

In the Lobachevsky–Bolyai geometry, a magnitude R is called *the Gauss–Schweikart constant* ($1/R = K$ characterizes the distortion with respect to a flat Euclidean space); iR is the radius of a "hypothetical Lambert–Taurinus imaginary sphere" , realized in 1907 by H. Minkowski as the upper sheet of his hyperboloid II. The J. Lambert's original idea and its development by F. Taurinus pointed out the simplest and natural way for realization of the whole hyperbolic non-Euclidean geometry on the complex-valued sphere of imaginary radius iR. This way became quite possible after introducing pseudo-Euclidean space by H. Poincaré in 1905 and later H. Minkowski as basis of STR.

A. Sommerfeld in 1909 established hyperbolic nature of velocities summation [62]. V. Varičak in 1910 conjectured that the velocities summation law is identical to the segments summing in Lobachevsky–Bolyai geometry [65]. F. Klein constructed the theoretical basis for this law, when he proved that the Lorentzian group in STR is equivalent to the group of motions in the Lobachevsky–Bolyai space. He interpreted the geometry in the large (1871) on the model inside the Cayley oval in the projective plane [42], which was anticipated by E. Beltrami in 1868 [41]. In addition, in 1928 F. Klein realized the projective model in pseudo-Euclidean space.

These various interpretations showed that quite different ways are possible for constructing and using the same non-Euclidean geometries in curved and flat spaces and space-time. Thus it is necessary to choose the most simplest and descriptive forms for studying and using the geometries and their applications in physical theories, what, for example, the the tensor trigonometry gives by its clear tools. Due to this all, as important applications, the tensor trigonometry interpretations of various motions in non-Euclidean geometries and models of kinematics and dynamics in the theory of relativity became possible and are exposed in Appendix to its exposed fundamentals.

APPENDIX

Trigonometric models of motions in Special Theory of Relativity and in non-Euclidean Geometries

Preface

In Appendix we consider a lot of concrete applications of the tensor trigonometry in its so called *elementary form* (with single eigen principal angle γ or φ and single secondary angle θ). It is useful for theoretical analysis of the motions in pseudo- and quasi-Euclidean spaces with $q = 1$ and in embedded into them metric spaces of constant radius (or Gaussian curvature) with non-Euclidean geometries. The main idea of the last consists in fact, that non-Euclidean geometries and the tensor trigonometry of pseudo- and quasi-Euclidean spaces at parameters n and R exist in one-to-one correspondence! Hence, results can be represented in the same trigonometric forms.

In Chapter 1A, for illustration of these opportunities, all the main postulates and notions of the Special Theory of Relativity (STR) from its Einsteinian *physical version* [48] are represented in scalar hyperbolic trigonometric forms. Further we use *original geometric group approach* of Poincaré in 1905 [47] and then of Minkowski in 1909 [49]. Stated due to this approach *isotropy* and *homogeneity* of the space-time of events allow to use of the tensor trigonometry in most wide aspects, than only in its scalar form. This was impossible before in the non-isotropic space-time of Lagrange. So, the Einsteinian postulates including his definition of events simultaneity, as this was shown here, are the trigonometric theorems in the Minkowskian pseudo-Euclidean space-time.

In the frame of trigonometric aspects, we give renewed and universal conception of the *covariant parallel angle* for both these types of non-Euclidean geometries in the hyperspaces of constant curvature, embedded respectively into quasi- and pseudo-Euclidean spaces. Due to this conception, initial definitions of both types of non-Euclidean geometries are realized through a choice of a parallel angle, whether spherical or hyperbolic one, with corresponding to their nature two variants of behavior of parallel lines. As it was demonstrated (the end of Ch. 1A), the classic Lobachevskian parallel angle is strictly correct in the case of a spherical geometry, because it has a spherical nature. The *universal parallel angle* (i. e., as the *motion angle* too!) is defined in the universal base $\tilde{E}_1$ of the enveloping or tangent space. In STR the parallel angle is defined also in the base $\tilde{E}_1$, which corresponds to the frame of reference for immovable Observer N_1 in Minkowskian space-time. Then it is identical to the *hyperbolic physical motion angle* γ, defined in its scalar form only by scalar velocity v as $\gamma = \text{artanh}\, v/c$.

The basic parameters of motions in the tensor trigonometric versions of the non-Euclidean geometries and STR are tensor angles of principal and secondary rotations in elementary forms (313, 314), (362, 363) and (497). A principal hyperbolic tensor angle Γ is argument of rotational matrix-function $roth\ \Gamma$ as the tensor of motion in STR. Also in STR the concomitant hyperbolic tensor of deformation in canonical matrix form (364, 365) is useful in the universal base $\tilde{E}_1$, as it expresses the *Lorentzian seeming to* N_1 *contraction* of geometric parameters of the moving objects. Tensor-hyperbolic interpretations of the so-called Einsteinian dilation of time and Lorentzian contraction of extent, and also of the *two concomitant to them relativistic effects* are realized as rotational and deformational transformations of coordinates. (Chapters 2A÷4A.)

One-to-one correspondence between physical characteristics of relativistic motion of a material object (inertially and uninertially) in the Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$ and trigonometric notions are considered in details. We constructed different trigonometric models of collinear motions. They correspond to rectilinear physical movements. The so called *hyperbolic motion* is exposed in these three trigonometric variants: on hyperbola, catenary, and tractrix. As passing results are the identical primary hyperbolic and analogous spherical equations for a tractrix in parametric and explicit forms with one parameter R. (In Chapter 10A we considered by $3D$ tensor trigonometric way another uniform relativistic *pseudoscrewed motion*.) The consideration of hyperbolic motion was used for trigonometric representation of the Beltrami pseudosphere only with one parameter R in the special quasi-Euclidean space $\langle \mathcal{Q}^{2+1} \rangle$. Furthermore, the result is inferred: *a hyperboloid* I *in* $\langle \mathcal{P}^{2+1} \rangle$ *is isometric with the Beltrami pseudosphere in* $\langle \mathcal{Q}^{2+1} \rangle$ *with same* R. At $n > 2$, these homeomorphic objects and their *n-dimensional geometries* are isometric too. In addition, the *hyperbolic relativistic analog* of the Ziolkovsky cosmic formula is obtained. As a concrete illustration, we exposed the trigonometric description of a hypothetical cosmic travel with hyperbolic reversible regime to the nearest Star. Its inference: these similar travels for contemporary man (non-robot) are practically unreal in reasonable times. (Chapters 5A, 6A).

The trigonometric general form of Lorentzian $4D$-transformations is given. Besides, the general law of summing two or multistep non-collinear principal motions (velocities) in STR and non-Euclidean geometries is inferred in trigonometric tensor, vector and scalar forms with the secondary orthospherical rotation angle. We represented the law in its noncommutative biorthogonal form with Big and Small Pythagorean Theorems. For Euclidean geometry, it is commutative. The equivalence of the orthospherical angle and the Harriot–Lambert (Gauss–Bonnet) angular deviation in figures on surfaces of constant curvature was established. We proposed an updated concept of the parallel angle for both types of non-Euclidean geometries, and gave the solution of a pseudo-Euclidean right triangle in a pseudoplane with connections of its complementary angles.

Besides, the trigonometric models for kinematics and dynamics of *general physical movements* (with oscillating orthospherical rotations) of a material body in $\langle \mathcal{P}^{3+1} \rangle$ are exposed with vector sine and scalar cosine projections into the space and the time.

The main *measureless* notion of the tensor trigonometry in STR is a *trigonometric tensor of motion roth* Γ, generated proportionally the relativistic dynamic tensors of momentum and energy. So, the tensor of momentum produces the pseudo-Euclidean interior right triangle from three momenta $P_0 = m_0c$, $P = mc$ and $p = mv$ with *pseudo-Pythagorean Theorem* in $\langle \mathcal{P}^{3+1} \rangle$. The own momentum P_0 as a hypotenuse is geometric invariant of Lorentzian transformations. The *tensor of deformation defh* Γ maps geometric parameters of moving objects observed in the base $\tilde{E}_1$. (Chapter 7A.)

With the use of spherical-hyperbolic analogies, a number of analogous notions, formulae and theorems are given and inferred in their spherical variants in a basis quasi-Euclidean space with index $q = 1$ and in the embedded into it hyperspheroid of radius R with the exterior variant of its spherical geometry. (Chapter 8A.)

In Chapter 9A, we discussed the question: does exist an opportunity for studying relativistic motions under gravitation using both Minkowskian and pseudo-Riemannian space-time in their roles? For a convincing resolution of this long-standing question, first of all, it is necessary to analyze the accompanying problem: does gravitation affect on the local speed of light or it only affect on the coordinate speed of light from the point of view of an external Observer? As well-known, Einsteinian GTR (1913-1916), is prevailing geometric theory of gravitation with full refuse the Minkowskian space-time, and it postulates such a bond of the local speed of light with a potential as $c\sqrt{-g_{44}}$. However over time, a rigorous analysis revealed that in GTR there are several essential problems, as some dependence of its equations solutions on the coordinate conditions and a violation of the Law of energy-momentum conservation (see in [59], [81], [85]). Moreover, numerous attempts to combine GTR with Quantum Mechanics have not yielded significant results. From where, some authors began to develop improvements of GTR as Bimetric-kinds Theories of Gravitation (BMT) with metric tensors of $\langle \mathcal{P}^{3+1} \rangle$ and $\langle \mathcal{R}^{3+1} \rangle$. First, BMT was outlined by Einstein's close colleague Nathan Rosen [75].

We showed on the two extreme by sense GR-effects, that in BMT the curvilinear pseudo-Riemannian space-time may be also interpreted as the *observational one*, where these effects are fixed, however real motions have place locally in the Minkowskian space-time. Then BMT-kinds conception divides events in a gravitational field on real local ones and observable ones, for example, in a weak field. The observed events may be identical to the real ones, and they may completely not correspond to them. The dualism of BMT may be used in descriptions of relativistic motions in a gravitational field, as local ones, in $\langle \mathcal{P}^{3+1} \rangle$, and, as observable ones, in $\langle \mathcal{R}^{3+1} \rangle$ or in flat $\langle \mathcal{P}^{9+1} \rangle$.

In Chapter 10A, with the use of tensor trigonometry, we developed the differential geometry of a world line with 4 vector-scalar parameters along it, completely defining its local orientation and configuration at each point; revealed its Riemann and pseudo-Riemann metric forms. They given the main physical characteristics of a moving particle. We create also the $4D$ analogue in $\langle \mathcal{P}^{3+1} \rangle$ of the $3D$ theory by Frenet–Serret with its representation in tensor trigonometry form in $\langle \mathcal{Q}^{2+1} \rangle$. A strict base for such one-valued mathematical–physical theories is isotropy and homogeneity of their space.

Additional notations

$\tilde{E}_1$ – universal (original) base for canonical trigonometric matrix forms and realization of concrete spherical-hyperbolic analogy (with immovable Observer),

$\mathbf{r}$ – $(n+1)\times 1$-radius-vector of the element or the world point in $\langle\mathcal{P}^{n+1}\rangle$ or $\langle\mathcal{Q}^{n+1}\rangle$,

$y^{(k)}$ or $ct^{(k)}$ and $\mathbf{x}^{(k)} \in \mathcal{E}^{n(k)}$ – orthoprojections of the elements in $\langle\mathcal{P}^{n+1}\rangle$ or $\langle\mathcal{Q}^{n+1}\rangle$,

$\tilde{E}_k^{(n)} \subset \tilde{E}_k$ – Cartesian subbase of the pseudo-Cartesian base $\tilde{E}_k$,

$\mathbf{e}_\alpha$ – unity $n\times 1$-vector of directional cosines $\cos\alpha_j$, $j=1,\ldots,n$ (here in $\tilde{E}_1^{(n)} \subset \tilde{E}_1$),

$\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ – same vectors for 1st and 2nd motion at two-steps non-collinear motions,

$\mathbf{e}_\sigma = \{\cos\sigma_k\}$, $\mathbf{e}_{\overset{\angle}{\sigma}} = \{\cos\overset{\angle}{\sigma}_k\}$, $k=1,2,3$ – same vectors for a direct and inverse order of a sum of the two motions in $\langle\mathcal{P}^{2+1}\rangle, \langle\mathcal{P}^{3+1}\rangle, \ldots, \langle\mathcal{P}^{n+1}\rangle$ or $\langle\mathcal{Q}^{2+1}\rangle, \ldots, \langle\mathcal{Q}^{n+1}\rangle$. (In $\langle\mathcal{P}^{1+1}\rangle$ and $\langle\mathcal{Q}^{1+1}\rangle$ it is rotations on a quadrohyperbola and a circle – Figure 3).

$\overrightarrow{ct}^{(k)}$ – arrow of the k-th coordinate time and ordinate axis in the base $\tilde{E}_k$ in $\langle\mathcal{P}^{3+1}\rangle$,

$x_j^{(k)}$ – j-th coordinate axis of the subbase $\tilde{E}_k^{(n)}$ in the k-th Euclidean space $\mathcal{E}_k^{n(k)}$,

$\overrightarrow{c\tau} = \overrightarrow{ct}^{(m)}$ – current or instantaneous arrow of the *proper time* under a binary hyperbolic angle Γ (inside internal cone) to the initial frame axis $\overrightarrow{ct}^{(1)}$, as ordinate axis of $\tilde{E}_{(m)}$ and with a chronometer of proper time τ in $\tilde{E}_{(m)}^{(3)}$),

$\overrightarrow{x^{(m)}}$ or $x^{(m)}$ – current or instantaneous *special abscissa axis* in the subbase $\tilde{E}_{(m)}^{(3)}$ of the base $\tilde{E}_{(m)}$ under a binary hyperbolic angle Γ (inside external cone) to the *mutual special frame abscissa axis* $\overrightarrow{\chi}$ or $\chi = x^{(1)} \in \mathcal{E}^{3(1)}$ in the initial subbase $\tilde{E}_1^{(3)} \subset \tilde{E}_1$.

Note. Greek symbols as τ and χ are used here for the *proper time* and *proper extent.* (The axes $\overrightarrow{c\tau}$ and $\overrightarrow{x^{(m)}}$ are symmetrical off the invariant asymptotic conic hypersurface! All the axes $\overrightarrow{ct}^{(1)}$, $\overrightarrow{c\tau}$, $\overrightarrow{x^{(m)}}$, $\overrightarrow{\chi}$ are situated in the current or instantaneous pseudoplane of the binary hyperbolic angle Γ with respect to the base $\tilde{E}_1$ in $\langle\mathcal{P}^{3+1}\rangle$ and with its vector of directional cosines $\mathbf{e}_\alpha = \langle\cos\alpha_j\rangle$, $j=1,2,3$ in the subbase $\tilde{E}_1^{(3)} \subset \tilde{E}_1$.)

η – hyperbolic angular velocity of γ, w_α – orthospherical angular velocity of $\mathbf{e}_\alpha$,

$\mathbf{c} = c\cdot\mathbf{i}$ – tangent 4×1-vector of a 4-velocity of absolute matter movement or of the material point M along the world line in $\langle\mathcal{P}^{3+1}\rangle$ (*4-velocity of Poincaré*),

$\mathbf{i}$ – time-like unity 4×1-vector in $\langle\mathcal{P}^{3+1}\rangle$ and the tangent to a world line,

$\mathbf{j}$ – space-like unity 4×1-vector in $\langle\mathcal{P}^{3+1}\rangle$ and the pseudonormal to a world line,

$\mathbf{u} = R\cdot\mathbf{i}$ and $\mathbf{v} = R\cdot\mathbf{j}$ – radius-vectors for a hyperboloid II with $\rho = iR$ and a hyperboloid I with $\rho = R$, where $\mathbf{i} = (\sinh\gamma\cdot\mathbf{e}_\alpha,\ \cosh\gamma)$, $\mathbf{j} = (\cosh\gamma\cdot\mathbf{e}_\alpha,\ \sinh\gamma)$,

$\mathbf{v} = d\mathbf{x}/dt = v \cdot \mathbf{e}_\alpha = c \cdot \tanh\gamma \cdot \mathbf{e}_\alpha$ – coordinate velocity of physical movement,
$\mathbf{v}^* = d\mathbf{x}/d\tau = v^* \cdot \mathbf{e}_\alpha = c \cdot \sinh\gamma \cdot \mathbf{e}_\alpha$ – proper velocity of physical movement,
$\mathbf{s} = s \cdot \mathbf{e}_\alpha = c \cdot \coth\ \gamma \cdot \mathbf{e}_\alpha$ – coordinate supervelocity at Lorentzian contraction,
$\mathbf{s}^* = s^* \cdot \mathbf{e}_\alpha = c \cdot csch\ \gamma \cdot \mathbf{e}_\alpha$ – proper supervelocity of moving along a tractrix,
$w = g/v$, $w^* = g/v^*$ – coordinate and proper angular velocity of physical movement;

$\mathbf{P}_0 = m_0\mathbf{c} = m_0 c \cdot \mathbf{i} = P_0 \cdot \mathbf{i}$ – own 4×1-momentum of a particle M on a world line,
$P = mc = P_0 \cdot \cosh\gamma$ – scalar cosine projection of $\mathbf{P}_0$ onto $\overrightarrow{ct}^{(1)}$ (total momentum),
$\mathbf{p} = m\mathbf{v} = P_0 \cdot \sinh\gamma \cdot \mathbf{e}_\alpha$ – 3×1-vector sine projection of $\mathbf{P}_0$ into $\mathcal{E}^{3(1)}$ (real momentum),
$E_0 = P_0 c = m_0 c^2$ – own Einstein energy for a material point M on a world line,
$E = Pc = mc^2$ – total Einstein energy for a moving material point M in $\tilde{E}_1$;

Γ and γ – tensor and scalar angles of the principal hyperbolic rotation in $\langle \mathcal{P}^{n+1} \rangle$,

Φ and φ – tensor and scalar angles of the principal spherical rotation in $\langle \mathcal{Q}^{n+1} \rangle$,

Θ and θ – tensor and scalar angles of the orthospherical rotation in the plane $\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle$ either in $\mathcal{E}^{3(1)}$ or in $\mathcal{E}^{3(2)}$ depending on the two variants of two-step motion,

ε or ϵ – angle between 1-st and 2-nd motions (or velocities),

$\mathbf{e}_\nu = (\mathbf{e}_\beta - \cos\varepsilon \cdot \mathbf{e}_\alpha)/\sin\varepsilon$ – unity vector of the orthogonal increment of the motion,

$\mathbf{F} = F \cdot \mathbf{p}_\beta = m_0\mathbf{g}$ – 4×1 inner force acting on material point M in $\langle \mathcal{P}^{3+1} \rangle$,
$\mathbf{g} = g \cdot \mathbf{p}_\beta$ – 4×1 inner acceleration of material point M in $\langle \mathcal{P}^{3+1} \rangle$,
(In the base $\tilde{E}_m$, in fact, $\mathbf{F}$ and $\mathbf{g}$ are 3×1-vectors, directed along $\mathbf{e}_\beta$ in $\mathcal{E}^{3(m)}$.)

$\overline{\overline{g}} = \cosh\gamma\ g \cdot \cos\varepsilon$, $\overset{\perp}{g} = g \cdot \sin\varepsilon$ – 3×1 proper tangential and normal 3-accelerations,

ζ – dimension of embedding the curve into the basis space $\langle \mathcal{P}^{3+1} \rangle$ or $\langle \mathcal{Q}^{2+1} \rangle$,

$\mathbf{k} = \overline{\overline{\mathbf{k}}} + \overset{\perp}{\mathbf{k}}$ – 4×1-vector of a pseudocurvature of a world line in $\langle \mathcal{P}^{3+1} \rangle$, proportional to the inner acceleration $\mathbf{g}$ in $\tilde{E}_m$, with its tangential and normal projections in $\mathcal{E}^{3(1)}$,

$\mathcal{K} = g/c^2$ – 4-pseudocurvature of a world line with orthoprojections K_α and K_ν;

$\mathbf{p}$ – unity 4×1-vector of a pseudonormal, $\mathbf{p}_\beta$ and $\mathbf{p}_\alpha$ are current and principal ones,

$\mathbf{t}$ – 4×1-vector of a torsion of a world line in $\langle \mathcal{P}^{3+1} \rangle$,

$\mathcal{T}$ – 4-torsion of a world line;

$\mathbf{p}_\nu$ and $\mathbf{p}_\mu$ – unity 4×1-vectors of sine and cosine binormals,

$\mathbf{y}$ – 4×1-vector of an orthoprecession of a world line in $\langle \mathcal{P}^{3+1} \rangle$,

$\mathcal{Y}$ – 4-orthoprecession of a world line,

$\mathbf{h}$ – unity 4×1-vector of the pseudoscrew,

$\Pi(a)$ – contravariant Lobachevsky parallel angle inferred in the universal base $\tilde{E}_1$,

$P(a) = \alpha$ – covariant parallel angle: spherical φ as motion angle in spherical geometry and in $\langle \mathcal{Q}^{n+1} \rangle$; hyperbolic γ as motion angle in hyperbolic geometry and in $\langle \mathcal{P}^{n+1} \rangle$.

Chapter 1A.

Space-time of Lagrange and space-time of Minkowski as mathematical abstractions and physical reality

At first, consider the *conditional trigonometric kinematic model* of a material point M physical movement in the 4-dimensional binary Lagrangian space-time $\langle\mathcal{L}^{3+1}\rangle$. Choose its universal base $\tilde{E}_1 = I$ as an original unity frame system. In this base, all the four coordinate axes $x_1,\ x_2,\ x_3,\ \overrightarrow{t^{(1)}}$ are defined as if Euclidean orthonormal ones. The three space axes $x_1,\ x_2,\ x_3$ form the Cartesian *space-like subbase* $\tilde{E}^{(3)}$. The time-arrow $\overrightarrow{t^{(1)}}$ is the directed affine *time-like ordinate axis*. The axes $x_1,\ x_2,\ x_3$ stay orthonormal under as if orthospherical rotations of the original base $\tilde{E}_1 = I$, they form a right-handed triple in $\tilde{E}^{(3)}$. Hence, 3-dimensional Euclidean trigonometry with dimensionless spherical functions is applicable in $\langle\mathcal{E}^3\rangle$. By definition, the base $\tilde{E}_1 = I$ corresponds to Observer N immobility! If the material point M moves with the vector velocity $\mathbf{v} = v\cdot\mathbf{e}_\alpha = \mathbf{const}$, then its proper base $\tilde{E}_m = V\tilde{E}_1$, where its new time-arrow $\overrightarrow{t}^{(m)}$ have the particular slopes, with respect to the three space coordinates axes of $\tilde{E}_1^{(3)}$. The ratios of the three space coordinates and the time ordinate are characterized by the tangent vector $\mathbf{tan}\ \nu$ (as a world-line slope) identical to the vector velocity $\mathbf{v}$ of the material point M (if frame center O corresponds to zero, $\mathbf{x} = \Delta\mathbf{x},\ t = \Delta t > 0$):

$$\mathbf{tan}\ \nu = \tan\ \nu\cdot\mathbf{e}_\alpha = \mathbf{x}/t \equiv \mathbf{v} = v\cdot\mathbf{e}_\alpha, \quad \tan\nu_j = x_j/t \equiv v_j,\ j = 1,2,3. \qquad (1A)$$

Admissible transformations in linear $\langle\mathcal{L}^{3+1}\rangle$ form the group $\langle V_G\rangle$ of the homogeneous Galilean transformations. It is the mathematical source of the Galilean Principle of relativity. The transformation V_G is continuous as *det* $V_G = +1$, this condition guarantees preserving base orientation. In Cartesian-affine bases $\tilde{E}_k$, the space-time $\langle\mathcal{L}^{3+1}\rangle$ is represented as the direct sum of an Euclidean space and an affine time-arrow:

$$\langle\mathcal{L}^{3+1}\rangle \equiv \langle\mathcal{E}^3\rangle \oplus \overrightarrow{t}^{(k)} \equiv \langle\mathcal{E}^3\rangle \oplus \overrightarrow{t}^{(1)} \equiv \text{CONST}, \quad (\Delta t > 0) \qquad (2A)$$

$$\langle\mathcal{E}^3\rangle \equiv \text{CONST}\,'. \qquad (3A)$$

Seems, there is paradox: const' + variable=const, but it is not valid for a direct sum!

There holds analogy with binary spaces of Ch. 11 ($q = 1$), but (2A) is not an orthogonal sum! All time-arrows form the complete set of affine axes $\langle\overrightarrow{t}\rangle$ consisting of time-like arrows with angular slopes to $\overrightarrow{t^{(1)}}$ ranging in $[0; \pm\pi/2]$. The invariant Euclidean space $\langle\mathcal{E}^3\rangle$ consists of space-like elements. All elements are real numbers. The space-time properties are preserved under Galilean transformations, because ones in general $\langle\mathcal{L}^{3+1}\rangle$ are reduced to following exactly three pure types:
1) automorphic orthospherical rotations *rot* Θ of the space $\langle\mathcal{E}^3\rangle$,
2) special *parallel* (or *middle*) rotations $f(\mathbf{tan}\ \nu)$ of $\overrightarrow{t}$, with respect to the space $\langle\mathcal{E}^3\rangle$,
3) linear space $\langle\mathcal{E}^3\rangle$ and $\overrightarrow{t}$ translations $\mathbf{p}$ due to this space-time homogeneity.

The general linear transformation V_G of a Cartesian-affine base $\tilde{E}_0$ is the following:

$$\overset{V_G}{\begin{bmatrix} R & \mathbf{a} \\ \mathbf{0}' & 1 \end{bmatrix}} \cdot \overset{\tilde{E}_0}{\begin{bmatrix} R_0 & \mathbf{a_0} \\ \mathbf{0}' & 1 \end{bmatrix}} = \overset{\tilde{E}}{\begin{bmatrix} R \cdot R_0 & R\mathbf{a_0} + \mathbf{a} \\ \mathbf{0}' & 1 \end{bmatrix}}, \; R \in \langle rot\ \Theta_{3\times3} \rangle. \qquad (4A)$$

For the matrices of the bases, their first three columns determine the constant space $\langle \mathcal{E}^3 \rangle$, the fourth column determines the variable time-arrow $\overrightarrow{t}$. If $\mathbf{a}_0 = \mathbf{0}$, then $\tilde{E}_0 = E_{1u}\rangle$ (the bases are universal), and in particular, if $R_0 = I$, then $\tilde{E}_0 = E_1$. In this case, the inverse matrix V_G^{-1} (of the same structure) maps a binary Cartesian-affine base $\tilde{E}$ into its simplest unity form, i. e., the original universal base $\tilde{E}_1$. The inverse matrix also realizes passive modal transformation of a linear element from $\tilde{E}_1$ into an admissible binary base $\tilde{E}$. A linear element of $\langle \mathcal{L}^{3+1} \rangle$ is represented in $\tilde{E}$ as the radius-vector:

$$\mathbf{r} = \mathbf{x} \oplus t = \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}.$$

Thus *homogeneous Galilean transformations* in their trigonometric form are the non-commutative products of parallel and orthospherical rotations in the *polar forms*:

$$\overset{V_G = F(\Theta_{3\times3}, \mathbf{tan}\ \nu)}{\begin{bmatrix} rot\ \Theta_{3\times3} & \mathbf{tan}\ \nu \\ \mathbf{0}' & 1 \end{bmatrix}} = \overset{f(\mathbf{tan}\ \nu)}{\begin{bmatrix} I_{3\times3} & \mathbf{tan}\ \nu \\ \mathbf{0}' & 1 \end{bmatrix}} \cdot \overset{rot\ \Theta}{\begin{bmatrix} rot\ \Theta_{3\times3} & \mathbf{0} \\ \mathbf{0}' & 1 \end{bmatrix}} = rot\ \Theta \cdot f[(\mathbf{tan}\ \nu)_\Theta], \; (5A)$$

where $det\ V_G = +1$, and $f(\mathbf{tan}\ \nu)$ is the 4×4-matrix of principal parallel rotations,

$$f[(\mathbf{tan}\ \nu)_\Theta] = rot\ (-\Theta) \cdot f(\mathbf{tan}\ \nu) \cdot rot\ \Theta, \quad but(!)\ (\mathbf{tan}\ \nu)_\Theta = rot\ (-\Theta_{3\times3}) \cdot \mathbf{tan}\ \nu.$$

An inverse and passive homogeneous Galilean transformation is represented as

$$V_G^{-1} = \begin{bmatrix} rot\ (-\Theta_{3\times3}) & rot\ (-\Theta_{3\times3}) \cdot (-\mathbf{tan}\ \nu) \\ \mathbf{0}' & 1 \end{bmatrix} =$$

$$= \overset{rot\ (-\Theta)}{\begin{bmatrix} rot\ (-\Theta_{3\times3}) & \mathbf{0} \\ \mathbf{0}' & 1 \end{bmatrix}} \cdot \overset{f[\mathbf{tan}\ (-\nu)]}{\begin{bmatrix} I_{3\times3} & -\mathbf{tan}\ \nu \\ \mathbf{0}' & 1 \end{bmatrix}} = f\{[\mathbf{tan}\ (-\nu)]_\Theta\} \cdot rot\ (-\Theta). \qquad (6A)$$

Formula (5A) is the Euclidean-affine analog of polar representations (474) and (475) in sect. 11.3. On the other hand, transformation of the base E_1 is similar to (480), (481):

$$\tilde{E} = V_G \cdot \tilde{E}_1 = f(\mathbf{tan}\ \nu) \cdot rot\ \Theta \cdot \tilde{E}_1 = rot\ \Theta \cdot f[(\mathbf{tan}\ \nu)_\Theta] \cdot \tilde{E}_1. \qquad (7A)$$

From the physical point of view, the subbase $\tilde{E}^{(3)}$ moves, with respect to the subbase $\tilde{E}_1^{(3)}$, at the velocity (1A). Inverse matrix (6A) transforms passively the coordinates of a world point $\mathbf{r} \in \langle \mathcal{L}^{3+1} \rangle$ as follows:

$$\mathbf{r} = V_G^{-1\text{`}} \cdot \mathbf{r}^{(1)} = F^{-1}(\Theta, \mathbf{tan}\ \nu) \cdot \mathbf{r}^{(1)} = \begin{bmatrix} rot\ (-\Theta_{3\times3}) \cdot (\mathbf{x}^{(1)} - \mathbf{tan}\ \nu \cdot t) \\ t \end{bmatrix}. \quad (8A)$$

If $\Theta = Z$ in (5A)–(8A), then we deal with pure parallel rotations in their conventional trigonometric and physical forms as the Galilean transformations of coordinates:

$$\left.\begin{array}{l} \mathbf{x} = \mathbf{x}^{(1)} - \mathbf{tan}\ \nu \cdot t = \mathbf{x}^{(1)} - \mathbf{v} \cdot t, \\ t = t^{(1)}. \end{array}\right\} \qquad (9A)$$

In $\langle \mathcal{L}^{3+1} \rangle$, the *scalar time* is invariant too and may be counted on the original axis $\overrightarrow{t^{(1)}}$ and $\overrightarrow{t^{(k)}}$ parallel to invariant $\langle \mathcal{E}^3 \rangle$. Due to this fact, so called *parallel rotation* $f(\mathbf{tan}\ \nu)$ of the time-arrow $\overrightarrow{t}$ (as the ordinate) is geometrically intermediate between spherical and hyperbolic ones! Note, that $f(\mathbf{tan}\ \nu)$ is expressed above as a 4×4-matrix with the variable 3×1-vector element $\mathbf{tan}\ \nu$. The latter is the tangent of the angle ν. Multistep parallel rotations lead to the *classical law of tangents* $\mathbf{tan}\ \nu$ *or velocities* $\mathbf{v}$ *commutative geometric summation in the projective Euclidean vectorial space* $\{\langle \mathcal{E}^3 \rangle\}$:

$$f(\mathbf{tan}\ \nu_{13}) = f(\mathbf{tan}\ \nu_{12}) f(\mathbf{tan}\ \nu_{23}) = f(\mathbf{tan}\ \nu_{23}) f(\mathbf{tan}\ \nu_{12}) = f(\mathbf{tan}\ \nu_{12} + \mathbf{tan}\ \nu_{23}) \rightarrow$$

$$\rightarrow f(\mathbf{tan}\ \nu) = f(\mathbf{tan}\ \overset{\angle}{\nu}) = \prod f(\mathbf{tan}\ \nu_{kj}) = f(\sum \mathbf{tan}\ \nu_{kj}), \ \ (\nu = \overset{\angle}{\nu}). \qquad (10A)$$

The set $\langle \mathbf{tan}\ \nu \rangle$ is the commutative group in the projective vectorial space of velocities, i. e., "tangents". The set of parallel rotations $\langle f(\mathbf{tan}\ \nu) \rangle$ is the *kinematic commutative subgroup* of the homogeneous Galilean group $\langle V_G \rangle$. Its another subgroup is the *non-commutative group of orthospherical rotations.* Note, that *rot* Θ is expressed above as a 4×4-matrix with the variable 3×3-matrix element *rot* $\Theta_{3\times 3}$. The group $\langle V_G \rangle$ consisting of these two subgroups is the subgroup of the *general* Galilean group.

The Lagrangian space-time is not isotropic (it is enough for this, that its space and time coordinates have different physical dimensions), but the space-time is homogeneous due to equivalence of all its point elements. In particular, any centralized 4×1-vector element in $\tilde{E}_1$ may be chosen as the new origin of a Cartesian-affine base admissible, and the admissibility does not depend on this choice. Parallel translations in $\langle \mathcal{L}^{3+1} \rangle$ form the *commutative translating subgroup* of the general Galilean group.

The vector structure of $\langle \mathcal{L}^{3+1} \rangle$ is direct sum (2A) of the two *independent subspaces* : the isotropic unoriented Euclidean space and the oriented affine time-arrow directed always from past to future. This determines affine nature of principal transformations and independence of space and time in (2A).

The Lagrangian space-time has a lot of applications in non-relativistic physics. However, as long ago as at the end of XIX century, Maxwell's electromagnetic wave equation were proved to change under changing Galilean inertial frame of reference. Thus Lorentz suggested in 1892 special transformations having no this essential defect (they were inferred formerly, in 1877, by famous theorist V. Voigt according to his light elasticity theory). In 1904, Lorentz, taking into account *the Poincaré physical principle of relativity* (valid for all physical phenomena), showed that these transformations follow from form-invariance of the electromagnetic wave equation [46]. The latter, according to classical Maxwell's theory, explains the nature and spreading of light.

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Further translate description of movements off $\langle \mathcal{L}^{3+1} \rangle$ into Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$. In the beginning of the 20-th century the revolutionary transformation of space and time theory into its more perfect, relativistic variant took place. *From the mathematical point of view*, the following quite new postulates were introduced.

<u>Postulate 1</u>: *By nature, space-time is homogeneous and isotropic* (the latter property is valid due to velocity-like scale factor "c" used for time-arrow ordinates).

<u>Postulate 2</u>: *By nature, space-time is a binary complex-valued quasi-Euclidean space with index $q = 1$, oriented by time-arrow ordinates $i \cdot \overrightarrow{ct}$ from past to future.*

The new conception of space-time as STR with these two postulates has no any more defects of the classical, non-relativistic space-time. Mainly, it realized the opportunity to transfer off the non-isotropic Euclidean-affine space-time $\langle \mathcal{L}^{3+1} \rangle$ into the isotropic binary either quasi- or pseudo- Euclidean space-time with its quadratic metric! Then, in the new space-time, we may use the opportunities of scalar, vectorial and tensor trigonometries for clear description and analysis of different types relativistic movement in brief clear forms. According to Postulate 1, formulae (1A) are transformed into

$$\left.\begin{array}{l} \mathbf{tan}\ \nu \rightarrow \mathbf{tan}\ \varphi_R = \mathbf{v}/c, \\ \mathbf{tan}\ \varphi_R \equiv \mathbf{sin}\ \varphi \equiv \mathbf{tanh}\ \gamma = \mathbf{v}/c. \end{array}\right\} \quad (t \rightarrow ct) \qquad (11A)$$

Here γ is a hyperbolic angle of principal motion, φ_R is its *visual* analog in $\tilde{E}_1$ (sect. 6.4); $\mathbf{tan}\ \varphi_R \equiv \mathbf{sin}\ \varphi \equiv \mathbf{tanh}\ \gamma$ express concrete spherical-hyperbolic analogies (355), (331), they are valid only *in the universal bases* $\tilde{E}_{1u}$ too. Due to Postulate 2, there hold:

$$\left.\begin{array}{l} \mathbf{tan}\ (-\varphi) = \mathbf{v}/ic \rightarrow \mathbf{tanh}\ (-i\varphi) = \mathbf{v}/c, \\ (1)\ \varphi \rightarrow i\gamma,\ \mathbf{tan}\ i\gamma = i\mathbf{v}/c;\ (2)\ -i\varphi \rightarrow \gamma,\ \mathbf{tanh}\ \gamma = \mathbf{v}/c. \end{array}\right\} \quad (t \rightarrow ict) \qquad (12A)$$

Relations $\varphi \rightarrow i\gamma$ (1) and $-i\varphi \rightarrow \gamma$ (2) correspond to process (323) and process (322) of abstract spherical-hyperbolic analogy *in quasi-Cartesian and pseudo-Cartesian bases* with the common reflector tensor $I^{\pm}$ of their spaces (see Ch. 6). Under further logical development, the Euclidean vectorial subspace of tangents or velocities was reduced into the hyperbolic tangent model or the Kleinian model inside Cayley oval (sect.12.1).

With Poincaré mathematical approach [47], STR was created with his principle of relativity in the Galilean frames of reference and the introduction of the new complex-valued isotropic and homogeneous space-time (sect.10.3). As a result, the Galilean transformations are replaced by Lorentzian ones, named so by Poincaré. What's more, he discovered their group nature, which opened up opportunities for multi-step using.

With Einsteinian physical approach [48], STR was created with the two main steps. At first, two Postulates – the principle of relativity and the principle of constancy of the speed of light, both in the Galilean frames of reference, were used. And in fact for introducing namely the quadratic metric of space-time, the definition of simultaneity was formulated and used. As a result, he came to the Lorentzian transformations too.

The Principle of relativity is traditionally applied only in its physical sense, although there exists its mathematical prototype, see in sect. 12.3. Any physical space-time (here $\langle \mathcal{L}^{3+1} \rangle$ and $\langle \mathcal{P}^{3+1} \rangle$) is only a certain mathematical abstraction, and its admissible coordinates may be used for describing objective laws of matter movement. *Adequate interpretation of these laws in the coordinates* maps the "reality" of the space-time.

On the whole, Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$ is isotropic. However its vectorial structure taking into account admissible directions to the *light*) cone contains three isotropic geometric parts with respect to their pseudo-Euclidean metric. They are: (1) the external conic cavity consisting of space-like elements with an Euclidean measure, (2) the internal conic cavity consisting of time-like elements with the anti-Euclidean (imaginary) measure, (3) the degenerated light conic dividing surface with the zero measure, it separates these external and internal cavities. Therefore rotational and deformational linear transformations in the space-time may be represented as 4×4 tensor trigonometric functions of 4×4 tensor angles Γ and Θ (Chs. 6 and 10–12).

Scalar trigonometric functions of $i\gamma$ in their *pseudospherical form* were first applied by H. Poincaré for representing Lorentzian transformations in the 2-dimensional form. Then H. Minkowski used the real-valued scalar functions of γ in the 2-dimensional *hyperbolic form* in $\langle \mathcal{P}^{1+1} \rangle$ for the same purpose. The authors used scalar trigonometry in a pseudo-plane for representing hyperbolic motions with 2×2 rotational matrices.

Tensor trigonometric functions of Γ, i. e., in their *hyperbolic form* in $\langle \mathcal{P}^{3+1} \rangle$ (they are partly described in Chs. 11 and 12) give us the pure trigonometric 4-dimensional tensor forms for kinematics and dynamics of STR – see in Chs. 5A, 7A and 10A. The original two *Einstein's postulates* used up to now in the pure physical version of STR have trigonometric prototypes due to *physical-mathematical isomorphism*, – sect. 12.3.

Pseudo-Euclidean trigonometric rotations correspond to homogeneous continuous Lorentzian transformations. Hyperbolic rotations with the pseudo-Euclidean invariant $\sinh^2\gamma - \cosh^2\gamma = i^2$, $\cosh\gamma > 1$, interpret clearly the Einsteinian dilation of time. Trigonometric hyperbolic deformations with the cross quasi-Euclidean invariant (in $\tilde{E}_1$) $\text{sech}^2\gamma + \tanh^2\gamma = 1$, $\text{sech}\gamma < 1$, interpret the Lorentzian contraction of extent. If the two phenomena are considered in the pseudoplane corresponding to the tensor angle Γ, a pseudo-Euclidean right triangle may be solved completely (sect. 6.4). The special *mathematical* principle of relativity for geometry of $\langle \mathcal{P}^{3+1} \rangle$ (see sect. 12.3) is in one-to-one correspondence with the Poincaré *physical* Principle of relativity. The Poincaré–Einstein Law of mutual dependence of the space and the time and their relativity may be explained with the fact that the relativistic Euclidean space and time-arrow are hyperbolically orthogonal complements of each other, they change always together under hyperbolic rotations, and both do not change under orthospherical rotations:

$$\langle \mathcal{P}^{3+1} \rangle \equiv \langle \mathcal{E}^3 \rangle^{(k)} \boxtimes \overrightarrow{ct}^{(k)} \equiv \text{CONST}. \tag{13A}$$

This space-time is the united indivisible 4-dimensional continuum. As a whole set, it is *an absolute*, consisting of these two variable together relative summands of index k.

What's more, the scaling coefficient "c", introduced by H. Poincaré for the time ordinates, is equal to the light velocity in the cosmic vacuum. This small, but great time modification led to identity of transformations in such isotropic space-time with Lorentzian transformations developed for covariancy of the Maxwell electromagnetic wave equation in the inertial systems [46, 47]. Later P. Dirac generalized the result in his relativistic covariant form of the Schrödinger quantum wave equation [55]. The fundamental Law of Energy and Momentum Conservation due to E. Noether Theorem are inferred in STR strictly from homogeneity and isotropy of this basis space-time.

Tensor trigonometric language (with hyperbolic and orthospherical functions) may be used for explaining all effects of STR connected with the time and Euclidean space.

Main Einstein's postulates on maximality of matter physical velocity due to $v < c$ and constancy of the light velocity c (only as scalar value) in all Galilean inertial frames of reference directly follows from properties of the hyperbolic tangent function module

$$||\mathbf{v}/c|| = ||\mathbf{tanh}\ \gamma|| < 1, \qquad (14A)$$

and from these properties of the hyperbolic angle for physical velocity

$$\pm\infty \pm \gamma = \pm\gamma \pm \infty = \pm\infty, \qquad (15A)$$

valid in any pseudo-Cartesian base $\tilde{E}$ of $\langle \mathcal{P}^{3+1} \rangle$ with relatively immobile Observer. Second rule (15A) implies that the light velocity does not depend on source movement. However, the instantaneous *proper velocity* v^* of a material object, from the point of view of Observer moving with it, changes due to $||\mathbf{v}^*|| < \infty$, as $\mathbf{v}^* = c \cdot \mathbf{sinh}\ \gamma$!

In Ch. 7A, we use a Rule of summing multistep motions with polar decomposition from Ch. 11 for inferring the relativistic non-commutative laws of summing velocities in STR and segments in hyperbolic geometries in the most general forms. Due to similar opportunities, we consider relativistic motions with their kinematics and dynamics in Galilean and non-Galilean local frames of reference (Chs. 5A, 6A, 7A, 10A).

The most difficult problem is similar considerations taking into account gravitation. The historically first and up to now prevailing *geometric-field conception* is based on the well-known Einsteinian GTR [51] with curved by gravitation pseudo-Riemannian space-time. Note also the alternative BMT conception (Bimetric Theory of Gravitation), which is based on the nature of gravitation as action of a tensor physical field in the basis Minkowskian space-time with conservating energy-impulse tensor of matter and field as a source of this field. Surprisingly, historically, the first version of BMT [75] was proposed by Nathan Rosen, an assistant to A. Einstein at Princeton University and later his close colleague! This shows how Albert Einstein was loyal to alternative points of view in science and even to his GTR. This is an example of the true and not just in words, attitude to the freedom of scientific thought. So later and up to now in a number of monographs, their authors quite convincingly show that all the well-known general relativistic effects are interpreted in the frame of BMT-kind theories, but with the elimination of some significant contradictions in GTR (see in discussional Ch. 9A).

* * *

Further, describe trigonometric approach to representation of physical relativistic movements in its simplest form. Choose the right universal, i. e., *inertial* base $\tilde{E}_1 = \{I\}$ with immovable Observer N_1. Other right universal bases $\tilde{E}_{1u}$ are linked as follows:

$$\tilde{E}_{1u} = rot\ \Theta \cdot \tilde{E}_1 = \{rot\ \Theta\}, \tag{16A}$$

where $rot'\ \Theta \cdot I^{\pm} \cdot rot\ \Theta = I^{\pm} = rot\ \Theta \cdot I^{\pm} \cdot rot'\ \Theta$ according to (470), and

$$I^{\pm} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{17A}$$

In the base $\tilde{E}_1$ and all universal bases, coordinate axes are originally quasi-Euclidean and pseudo-Euclidean orthonormal. Hence, the concrete spherical-hyperbolic analogy of Ch. 6 may be used, and this is important from theoretical point of view. Bases that are not universal ones are only pseudo-Euclidean orthonormal:

$$\tilde{E}' \cdot I^{\pm} \cdot \tilde{E} = I^{\pm} = (\sqrt{I^{\pm}} \cdot \tilde{E})' \cdot (\sqrt{I^{\pm}} \cdot \tilde{E}), \tag{18A}$$

where $\sqrt{I^{\pm}}$ is the arithmetic root of type (443). The latter gives a variant without $I^{\pm}$. A new base, according to polar representations (480), (481), is the result of a unique combination of a hyperbolic rotation (in $\tilde{E}_1$) and orthospherical one (in $\tilde{E}_{1h}$), or in the reverse order, where the matrices are compatible with the reflector tensor $I^{\pm}$:

$$\tilde{E} = roth\ \Gamma \cdot rot\ \Theta \cdot \tilde{E}_1 = \{rot\ \Theta\}_{\tilde{E}_{1h}} \cdot \tilde{E}_{1h}. \tag{19A}$$

Suppose that a new pseudo-Cartesian base is the result of a pure hyperbolic rotation

$$\tilde{E}_{1h} = roth\ \Gamma \cdot \tilde{E}_1 = \{roth\ \Gamma\}. \tag{20A}$$

Then the new coordinate axes in $\tilde{E}_{1h}$ are, according to (363), completely spherically non-orthogonal to each other, and their scales in the Euclidean metric are distorted (this holds for at least two of the axes, one of them is the time-arrow). Pure hyperbolic base rotation (20A) has the physical sense of uniform rectilinear movement of $\tilde{E}_{1h}^{(3)}$ with its N_{1h} relatively of $\tilde{E}_1^{(3)}$ with its N_1 at the velocity $v = c \cdot \tanh\gamma$. Hyperbolic rotation is elementary, it is performed in the rotation eigen pseudoplane $\langle \mathcal{P}^{1+1} \rangle \subset \langle \mathcal{P}^{3+1} \rangle$ determined here by the time-arrow $\overrightarrow{ct}^{(1)}$ and the vector $\mathbf{v} = c \cdot \mathbf{tanh}\ \gamma$ in $\langle \mathcal{E}^3 \rangle^{(1)}$.

In the simplest case of 2×2-dimensional matrix (324), we have in the pseudoplane

$$\tilde{E}_{II} = \{roth\ \Gamma\}_{2\times 2} \cdot \tilde{E}_I = \begin{bmatrix} \cosh\gamma & \sinh\gamma \cdot \cos\alpha \\ \sinh\gamma \cdot \cos\alpha & \cosh\gamma \end{bmatrix}, \quad \cos\alpha = \pm 1. \tag{21A}$$

It is a hyperbolic rotation of the axes $x^{(1)}$ and $ct^{(1)}$ at the angle γ to the bisectrix of the 1-st quadrant if $\cos\alpha = +1$ and to the bisectrix of the 2-nd quadrant if $\cos\alpha = -1$).

Consider in details the physical uniform rectilinear movement of a material point M. At the moment $t = 0$ the point passes through the origin O of the frame of reference $\tilde{E}_1$, which here is the common origin for all centralized bases $\langle \tilde{E}_k \rangle$. Then the *world line* of M is a straight line inside the invariant light cone [49]. The light cone is the locus of all central light rays proceeding from O. A certain pseudo-Cartesian base $\tilde{E}$, where M is immobile, has its time-arrow $\overrightarrow{ct}$ coinciding with the straight world line of M mapped in the original base $\tilde{E}_1$. (In general, all the new coordinate axes are determined by columns of the matrix for a new base.) This new time-arrow $\overrightarrow{ct}$ is completely determined in $\tilde{E}_1$ by the hyperbolic angle γ with $\overrightarrow{ct}^{(1)}$ and the fixed directional cosines of the vector $\mathbf{tanh}\ \gamma \in \langle \mathcal{E}^3 \rangle^{(1)}$ or the point M velocity $\mathbf{v} = v{\cdot}\mathbf{e}_\alpha = c{\cdot}\mathbf{tanh}\ \gamma = \mathbf{const}$. Also it is completely determined by the fourth column of rotational matrix *roth* Γ in (20A) with its canonical structure (363) expressed in the initial base $\tilde{E}_1$.

A world line may be, of course, arbitrary curvilinear one (as *geometric invariant*), but its slope must be less than the slope of the light cone, i. e., of rays of light relatively to the time-arrow $\overrightarrow{ct}^{(1)}$. We represent world lines in a universal base $\tilde{E}_1 = \{I\}$ only for its geometric visuality and comparison with other world lines, as well as all the other pseudo-Cartesian bases $\tilde{E}$ are expressed also with respect to $\tilde{E}_1$! With these arguments, the base $\tilde{E}_1$ is defined initially as if Cartesian one too! Such approach was used before in Ch. 12 for representing the two Minkowskian Hyperboloids with the same purpose. (An universal base $\tilde{E}_1$ is the relative notion defined by inertial Observer N_1.) Laws of movements discussed in Chs. 5A, 7A, 8A and 10A are interpreted as a rule also in $\tilde{E}_1$.

In trigonometric kinematics of STR, the angles γ and Γ of motion tensor in (20A) for transformations of coordinates always have the sign $+$. The sign $-$ for the angles is possible only in *mental* motions to past with the use of *antipodal* hyperbolic geometry (sect. 12.1.) This is equivalent to the *principle of determinism* for material phenomena. These facts distinguish to a some extent hyperbolic kinematics of STR and the laws of hyperbolic motions in the Lobachevsky–Bolyai geometry. The same time-arrow $\overrightarrow{ct}$ (and the same world straight lines) in the two cavities of the light cone are determined with the same matrices *roth* Γ corresponding, from the physical point of view, to the same velocity vector and, from the geometrical point of view, to the same motion:

$$roth\ \Gamma = F(\gamma, \mathbf{e}_\alpha) \equiv F(-\gamma, -\mathbf{e}_\alpha). \qquad (22A)$$

The last expression here is valid only in *antipodal* hyperbolic geometry. Another time-arrow that is symmetric to original one with respect to $\overrightarrow{ct}^{(1)}$ (and the parallel to it world straight line) is determined with the inverse matrix. It has the physical sense of an additively opposite velocity vector and the corresponding to it geometric sense:

$$roth^{-1}\ \Gamma = F(\gamma, -\mathbf{e}_\alpha) = roth\ (-\Gamma) \equiv F(-\gamma, \mathbf{e}_\alpha). \qquad (23A)$$

And here the last expression is valid only in antipodal hyperbolic geometry. In (22A) and (23A), the angle γ is formally positive for directions of material points movements along the time arrow to future, it is formally negative for mental motions to past.

Formulae (20A), (21A) imply that due to hyperbolic rotations the *coordinate velocity* of physical movement v along $x^{(1)}$ is expressed trigonometrically from this relation:

$$\frac{v}{c} = \frac{\Delta x}{c \cdot \Delta t} = \frac{\sinh\gamma \cdot \cos\alpha}{\cosh\gamma} = \tanh\gamma \cdot \cos\alpha, \quad (\cos\alpha = \pm 1). \qquad (24A)$$

Generally, for $\langle \mathcal{P}^{3+1} \rangle$, the *vector of coordinate velocity* $\mathbf{v}$ in $\langle \mathcal{E}^3 \rangle^{(1)}$ is determined by its absolute value $||\mathbf{v}||$ and the directional cosines $\cos\alpha_j$, $j = 1, 2, 3$; its three Euclidean projections onto the axes have also physical and trigonometric forms:

$$\frac{v_j}{c} = \frac{\Delta x_j}{c \cdot \Delta t} = \tanh\gamma \cdot \cos\alpha_j, \ j = 1, 2, 3, \quad (\mathbf{v} = \{v_j\} = v \cdot \mathbf{e}_\alpha = c \cdot \mathbf{tanh}\ \gamma), \quad (25A)$$

where $-1 \leq \cos\alpha_j \leq +1$ and $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1$.

For describing the physical uniform rectilinear movement at velocity v, according to scalar pseudo-Euclidean trigonometry (sect. 6.4), in the pseudoplane of rotation $\langle \mathcal{P}^{1+1} \rangle$ the new coordinate axes x and $\overrightarrow{ct}$ in the base $\tilde{E}_1 = \langle x^{(1)}, \overrightarrow{ct}^{(1)} \rangle$ are hyperbolically rotated, as in (21A), at the angle $\gamma = \text{artanh}\ (v/c)$ to the bisectrix of the 1-st quadrant, i. e., to the light cone. (Recall, that the positive spherical rotation at angle φ is realized in the right bases $\tilde{E}_k$ as the counter-clockwise angle!) The concrete spherical–hyperbolic analogy between γ and φ in the universal base $\tilde{E}_1$ are here either *sine-tangent (they are very important further*) or visual tangent-tangent (sect. 6.4):

$$dx^{(1)}/(dct^{(1)}) = v/c = \tanh\gamma \equiv \sin\varphi = \tan\varphi_R \ \ in\ \tilde{E}_1, \quad (\gamma > \varphi(\gamma) > \varphi_R(\gamma)).$$

There is no *infinitesimal distinction* between the angles $\varphi, \varphi_R, \gamma$ if $\gamma \to 0$ ($v \ll c$). If we analyze in $\tilde{E}_{1u}$ (with respect to immovable Observer) one-step physical movements, then spherical geometry and hyperbolic one are equally applicable. But if we deal with combined non-collinear principal movements, for example, with respect to moving Observer, as well as multistep or integral movements, then only hyperbolic geometry should be applied. This holds for motions in spherical and hyperbolic geometries too!

So, the *spherical* parallel angle of N. Lobachevsky $\Pi(a)$ [14, p. 186; 69] up to now is the fundament of *hyperbolic* non-Euclidean geometry. However, from the point of view of an enveloping space $\langle \mathcal{P}^{n+1} \rangle$ in its interpretation on a hyperboloid II the angular argument may have a certain geometric sense only in the universal bases $\tilde{E}_{1u}$ and only for one-step motions. On the contrary, the hyperbolic angular argument $\gamma = a/R$ is consistent in any pseudo-Cartesian bases (with $\{I^{\pm}\}$), see in sect. 6.4, ch. 12. Further,

$$\left.\begin{array}{l} \Pi(a) \equiv \pi/2 - \varphi(\gamma) = \xi(\gamma) = \pi/2 - \arcsin(\tanh\gamma) = 2\arctan[\exp(-\gamma)], \\ \varphi, \gamma: \ \sin\varphi \equiv \tanh\gamma \ \Leftrightarrow \ \tan\varphi \equiv \sinh\gamma \ \ (\text{but } \sin\Pi(a) \equiv \text{sech}\ \gamma), \\ \xi, \upsilon: \ \sin\xi \equiv \tanh\upsilon \ \Rightarrow \upsilon = \ln\coth\gamma/2, \ \underline{d\upsilon = -d\gamma/\sinh\gamma, \ d\xi = -d\varphi}. \end{array}\right\} \quad (26A)$$

And we have the two alternative *covariant* parallel angles for both types geometry:
$\varphi = a/R$ – is the *covariant parallel angle* in spherical type non-Euclidean geometries,
$\gamma = a/R$ – is the *covariant parallel angle* in hyperbolic type non-Euclidean geometries.

Both parallel angles are correct for the principal motions in their geometries, because they have the same nature and change covariantly to the motions directions! But the *countervariant angle* $\Pi(a)$ changes contrary to them and is only spherical. It takes place, because the parallel angle $\Pi(a)$ is the complement to the principal motion angle φ in the spherical case or the angle $\varphi(\gamma)$ in the hyperbolic case. Choice of $\Pi(a)$ is reduced to concrete *countervariant* analogy of parallel and motion angles (sect. 6.4).

In order to get absolute (not depending on the 5-th Euclid's postulate) geometry, the spherical or hyperbolic nature of the parallel angle α should not be fixed! Initially put $\alpha = \pm P(a)$ (if $\alpha \neq 0$) as the angle between abstract and Euclidean parallels in the universal base $\tilde{E}_1$. (The angle α is complementary to $\pi/2$ for $\Pi(a)$). Only after this step, we come to the dilemma: what nature of the principal angle α should be chosen? If $P(a) > 0$ and $\alpha = P(a)$ is chosen as a spherical angle, then non-Euclidean geometry of spherical type is obtained, and its parallels are intersected due to G. Saccheri [32]. If $P(a) < 0$ and $\alpha = -P(a)$ is chosen as a hyperbolic angle, then non-Euclidean geometry of hyperbolic type is obtained, and its parallels converge into ∞ on the side of α due to N. Lobachevsky [37, 38]. ($P(a) = 0$ corresponds to Euclidean geometry.) Moreover, if in the universal base $\tilde{E}_1$ a geodesic motion is realized from the center C of a hyperboloid II along a hyperbola or a hyperspheroid along a circle – Ch. 12, Figure 4, then its principal angle changes covariantly to the motion direction as follows:
$-P_\gamma(a) = \gamma \in [0 \cdots \pm \infty)$, $+P_\varphi(a) = \varphi \in [0 \cdots \pm \pi/2]$ – see descriptively in Ch. 12.

Conclude this chapter with the following essential remark. The initial *mathematical approach* of H. Poincaré in 1905 [47] to constructing Theory of Relativity is logically quite perfect, contrary to the initial *physical approach* of A. Einstein based on his two postulates (see also in [61, p. 42–44]). Similarly, only the extrema $|\tanh\gamma|_{max} = 1$ in all frames of reference and the mathematical principle of relativity (sect. 12.3) are not sufficient for constructing pseudo-Euclidean trigonometry (with $q = 1$). These two mathematical statements are equivalent to the both Einstein's postulates and lead logically only to constructing an infinite set of "trigonometries" and their quasiphysical isomorphisms with pseudo-Hölderian metrics of powers p (non-quadratic if $p \neq 2$):

$$|da|^p = |dx_1|^p + |dx_2|^p + |dx_3|^p - |dy|^p, \qquad 1 \leq p < \infty.$$

However, A. Einstein proposed the graceful physical manner for clear definition of simultaneity of events with the use of two light rays. This *axiomatic definition of simultaneity* introduced implicitly the quadratic pseudo-Euclidean metric ($p = 2$) in the space-time of STR. But the Einsteinian definition is only a beautiful theorem of Minkowskian pseudo-Euclidean geometry, see more in Ch. 4A. As it is well-known, H. Minkowski in 1909 renovated the foundation of STR with the use of pseudo-Euclidean space-time and geometry with the index $q = 1$, i. e., factually he regenerated the initial mathematical approach of H. Poincaré. In the Minkowskian space-time the notion of events simultaneity, with respect to the given frame of reference, is defined geometrically very simply and clearly – see in Ch. 4A and at Figure 1A, Ch. 3A.

Chapter 2A

The tensor trigonometric model of Lorentzian homogeneous principal transformations

Let a particle M moves absolutely in the space-time $\langle \mathcal{P}^{3+1} \rangle$ uniformly and rectilinearly along its straight world line passing through the center O. Then, according to (21A), its 4 coordinates in the original base $\tilde{E}_1$ and in the base $\tilde{E}$ tied with M are expressed in the simplest trigonometric form by the following *passive* linear transformation in the hyperbolic angle Γ:

$$\begin{matrix} roth\ (-\Gamma) & \mathbf{r}\{\tilde{E}_1\} & & \mathbf{r}\{\tilde{E}\} \\ \left[\begin{matrix} \cosh\gamma & 0 & 0 & -\sinh\gamma\cdot\cos\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\gamma\cdot\cos\alpha & 0 & 0 & \cosh\gamma \end{matrix}\right] \cdot & \left[\begin{matrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ ct^{(1)} \end{matrix}\right] & = & \left[\begin{matrix} \cosh\gamma\cdot x_1^{(1)} - \sinh\gamma\cdot\cos\alpha\cdot ct^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ \cosh\gamma\cdot ct^{(1)} - \sinh\gamma\cdot\cos\alpha\cdot x_1^{(1)} \end{matrix}\right]; \end{matrix}$$

Represent the *hyperbolic* transformation in the 4-dimensional system $\{t = 0, \mathbf{x} = \mathbf{0}\}$:

$$\left.\begin{aligned} x_1 &= \cosh\gamma\cdot x_1^{(1)} - \sinh\gamma\cdot\cos\alpha\cdot ct^{(1)} = \frac{x_1^{(1)} - \tanh\gamma\cdot\cos\alpha\cdot ct^{(1)}}{\mathrm{sech}\ \gamma}, \\ x_2 &= x_2^{(1)}, \qquad x_3 = x_3^{(1)}, \\ ct &= \cosh\gamma\cdot ct^{(1)} - \sinh\gamma\cdot\cos\alpha\cdot x_1^{(1)} = \frac{ct^{(1)} - \tanh\gamma\cdot\cos\alpha\cdot x_1^{(1)}}{\mathrm{sech}\ \gamma}. \end{aligned}\right\} \qquad (27A)$$

This is the initial trigonometric form of Poincaré–Minkowski (in fact 2-dimensional) of the *Lorentz homogeneous (linear) transformations for space and time* in $\tilde{E}_1$ and $\tilde{E}$. The multiplier $\cos\alpha = \pm 1$ determines two directions of the sine and tangent vectors. If (24A) are taken into account, they may be expressed in the *physical* form of [46,47]:

$$x_1 = \frac{x_1^{(1)} - v\cdot t^{(1)}}{\sqrt{1 - v^2/c^2}}, \quad x_2 = x_2^{(1)}, \quad x_3 = x_3^{(1)}, \quad ct = \frac{ct^{(1)} - (v/c)\cdot x_1^{(1)}}{\sqrt{1 - v^2/c^2}}.$$

Take advantage of the hyperbolic rotational matrix with general canonical structure (363) in the base $\tilde{E}_1$, then we obtain the *general trigonometric linear transformations* (pure hyperbolic) of the four coordinates of M as the three scalar space-orthoprojections (at $i = 1, 2, 3$) and the time-orthoprojection

$$\left.\begin{aligned} x_i = {} & \cos\alpha_i\cdot[\cosh\gamma\cdot S - \sinh\gamma\cdot ct^{(1)}] + [x_i^{(1)} - \cos\alpha_i\cdot S], \\ ct = {} & \cosh\gamma\cdot ct^{(1)} - \sinh\gamma\cdot S, \\ \text{where: } S = {} & \cos\alpha_1\cdot x_1^{(1)} + \cos\alpha_2\cdot x_2^{(1)} + \cos\alpha_3\cdot x_3^{(1)}, \end{aligned}\right\} \qquad (28A)$$

and their vectorial-scalar form with an arbitrary direction of sine and tangent vectors

$$\left.\begin{aligned} \mathbf{x} &= [\cosh\gamma\cdot\mathbf{e}_\alpha\mathbf{e}'_\alpha\cdot\mathbf{x}^{(1)} - \sinh\gamma\cdot\mathbf{e}_\alpha\cdot ct^{(1)}] + (I - \mathbf{e}_\alpha\mathbf{e}'_\alpha)\cdot\mathbf{x}^{(1)} = \\ &= [\cosh\gamma\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{x}^{(1)} - \sinh\gamma\cdot\mathbf{e}_\alpha\cdot ct^{(1)}] + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{x}^{(1)}, \\ ct &= \cosh\gamma\cdot ct^{(1)} - \sinh\gamma\cdot\mathbf{e}'_\alpha\cdot\mathbf{x}^{(1)}. \end{aligned}\right\} \qquad (29A)$$

In its general form, the vector of the directional cosines $\mathbf{e}_\alpha = \{\cos\alpha_i\}$ determines the direction of the sine and tangent vectors in $\tilde{E}_1^{(3)}$ of $\tilde{E}_1$ as well as of the velocity, and:

$$\overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha} = \mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{v}\mathbf{v}'} = \mathbf{v}\mathbf{v}'/|\mathbf{v}'\mathbf{v}| = \mathbf{v}\mathbf{v}'/||\mathbf{v}||^2, \quad I - \mathbf{e}_\alpha\mathbf{e}'_\alpha = \overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha} = \overrightarrow{\mathbf{v}\mathbf{v}'}$$

are the orthoprojectors in $\tilde{E}_1^{(3)}$ (see in sect. 2.5) into $\langle im\ \mathbf{v}\rangle$ and $\langle im\ \mathbf{v}\rangle^\perp$ in $\langle\mathcal{E}^3\rangle$.

Transformations equivalent to (29A) were derived by G. Herglotz [62; 53, p. 27] as

$$\mathbf{x} = \mathbf{x}_\mathbf{v} + (\mathbf{x}^{(1)} - \mathbf{x}_\mathbf{v}{}^{(1)}) = \frac{\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha{}'} \cdot \mathbf{x}^{(1)} - \mathbf{v}\cdot t^{(1)}}{\sqrt{1-||\mathbf{v}||^2/c^2}} + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha{}'}\cdot\mathbf{x}^{(1)}, \quad ct = \frac{ct^{(1)} - (\mathbf{v}/c)'\cdot\mathbf{x}^{(1)}}{\sqrt{1-||\mathbf{v}||^2/c^2}}.$$

He decomposed $\mathbf{x}^{(1)}$ in $\langle\mathcal{E}^3\rangle$ as the relativistic and non-relativistic projections onto $\mathbf{v}$ (*the Principle of Herglotz*). They are turned into the form (29A) with $\mathbf{v}/c = \mathbf{tanh}\gamma$.

The clear interpretation of these general trigonometric and physical transformations follows from their comparison with (27A). When the base $\tilde{E}_1$ is hyperbolically rotated in the pseudoplane $\langle\mathbf{v}, ct^{(1)}\rangle$, then only the time projection and the relativistic space projection $\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha{}'}\mathbf{x}^{(1)}$ are subjected to the modal transformation. The space-projection $\overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha{}'}\mathbf{x}^{(1)}$ orthogonal to $\mathbf{v}$ is invariant under Lorentzian and Galilean transformations. In the projective non-Euclidean vectorial tangent subspace of radius $R = 1$ there hold: $||\mathbf{tanh}\ \gamma|| = \tanh\gamma = ||\mathbf{v}||/c = \sqrt{\tanh^2\gamma_1 + \tanh^2\gamma_2 + \tanh^2\gamma_3}$ $(\gamma \geq 0)$, and

$$\mathbf{tanh}\ \gamma = \tanh\gamma\cdot\mathbf{e}_\alpha = \mathbf{v}/c \rightarrow \tanh\gamma_i = \cos\alpha_i\cdot\tanh\gamma = v_i/c, \quad (i = 1,2,3), \quad (30A)$$

where γ_i are the partial angles with their values in the Euclidean orthoprojections $\tanh\gamma_i = \cos\alpha_i\cdot\tanh\gamma$ of the vector $\mathbf{tanh}\ \gamma$ in the subbase $\tilde{E}_1^{(3)}$. The same for sine is $\sinh\gamma_i = \cos\alpha_i\cdot\sinh\gamma = \cosh\gamma\cdot\tanh\gamma_i$. But the projective vectorial sine space is Euclidean one, because for it $R\rightarrow\infty$. In both these especial vectorial spaces (of tangents and sines), the Pythagorean Theorem for moduli of the projections is inferred. (By multiplier c, they are transformed in the velocities spaces – see in Ch. 3A).

In these transformations of a material or a world point coordinates as a rule two kinds of bases are used: $\tilde{E}_1 = \{I\}$ and $\tilde{E} = roth\ \Gamma\cdot\tilde{E}_1 = \{roth\ \Gamma\}$. The first base is universal one (16A). In STR, the universal base $\tilde{E}_1 = \{I\}$ is also a relative notion. However it is tied to the immovable inertial Observer, say N_1 in the subbase $\tilde{E}_1^{(3)}$. Canonical trigonometric matrix forms are expressed initially in terms of the base $\tilde{E}_1$! The base determines a relation between Observer N_1 and other pseudo-Cartesian base $\tilde{E}_k = T_{1k}\cdot\tilde{E}_1$ with Observer N_k. The following two pure variants are possible.
(1) $T'_{1k}\cdot T_{1k} = I$. Then $\tilde{E}_{1k}\in\langle rot\ \Theta\rangle$, it is another universal base, but for N_{1k}.
(2) $T_{2k} = T'_{2k}$. Then $\tilde{E}_{2k}\in\langle roth\ \Gamma\rangle$, it is a certain base for inertially moving N_{2k}.

In variant (1), the subbase $\tilde{E}_k^{(3)}$ is immovable with respect to N_1, it is the result of orthospherical rotating $\tilde{E}_1^{(3)}$ at the angle Θ_{1k}. In variant (2), the subbase $\tilde{E}_k^{(3)}$ is moving at the velocity $\mathbf{v} = c\cdot\mathbf{tanh}\ \gamma$ with respect to N_1. Any general homogeneous Lorentzian transformation of bases in $\langle P^{3+1}\rangle$ may by represented as the product of the two pure types transformation (1) and (2) due to the polar decomposition (19A).

Lorentzian transformations are applied *actively* to pseudo-Cartesian bases. Ones in their *passive* (inverse) form, as in (27A), are applied to a material or a world point coordinates. Lorentzian transformations, in that number, in their space and time projections are used as *instantaneous* with changes of space-time coordinates differences by differentials – see more in Chs. 5A–7A and 10A.

The special physical-mathematical principle of relativity (sect. 12.3) takes place for them too. It consists here in form-invariance of expressions for transformations of coordinates in any pseudo-Cartesian bases for a moving uniformly and rectilinearly material point or a world point on a straight world line. Of course, it is the simplest case.

Due to homogeneity and isotropy of the Minkowskian space-time, all Lorentzian transformations may be expressed in the clear *trigonometric forms*. However, if we deal with a moving *non-point* geometric object, then, in addition, the quite another trigonometric type of relativistic transformations may be used. It determines relativistic contraction of the object with geometric parameters in the direction of its physical movement. Generally, in scalar and tensor variants of a trigonometry, projective characteristics of two kinds, either sine–cosine or tangent–secant, are evaluated. Their kind depends on a problem being solved. So, in tensor trigonometry of the space-time, the rotational as deformational elementary trigonometric matrix-functions are used. Their canonical forms with respect to the base $\tilde{E}_1$ were given by formulae (362), (363) and (364), (365), for example, generally in the *fourth-block forms*:

$$\begin{array}{cc} roth\ \Gamma & defh\ \Gamma \end{array}$$

$$\left\| \begin{array}{c|c} \cosh\gamma\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' & \sinh\gamma\cdot\mathbf{e}_\alpha \\ \hline \sinh\gamma\cdot\mathbf{e}'_\alpha & \cosh\gamma \end{array} \right\| \begin{array}{c} \cdots \\ \cdots \end{array} \left\| \begin{array}{c|c} \text{sech}\ \gamma\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' & -\tanh\gamma\cdot\mathbf{e}_\alpha \\ \hline +\tanh\gamma\cdot\mathbf{e}'_\alpha & \text{sech}\ \gamma \end{array} \right\|. \qquad (31A)$$

Rotational hyperbolic matrix (31A) and orthospherical matrix (497) from the sect. 12.2 in these elementary forms are the two pure types of the homogeneous Lorentz transformations in their canonical forms with respect to the universal base $\tilde{E}_1$. All their compositions in pseudo-Cartesian bases admissible with reflector tensor (17A) form the group of continuous homogeneous Lorentz transformations. Such transformations may be reduced to their polar forms as products of these two matrices of pure types. All orthospherical rotations form their proper subgroup of the Lorentz group. (In STR and in non-Euclidean hyperbolic geometry, these two pure types of rotations are used only in elementary forms with $q = 1$, and, more clearly, as (362), (363), (497).

The term "Lorentz transformations group" was introduced by H. Poincaré in his pioneer papers on relativity theory [47]. The rotational homogeneous transformations play the essential role in his previously suggested Physical Principle of Relativity as development of classical Galilean one.

In two next chapters 3A and 4A, we give trigonometric interpretations (sine–cosine and tangent–secant) of four space-time relativistic effects of STR. They take place in the internal and external cavities of the light cone with respect to the original base $\tilde{E}_1$.

Chapter 3A

Einsteinian dilation of time as a consequence of the time-arrow hyperbolic rotation

A world line in $\langle\mathcal{P}^{3+1}\rangle$ is connected at each point M with the instantaneous light cone with its center – a world point M, where two internal cavities of the cone diverge as these *cone of past* and *cone of future*. Any physical movement is directed along the proper time-arrow from past to future. Hence, it is performed inside the light *cone of future*, where a slope of a world line at any point satisfies inequalities $0 \leq |\tanh\gamma| \leq 1$ – Figure 1A(1). In $\langle\mathcal{P}^{3+1}\rangle$, all physical movements are represented by world lines in homogeneous coordinates [47, 49] according to Poincaré and Minkowski. Further the straight lines represent uniform rectilinear physical movements, because the relativistic effects of STR mentioned in Chs. 1A, 2A need in differentiation of 1-st order with linear part as 1-st differentials of increments of space-time coordinates along a world line!

The material point representing a real lengthy object is the object inertia center (the barycenter), i. e., as a particle. A material point M (see Figure 1A) in $\langle\mathcal{P}^{3+1}\rangle$ is physically immovable with respect to a certain frame of reference $\tilde{E}_2$ and is physically moving with respect to $\tilde{E}_1$. The straight world line of the particle M in $\langle\mathcal{P}^{3+1}\rangle$ with respect to $\tilde{E}_1$ is its time-arrow parallel to $\overrightarrow{ct}^{(2)}$ (the light cone inclination does not depend on the base chosen, as it is invariant). For the movement, the bases $\tilde{E}_1$ and $\tilde{E}_2$ are connected by the hyperbolic rotation $\tilde{E}_2 = roth\ \Gamma_{12} \cdot \tilde{E}_1$. From the point of view of Observer N_1, the particle M is moving in $\langle\mathcal{E}^3\rangle^{(1)}$ at velocity $v_{12} = c \cdot \tanh\gamma_{12}$. In a neighborhood of M, a certain process may take place. By the clock of Observer N_2, the process takes time interval $\Delta t^{(2)}$ determined by segment $M'M''$ of the world line parallel to $\overrightarrow{ct}^{(2)}$ with taking into account the scale in the time-arrow. It is, according to STR, the *proper time* $\Delta\tau = \Delta t^{(2)}$ of the process, as it is counted by a relatively immovable clock. Proper time in any moving object is its absolute characteristic, or a pseudo-Euclidean metric invariant inside the cone of future. With respect to its rest base $\tilde{E}_2$, it is identical to coordinate time $\Delta t^{(2)}$. With respect to $\tilde{E}_1$, coordinate time of the process counted by Observer N_1 is determined by projection of the segment $M'M''$ into $ct^{(1)}$ with taking into account the scale, it is equal to $\Delta t^{(1)}$ [53, p. 109]. *Coordinate time* $\Delta t^{(1)}$ of the process in moving object is its *relative characteristic* [48]. For example, with respect to $\tilde{E}_1$, this time is evaluated with the use of passive rotational transformation as well as one in the hyperbolic angle Γ_{12} of $\tilde{E}_1$ into $\tilde{E}_2$:

$$\Delta\mathbf{r}^{(1)} = roth\ \Gamma_{12} \cdot \Delta\mathbf{r}^{(2)} = roth\ \Gamma_{12}) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta c\tau \end{bmatrix} = \begin{bmatrix} \sinh\gamma_{12} \cdot \cos\alpha_1 \cdot \Delta c\tau \\ \sinh\gamma_{12} \cdot \cos\alpha_2 \cdot \Delta c\tau \\ \sinh\gamma_{12} \cdot \cos\alpha_3 \cdot \Delta c\tau \\ \cosh\gamma_{12} \cdot \Delta c\tau \end{bmatrix} = \begin{bmatrix} \Delta x_1^{(1)} \\ \Delta x_2^{(1)} \\ \Delta x_3^{(1)} \\ \Delta ct^{(1)} \end{bmatrix}, \qquad (32A)$$

where $\Delta c\tau = \Delta ct^{(2)}$, and from the matrices fourth rows we obtain:

$$\Delta ct^{(1)} = \cosh\gamma_{12} \cdot \Delta c\tau \rightarrow \Delta c\tau = \Delta ct^{(1)}/\cosh\gamma_{12} < \Delta ct^{(1)}. \qquad (33A)$$

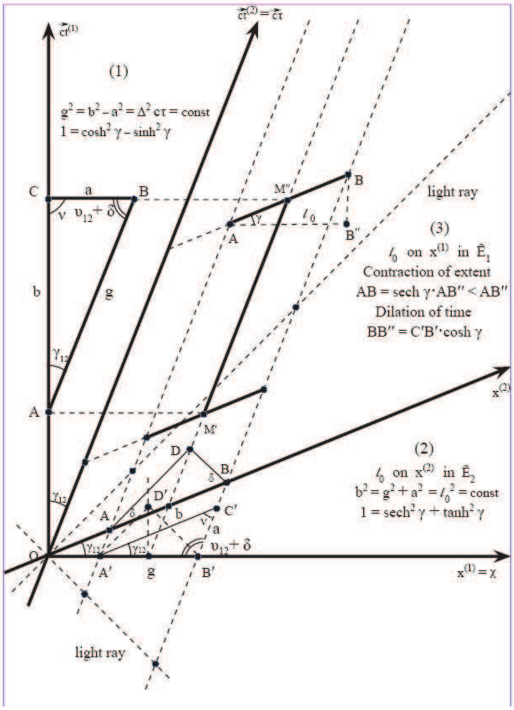

Figure 1A. Trigonometric interpretations of the four relativistic effects inside and outside the light cone in coordinates $\{\mathbf{x}, ct\}$ of the metric space-time according to the Poincaré and Einstein different approaches.

(1). Einstein's dilation of time of a moving object with its interpretation in the pseudo-Euclidean interior right triangle ABC; coordinate and proper velocities:

$$g^2 = b^2 - a^2 = \Delta^2 c\tau = \text{const} \ \sim 1 = \cosh^2\gamma - \sinh^2\gamma,$$
$$b = \Delta ct^{(1)} = \cosh\gamma \cdot g > g = ct^{(2)} \ \rightarrow \ \Delta ct^{(2)} = \Delta c\tau = \Delta ct^{(1)}/\cosh\gamma < \Delta ct^{(1)},$$
$$a = \sinh\gamma \cdot g = \tanh\gamma \cdot b = \Delta x^{(1)} = \Delta\chi,$$
$$v = \Delta x^{(1)}/\Delta t^{(1)} = \Delta\chi/\Delta t^{(1)} = c \cdot \tanh\gamma, \ v^* = \Delta\chi/\Delta\tau = c \cdot \sinh\gamma \ \rightarrow v^* > v.$$
$$v < c \ \text{ and } \ v < v^* < \infty.$$

(2). Lorentz's contraction of a moving rod extent with its interpretation in the pseudo-Euclidean exterior right triangle $A'B'C'$; supervelocity of two moving rods contacts:

$$b^2 = g^2 + a^2 = l_0^2 = \text{const} \ \sim 1 = \text{sech}^2\gamma + \tanh^2\gamma \equiv \cos^2\varphi(\gamma) + \sin^2\varphi(\gamma),$$
$$g = l = \text{sech}\ \gamma \cdot b < b = l_0 \ \rightarrow \ l = \text{sech}\ \gamma \cdot l_0 \equiv \cos\varphi(\gamma) \cdot l_0 < l_0,$$
$$a = \tanh\gamma \cdot b = \tanh\gamma \cdot l_0 = \Delta ct^{(2)} \neq 0, \ w = l_0/\Delta t^{(2)} = c \cdot \coth\gamma = c^2/v > c.$$

(3). The Einsteinian approach to creation of STR on the basis of his definition of simultaneity (Ch. 4A) with validation of the same relativistic effects in $\tilde{E}_1$ and $\tilde{E}_2$.

In STR relativistic effect (33A) is called *Einsteinian dilation of time* [53, p. 30, 48]. The segment $\Delta c\tau$ of the straight world line, i. e., of the process time in M, is expressed in the coordinates of its base $\tilde{E}_2 = \{x^{(2)}, \overrightarrow{ct}^{(2)}\}$. Geometrically this segment of the world line is a linear tensor element as the time-like oriented vector in $\langle \mathcal{P}^{3+1} \rangle$.

Its quadratic pseudo-Euclidean imaginary invariant in the general four-dimensional form of coordinates with respect to any pseudo-Cartesian base $\tilde{E}$ is

$$-(\Delta c\tau)^2 = -(\Delta ct)^2 + (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 = \text{const}, \qquad (34A)$$

where $\Delta t > 0,\ \Delta\tau > 0$. The invariant can be interpreted as radius $+iR = i\Delta c\tau$ of a Minkowskian hyperboloid II, $\Delta\mathbf{r}^{(2)}$ is its certain radius-vector (see in Ch. 12). Since $\Delta c\tau = const$, invariant (34A) can be reduced, say in the base $\tilde{E}_1$, to its sine-cosine form-invariant trigonometric expression varied by the radius of a unity hyperboloid II:

$$(i)^2 = -1 = -\cosh^2\gamma + (\sinh^2\gamma'_1 + \sinh^2\gamma'_2 + \sinh^2\gamma'_3) =$$
$$= -\cosh^2\gamma + ||\mathbf{sinh}\ \gamma||^2 = -\cosh^2\gamma + \sinh^2\gamma, \qquad (35A)$$

where γ'_j (at $j = 1, 2, 3$) are the particular hyperbolic angles with their values in the Euclidean orthoprojections $\sinh\gamma'_j = \cos\alpha_j \cdot \sinh\gamma$ of the space-like sine vector $\mathbf{sinh}\ \gamma$ in the base $\tilde{E}_1$. Formula (35A) gives trigonometric quadratic invariant -1 under Lorentzian transformations of an unit time-like linear element $\Delta\mathbf{i}$.

Invariant scalar proper time is expressed in any pseudo-Cartesian base $\tilde{E}$ as

$$\Delta\tau = \Delta t/\cosh\gamma = \min\langle\Delta t^{(k)}\rangle. \qquad (36A)$$

When one deals with a *curvilinear world line*, the similar rotational transformation is *instantaneous*, and (32A) is applied to its arc differential as a linear element:

$$d\mathbf{r}^{(1)} = \{roth\ \Gamma\}^{(m)} d\mathbf{r}^{(m)} = \{roth\ \Gamma\}^{(m)} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ dc\tau^{(m)} \end{bmatrix} = \begin{bmatrix} dx_1^{(1)} \\ dx_2^{(1)} \\ dx_3^{(1)} \\ dct^{(1)} \end{bmatrix}. \qquad (37A)$$

Here the linear element $d\mathbf{r}^{(m)}$ is expressed also in coordinates of the *instantaneous base* $\tilde{E}_m = \{x^{(m)}, \overrightarrow{ct}^{(m)}\}$. In STR the instantaneous bases, *on the differential level, are always inertial*, but only from the point of view of inertial Observer, say N_1 in $\tilde{E}_1$. This has place, because the axes $\overrightarrow{ct}^{(m)}$ and $x^{(m)}$ are *instantaneous tangent and pseudo-normal* to a world line at a point M. Hence, the differential form similar to (36A) is

$$d\tau^{(m)} = dt^{(1)}/\cosh\gamma = d\lambda^{(m)}/(ic) = \min\langle dt^{(k)}\rangle. \qquad (38A)$$

Integrating (38A), one obtains $\Delta\tau = \Delta\lambda/(ic)$, where $\Delta\lambda$ is the pseudo-Euclidean length of a world line segment [53, p. 110]. Formulae (36A), (38A) express in the clear trigonometric form the relativistic effect of Einsteinian dilation of time in a moving object with respect to immovable Observer, for example, of some process time in the object [48]. *The effect may be easily interpreted as a consequence of the hyperbolic rotation of* $\overrightarrow{ct}^{(m)}$! Formally, this effect of time dilation was first established by V. Voight in 1887 [57] (in his light elasticity theory) and independently by H. Lorentz in 1892.

This segment of a world line $\Delta\mathbf{r}^{(2)}$ in $\tilde{E}_1$, according to (32A), has else the space-like orthoprojection $a = \Delta\chi$ into $\langle\mathcal{E}^3\rangle^{(1)}$ – Figure 1A. (But such space projection of $\Delta\mathbf{r}^{(2)}$ into $\langle\mathcal{E}^3\rangle^{(2)}$ is zero.) It is the non-relativistic space trajectory of the object M corresponding to a world line segment $\Delta\mathbf{r}^{(2)}$. It can be expressed *in terms of coordinate time as well as Einsteinian proper time* with the two different definitions of velocity:

$$\Delta\,\chi = \sqrt{\Delta^2 x_1^{(1)} + \Delta^2 x_2^{(1)} + \Delta^2 x_3^{(1)}} = \tanh\gamma\,\Delta ct^{(1)} = v\cdot\Delta t^{(1)} = \sinh\gamma\,\Delta c\tau = v^*\cdot\Delta\tau.$$

The *proper velocity* v^* is defined in addition to a *coordinate velocity* v as a concomitant relativistic effect. It is measured in proper distance $d\chi = dx^{(1)}$ by proper time $d\tau$:

$$\left.\begin{array}{l} v^* = c\cdot\sinh\gamma = d\chi/d\tau = v\cdot\cosh\gamma = d\chi/d\tau > v = c\cdot\tanh\gamma; \\ v_j^* = c\cdot\sinh\gamma_j' = c\cdot\cos\alpha_j\cdot\sinh\gamma > v_j = c\cdot\tanh\gamma_j\ (j=1,2,3). \end{array}\right\} \qquad (39A)$$

The four vectors $\mathbf{v}, \mathbf{v}^*, \mathbf{tanh}\gamma, \mathbf{sinh}\ \gamma$ are collinear. The hyperbolic angles γ_j and γ_j' in (30A) and (35A) are related as follows:

$$(v_j = c\cdot\tanh\gamma_j = v\cdot\cos\alpha_j,\ \ v_j^* = c\cdot\sinh\gamma_j' = v^*\cdot\cos\alpha_j)\ \rightarrow\ \sinh\gamma_j' = \cosh\ \gamma\cdot\tanh\gamma_j.$$

In the pseudoplane of hyperbolic rotation, the given problem is reduced to solving an "interior hyperbolically right pseudo-Euclidean triangle" (see in sect. 6.4), where $\Delta c\tau$ is similar to the hypotenuse g, and $\Delta\chi$, $\Delta ct^{(1)}$ are similar to the legs a, b.

In products (32A), (37A), the hyperbolic rotational matrix is formally truncated, only its last row is used, because the original linear element $\Delta\mathbf{r}^{(2)}$ is parallel to its time-arrow $\overrightarrow{ct}^{(2)}$, and all its points in $\tilde{E}_2$ have *zero abscissa*. The whole matrix is used if the original element is on another time-arrow $\overrightarrow{ct}^{(3)}$ under an additional angle γ_{23} from the time-arrow $\overrightarrow{ct}^{(2)}$. It is valid for two- and multistep motions (see in Ch. 7A).

Another important theorem of STR and Minkowskian Geometry is the following.

Let M' and M'' be two causally-connected world points in $\langle P^{3+1}\rangle$. Then the straight-line segment $M'M''$ inside the light cone of future has the maximal pseudo-Euclidean length (proper time) among all continuous world lines in $\tilde{E}_1$ connecting M' and M'':

$$ct_2 - ct_1 = ct|_{t_1}^{t_2} = \Delta ct\ > \Delta ct' = \int_{t_1}^{t_2} dct/\cosh\gamma(t) = \int_{t_1'}^{t_2'} dct' \geq 0,$$

where t is the time of relatively immovable Observer, t' is the time in a moving object. Such continuous world line has the minimal length $\lambda = 0$ if the points M' and M'' are connected by light segments, and only two light segments are enough. The inequality (divided by c) is the descriptive trigonometric illustration to the well-known relativistic "twins paradox", as its left part is so called *earth time* t, and its right part is time t' counted by astronauts. The notion "proper velocity" as $v^* = c\cdot\sinh\gamma > v$ gives the simplest its interpretation from (39A). Similar effects of STR, with real difference of time in different frames of reference, are possible only under action of the two great basis Poincaré Relativity Principle and Mach Principle. Moreover, in Ch. 5A we shall prove, that for men (non-robots) the flights even to nearest stars are Utopia.

Chapter 4A

Lorentzian seeming contraction of moving object extent as a consequence of the moving Euclidean subspace hyperbolic deformation

The Lorentzian seeming contraction of extent, as opposed to dilation of space and time coordinates in a moving system with the coefficient $\cosh^{-1}\gamma(v)$ – see Figure 1A (2), (3), is interpreted correctly on the basis of Einsteinian physical definition of simultaneity. The latter is a geometric theorem in $\langle\mathcal{P}^{3+1}\rangle$, but only due to its quadratic metric!

In the external cavity of the light cone in $\langle\mathcal{P}^{3+1}\rangle$ – see at Figure 1A(2), one usually considers some sets of world points belonging on the whole to a certain Euclidean space $\langle\mathcal{E}^3\rangle^{(k)}$, mapping in the base $\tilde{E}_k$ with its own time coordinate $t^{(k)}$. In the simplest practical variant, the set consists of two world points as some events with a space-like interval between them. In another variant, important for subject of this Chapter, the set consists of points of a concrete geometric object immovable in a certain Euclidean space $\langle\mathcal{E}^3\rangle^{(j)}$ and moving with its projective map in a certain base $\tilde{E}_i$ from point of view of Observer N_i. Of course, in the base $\tilde{E}_j$ all the geometric object points are simultaneous, as they have the same time coordinate on its own time-arrow $\overrightarrow{ct}^{(j)}$.

From the other hand, all world points of a given geometric object belong to their world lines in $\langle\mathcal{P}^{3+1}\rangle$. If the object is immovable with respect to the base $\langle\mathcal{E}^3\rangle^{(j)}$ and it is in uniform rectilinear movement with respect to the base $\tilde{E}_i$, then the world lines of all its points are parallel to the time-arrow $\overrightarrow{ct}^{(j)}$. Observer N_i fixes the moving object points in his own $\langle\mathcal{E}^3\rangle^{(i)}$ at a certain value of time on his own time-arrow $\overrightarrow{ct}^{(i)}$ although simultaneously, but with the object sizes distortion along the moving direction. This space-like phenomenon is defined as *a improper world fixation* of the world points or of the moving object (as a set of world points fixed in $\langle\mathcal{E}^3\rangle^{(i)}$).

$\langle\mathcal{E}^3\rangle^{(i)}$ and $\overrightarrow{ct}^{(j)}$ are hyperbolically orthogonal in $\langle\mathcal{P}^{3+1}\rangle$ iff the object is physically immovable just in $\langle\mathcal{E}^3\rangle^{(i)}$. Then $i = j$ and *the world fixation of the object is proper.* It corresponds to true sizes of the object as immovable one. This graphical way for constructing fixations defines simultaneity of the world points set in a certain base.

The Einstein's definition of simultaneity is a graceful geometric theorem in $\langle\mathcal{P}^{3+1}\rangle$. In the 2-, 3- and 4-dimensional cases, it is expressed as follows.

Theorem 1. *If a triangle ABC* (see Figure 1A) *is formed by a space-like segment AB and two light segments AC and BC (i. e., isotropic zero legs) coming from the opposite directions, then its median and height passing through the point C are identical.*

Corollary. If ABC is such a light triangle in a certain pseudoplane, then its median (a height) and its base (a hypotenuse) are the time-arrow $\overrightarrow{ct}^{(k)}$ and the space axis $x^{(k)}$.

Theorem 2. *In the cone obtained with any elliptic cut of a light cone, the median passing through its apex C and* 2- *or* 3-*dimensional base are hyperbolically orthogonal to each other, hence its height and median passing through the point C are identical.*

The theorems, on the base of Einstein's physical definition of simultaneity, motivate the *quadratic* metric in his physical version of STR! Simultaneity of events as a world points fixation is a relative notion too. It is defined with respect to a certain Euclidean space $\langle\mathcal{E}^3\rangle^{(k)}$ with a time-arrow $\overrightarrow{ct}^{(k)}$ of the pseudo-Cartesian base $\tilde{E}_k$ in $\langle\mathcal{P}^{3+1}\rangle$.

This is illustrated clearly at Figure 1A(2). Here a rod as a geometric object is immovable on the axis x_2 (i. e., $j = 2$), and it is moving physically along the axis x_1 (i. e., $i = 1$) at velocity $\pm v$ ($\tanh\,\gamma = ||v||/c$). This rod points world lines are parallel to the time-arrow $\overrightarrow{ct}^{(j)}$. That is why, Observer N_i fixes this rod points on its axis x_1 as their projections in parallel to the time-arrow $\overrightarrow{ct}^{(j)}$. From mathematical point of view, the improper fixation is *a cross projection* onto x_1 parallel to $\overrightarrow{ct}^{(j)}$ – see definition of a cross projection as *tensor trigonometric deformation* in sect. 5.10. But here we have the tensor deformation of the hyperbolic type, i. e., in terms of *hyperbolic* secant-tangent transformation. Due to this, the moving rod contraction seems to Observer N_i.

In general, an improper world fixation, with respect to a certain pseudo-Cartesian base $\tilde{E}_i$, is defined here as a graphically *simultaneous* cut of a geometric object world trajectory in parallel to $\langle\mathcal{E}^3\rangle^{(i)}$ at a certain moment of time $t^{(i)}$. If the object is physically immovable in $\langle\mathcal{E}^3\rangle^{(j)}$, then its world trajectory in $\langle\mathcal{P}^{3+1}\rangle$ is parallel to the time-arrow $\overrightarrow{ct}^{(j)}$. Therefore definition of a world fixation of an object in the base $\tilde{E}_i$ is reduced to its projecting into $\langle\mathcal{E}^3\rangle^{(i)}$ parallel to $\overrightarrow{ct}^{(j)}$, i. e., to a space projection in the cross base $\tilde{E}_{i,j} \equiv \{x_k^{(i)}, \overrightarrow{ct}^{(j)}\}$ (sect. 5.10). Single cross projecting is expressed trigonometrically as hyperbolic deformation in the pseudoplane of rotation. The pseudoplane at cross projecting has some properties of a quasi-Euclidean plane, but only of the universal base $\tilde{E}_i$, as then the *cross quasi-Euclidean invariant* under trigonometric deformations is valid in this pseudoplane, see sect. 5.10 and 12.2. For a given geometric object, the volume of its fixation is maximal iff the fixation is proper:

$$V = v^{(i,j)}/\mathrm{sech}\ \gamma = \max\langle v^{(i,j)}\rangle = const. \qquad (40A)$$

If a k-dimensional ($k \leq n$) geometric object is moving rectilinearly and uniformly, then exactly four variants of its world trajectory are possible:
1) a **line** if $k = 0$, the object is a particle as a world point;
2) a **band** if $k = 1$, the object is a rod as a directed segment (a vector);
3) a **3-dimensional band** if $k = 2$, the object is a triangle or a parallelogram;
4) a **4-dimensional band** if $k = 3$, the object is a tetrahedron or a parallelepiped.
We consider only simplest objects, they are represented by $4 \times$ *k-lineors*, see sect. 5.1.

The set of all world fixations for a given object is, from geometrical point of view, equivalent to the set of all space-like cuts of its *world trajectory*. So, relatively immovable Observer N_1 fixes a rod simultaneously as its projection into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(2)}$ (see Figure 1A). A world fixation, as well as a world trajectory, is a tensor notion, their valency is 1. World fixations of objects pointed out above are expressed as either 4×1-vectors, or 4×2-lineors, or 4×3-lineors. If an object is immovable in $\langle\mathcal{E}^3\rangle^{(j)}$, then its proper world fixation is defined with respect to $\tilde{E}_j$.

In the base $\tilde{E}_j$, these one-, two-, and three-dimensional immovable geometric objects reduced to a current world point (the barycenter of a material body) are expressed initially as the following space-like $4 \times k$-*lineors* in the Minkowskian linear space-time:

$$\mathbf{a}^{(j)} = \begin{bmatrix} \Delta x_1^{(j)} \\ \Delta x_2^{(j)} \\ \Delta x_3^{(j)} \\ 0 \end{bmatrix}; \quad A_{4\times 2}^{(j)} = \begin{bmatrix} \Delta x_{11}^{(j)} & \Delta x_{12}^{(j)} \\ \Delta x_{21}^{(j)} & \Delta x_{22}^{(j)} \\ \Delta x_{31}^{(j)} & \Delta x_{32}^{(j)} \\ 0 & 0 \end{bmatrix}; \quad A_{4\times 3}^{(j)} = \begin{bmatrix} \Delta x_{11}^{(j)} & \Delta x_{12}^{(j)} & \Delta x_{13}^{(j)} \\ \Delta x_{21}^{(j)} & \Delta x_{22}^{(j)} & \Delta x_{23}^{(j)} \\ \Delta x_{31}^{(j)} & \Delta x_{32}^{(j)} & \Delta x_{33}^{(j)} \\ 0 & 0 & 0 \end{bmatrix}. \tag{41A}$$

With respect to the cross base $\tilde{E}_{j,i}$, we take out only Euclidean images in $\langle \mathcal{E}^3 \rangle^{(j)}$ of the lineors as their proper fixations, because they are immovable with respect to $\tilde{E}_j$:

$$\mathbf{a}^{(j,i)} = \mathbf{a}^{(j)}; \quad A_{4\times 2}^{(j,i)} = A_{4\times 2}^{(j)}; \quad A_{4\times 3}^{(j,i)} = A_{4\times 3}^{(j)}. \tag{42A}$$

If the coordinates of these tensors are subjected to deformational transformation from $\tilde{E}_{j,i}$ into another cross base $\tilde{E}_{i,j}$, then *one-time pseudo-Euclidean quasi-invariant* holds (i. e., for the one-time transformation). The invariant is expressed as follows:

$$[\mathbf{a}^{(j)}]' \cdot \mathbf{a}^{(j)} = [\mathbf{a}^{(i,j)}]' \cdot \mathbf{a}^{(i,j)} = ||\mathbf{a}||_E^2 = l_0^2 = \text{const} > 0, \tag{43A}$$

$$[A^{(j)}]' \cdot A^{(j)} = [A^{(i,j)}]' \cdot A^{(i,j)} = |A|^2 = \text{Const}, \tag{44A}$$

where $|A|$ is the $k \times k$-matrix Euclidean module of the $4 \times k$-lineor A (sect. 9.4). This is similar to the Euclidean invariant due to one-time *spherical-hyperbolic analogy* (341) with respect to the *universal* base $\tilde{E}_i$ for Observer N_i fixed the Lorentzian contraction:

$$\tilde{E}_j = roth\, \Gamma_{ij} \tilde{E}_i \to \tilde{E}_{i,j} = defh\, \Gamma_{ij} \cdot \tilde{E}_{j,i},\ defh\, \Gamma_{ij} \equiv rot\, \Phi(\Gamma_{ij}) \equiv defh^{-1} \Gamma_{ji}. \tag{45A}$$

Express with the passive modal transformation the new coordinates of lineors (41A) with initial equalities (42A) in terms of both the modal matrices:

$$\mathbf{a}^{(i,j)} = defh\ \Gamma_{ij} \cdot \mathbf{a}^{(j)} = rot\ \Phi_{ij} \cdot \mathbf{a}^{(j)} = \begin{bmatrix} \Delta x_1^{(i,j)} \\ \Delta x_2^{(i,j)} \\ \Delta x_3^{(i,j)} \\ \Delta ct^{(j,i)} \end{bmatrix}, \tag{46A}$$

$$A_{4\times 2}^{(i,j)} = defh\ \Gamma_{ij} \cdot A_{4\times 2}^{(j)} = rot\ \Phi_{ij} \cdot A_{4\times 2}^{(j)} = \begin{bmatrix} \Delta x_{11}^{(i,j)} & \Delta x_{12}^{(i,j)} \\ \Delta x_{21}^{(i,j)} & \Delta x_{22}^{(i,j)} \\ \Delta x_{31}^{(i,j)} & \Delta x_{32}^{(i,j)} \\ \Delta ct_1^{(j,i)} & \Delta ct_2^{(j,i)} \end{bmatrix}, \tag{47A}$$

$$A_{4\times 3}^{(i,j)} = defh\ \Gamma_{ij} \cdot A_{4\times 3}^{(j)} = rot\ \Phi_{ij} \cdot A_{4\times 3}^{(j)} = \begin{bmatrix} \Delta x_{11}^{(i,j)} & \Delta x_{12}^{(i,j)} & \Delta x_{13}^{(i,j)} \\ \Delta x_{21}^{(i,j)} & \Delta x_{22}^{(i,j)} & \Delta x_{23}^{(i,j)} \\ \Delta x_{31}^{(i,j)} & \Delta x_{32}^{(i,j)} & \Delta x_{33}^{(i,j)} \\ \Delta ct_1^{(j,i)} & \Delta ct_1 2^{(j,i)} & \Delta ct_3^{(j,i)} \end{bmatrix}. \tag{48A}$$

Thus we have two equivalent trigonometric definitions of a general world fixation with one-time cross projecting, and respectively two kinds of the modal matrices in (45A): hyperbolic deformational one and spherical rotational one. In the spherical rotational variant, the angle Γ should be transformed into its analog $\Phi(\Gamma)$ by the analogy. The second variant is used for visual graphical interpretation of the Lorentz contraction. We choose mainly the first variant with angle Γ_{ij} connected simply with velocity $\mathbf{v}$. For example, express by passive modal transformation (46A) the new coordinates of the rod in terms of original ones from (41A), (42A) with the use of canonical structure (364) for the hyperbolic deformational modal matrix:

$$\mathbf{a}^{(i,j)} = \begin{bmatrix} \Delta x_1^{(i)} \\ \Delta x_2^{(i)} \\ \Delta x_3^{(i)} \\ \Delta ct^{(j)} \end{bmatrix} = \begin{bmatrix} \Delta x_1^{(j)} - \cos\alpha_1 \cdot \cos\varepsilon \cdot l_0 \cdot (1 - \operatorname{sech}\gamma) \\ \Delta x_2^{(j)} - \cos\alpha_2 \cdot \cos\varepsilon \cdot l_0 \cdot (1 - \operatorname{sech}\gamma) \\ \Delta x_3^{(j)} - \cos\alpha_3 \cdot \cos\varepsilon \cdot l_0 \cdot (1 - \operatorname{sech}\gamma) \\ \cos\varepsilon \cdot l_0 \cdot \tanh\gamma \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{e}_\alpha \cdot [1 - \cos\varepsilon \cdot (1 - \operatorname{sech}\gamma)] \cdot l_0 \\ \cos\varepsilon \cdot \tanh\gamma \cdot l_0 \end{bmatrix}, \qquad (49A)$$

where in the rod fixation, the first three rows determine its new Cartesian coordinates in the base $\tilde{E}_i$, the fourth row determines its non-zero time-like projection onto $\overrightarrow{ct}^{(j)}$ as the additional *time-like effect* (explanation in details will be lower);
$l_0 = ||\mathbf{a}^{(j)}||$ is the Euclidean length of the rod in its rest state in the subbase $\tilde{E}_j^{(3)}$, ε is the angle in the same subbase $\tilde{E}_j^{(3)}$ between the rod and the antivelocity vector $\mathbf{v}_{ji} = (-\mathbf{e}_\alpha \cdot v_{ij})^{(j)}$ with the unity vector of directional cosines (formally these cosines are equal to ones of $\mathbf{v}_{ij}$ in $\tilde{E}_i^{(3)}$). And there holds

$$\cos\alpha_1 \cdot \Delta x_1^{(j)} + \cos\alpha_2 \cdot \Delta x_2^{(j)} + \cos\alpha_3 \cdot \Delta x_3^{(j)} = \mathbf{e}'_\alpha \cdot \mathbf{a}^{(j)} = \cos\varepsilon \cdot l_0 = ||\overleftarrow{\mathbf{v}\mathbf{v}'} \cdot \mathbf{a}^{(j)}||. \quad (50A)$$

Note one more relativistic effect: *the hyperbolic angle between the velocity and antivelocity is non-zero and equal to* γ_{ij}. If the velocity and the axis x_1 are parallel, then $\cos\alpha_1 = 1 = \cos\varepsilon$, $\cos\alpha_2 = \cos\alpha_3 = 0$, and the new rod coordinates are

$$\mathbf{a}^{(i,j)} = \begin{bmatrix} \Delta x_1^{(i)} \\ \Delta x_2^{(i)} \\ \Delta x_3^{(i)} \\ \Delta ct^{(j)} \end{bmatrix} = \begin{bmatrix} 0 + \operatorname{sech}\gamma \cdot \Delta x_1^{(j)} \\ \Delta x_2^{(j)} \quad + 0 \\ \Delta x_3^{(j)} \quad + 0 \\ 0 + \tanh\gamma \cdot \Delta x_1^{(j)} \end{bmatrix}, \quad (\Delta x_1^{(j)} = \cos\varepsilon \cdot l_0 = l_0). \quad (51A)$$

Here the non-relativistic and relativistic parts are pointed out as the summands from the left and from the right respectively. More generally, if in (49A) also the rod and the velocity are formally coaxial ($\cos\varepsilon = 1$) in $\tilde{E}_j^{(3)}$, then there holds

$$\mathbf{a}^{(i,j)} = \begin{bmatrix} \mathbf{e}_\alpha \cdot \operatorname{sech}\gamma \cdot l_0 \\ \tanh\gamma \cdot l_0 \end{bmatrix}. \qquad (52A)$$

The Cartesian coordinates in (51A, 52A) express the relativistic effect of so-called *Lorentzian contraction of extent* [46; 53, p. 109], which realizes coaxially to velocity:

$$l^{(i,j)} = l^{(i)} = \text{sech}\ \gamma_{ij} \cdot l_0 = \sqrt{1-(v/c)^2} \cdot l_0 < l_0. \qquad (53A)$$

Other coordinates are normal to the velocity, they do not change. The original and new *four* coordinates of the rod in (49A) and special cases (51A) satisfy (43A), i. e., they form quasi-Euclidean invariant, this follows from (45A). The sum of all *three space* coordinates squares is the squared Euclidean length module of the rod contracted. In this most general case, for the Lorentzian contracted oriented rod, there holds:

$$l^{(i,j)} = l^{(i)} = ||\Delta\mathbf{x}^{(i)}|| = l_0\sqrt{\cos^2\varepsilon \cdot \text{sech}^2\gamma_{ij} + \sin^2\varepsilon} =$$

$$= l_0\sqrt{1-\cos^2\varepsilon \cdot \tanh^2\gamma_{ij}} = l_0\sqrt{1-\cos^2\varepsilon \cdot (v/c)^2} < l_0. \qquad (54A)$$

Apply the Herglotz Principle and evaluate its relativistic and non-relativistic summands. The non-relativistic Cartesian part of a fixation (that is normal to the velocity vector) is the Euclidean invariant:

$$\mathbf{a}^{(i,j)}_{inv} = \mathbf{a}^{(j)} - \cos\varepsilon \cdot l_0 \cdot \begin{bmatrix} \mathbf{e}_\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{x}^{(j)} - \mathbf{e}_\alpha \cos\varepsilon \cdot l_0 \\ 0 \end{bmatrix}. \qquad (55A)$$

Subtracting (49A) and (55A) gives the relativistic part:

$$\mathbf{a}^{(i,j)}_{rel} = \begin{bmatrix} \mathbf{e}_\alpha \cdot \cos\varepsilon \cdot\ \text{sech}\ \gamma \cdot l_0 \\ \cos\varepsilon \cdot \tanh\gamma \cdot l_0 \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{x}^{(i)}_{rel} \\ \Delta ct^{(j)} \end{bmatrix}. \qquad (56A)$$

Apply the Pythagorean Theorem to its Cartesian part and obtain the relativistic part $|\cos\varepsilon \cdot \text{sech}\ \gamma \cdot l_0|$ for the Euclidean length of a moving rod. From (55A) and (50A) the non-relativistic part $|\sin\varepsilon \cdot l_0|$ is evaluated too. This is the algebraic way for explaining structure of (54A), another way is graphical. The Euclidean length of a moving rod is, due to (54A), the orthogonal sum in $\langle\mathcal{E}^3\rangle^{(i)}$ of non-relativistic projection $\sin\varepsilon \cdot l_0$ and relativistic projection $\cos\varepsilon \cdot \text{sech}\ \gamma \cdot l_0$. *The first summand* is normal projection of the rod relatively to the antivelocity $\mathbf{v}_{ji}$. It is invariant under cross projecting (hyperbolic deformation). That is why, this part of the rod fixation is spherically orthogonal to both the vectors $\mathbf{v}_{ij}$ in $\langle\mathcal{E}^3\rangle^{(i)}$ and $\mathbf{v}_{ji}$ in $\langle\mathcal{E}^3\rangle^{(j)}$. *The second summand* (relativistic projection) is obtained from parallel projection of the rod with its cross projecting into $\langle\mathcal{E}^3\rangle^{(i)}$ parallel to $\overrightarrow{ct}^{(j)}$, i. e., under condition in (52A) onto velocity $\mathbf{v}_{ij}$.

Squared quasi-Euclidean lengths of relativistic fixations (52A) and (56A) for the rod, due to (43A) and (45A), are invariants under one-time hyperbolic deformation. More exactly, they are space-like quadratic *cross quasi-Euclidean invariants*:

$$[l^{(j)}]^2 = ||\Delta\mathbf{x}^{(i,j)}||^2 + \Delta^2 ct^{(j,i)} = [l^{(i,j)}]^2 + \Delta^2 ct^{(j,i)} = l_0^2 = \text{const}, \qquad (57A)$$

$$[l^{(j)}]^2_{rel} = ||\Delta\mathbf{x}^{(i,j)}||^2_{rel} + \Delta^2 ct^{(j,i)} = [l^{(i,j)}]^2_{rel} + \Delta^2 ct^{(j,i)} = l_0^2\cos^2\varepsilon = \text{const}. \qquad (58A)$$

The trigonometric secant-tangent form of invariant (58A) is

$$(\mathrm{sech}^2\gamma_1'' + \mathrm{sech}^2\gamma_2'' + \mathrm{sech}^2\gamma_3'') + \tanh^2\gamma = ||\mathbf{sech}^2\gamma|| + \tanh^2\gamma = 1, \qquad (59A)$$

where γ_k'' is the hyperbolic angle between vector $-\mathbf{v}_{ji}$ in the subbase $\tilde{E}_j^{(3)}$ and the axis x_k in the subbase $\tilde{E}_i^{(3)}$ and sech $\gamma_k'' = \cos\alpha_k \cdot$ sech γ. This is an invariant for a unit space-like linear element. The proper length of a rod (in the rest state) is a quasi-Euclidean metric invariant in all other cross bases $\tilde{E}_{kj}$, in particular, in $\tilde{E}_{ij}$:

$$l_0 = \frac{l^{(i,j)}}{\sqrt{1 - \cos^2\varepsilon \cdot \tanh^2\gamma_{ij}}} = \max\langle l^{(i,j)}\rangle. \qquad (60A)$$

This follows from (54A). The Lorentzian contraction as the relativistic effect has coordinate nature, i. e., it does not lead to any mechanical stretch. Formally, contraction of moving objects as in (53A) was first established by G. Fitzgerald in 1892 [58].

The set of all world fixations of a moving rod is semiopen, as it does not contain extremal cuts of its world trajectory by the hypersurface of the light cone, see Figure 1A. These extremal cuts for a rod have zero Euclidean length of the relativistic space cross projection, ones for objects of rank greater than 1 have zero Euclidean norms of order 1 and 2 for their relativistic space cross projection and order 3 for their space volume fixation. These cuts correspond to objects as if moving at the velocity c.

Furthermore, this rod, in addition, has the time-like projection in the same cross base $\tilde{E}_{ij}$, this follows from (56A). Projecting is performed into the time-arrow $\overrightarrow{ct}^{(j)}$, thus it is expressed in the base $\tilde{E}_j$. This effect has the following relativistic explanation. Observer N_j can see the analogous rod as immovable on the axis $x_1^{(i)}$ and moving at the same velocity $\mathbf{v}_{ji}$ in $\tilde{E}_j$, with seeming Euclidean length (54A). In the general case, when the two identical rods meet, their two left ends and two right ends considered separately meet, according to (56A), with the following time lag:

$$\Delta ct^{(i,j)} = \Delta ct^{(j)} = \cos\ \varepsilon \cdot l_0 \cdot \tanh\gamma_{ij} \neq 0. \qquad (61A)$$

It is the *relativistic effect of non-synchronous meeting of two identical immovable and moving coaxial rods paired points.* Contact of the points pairs of meeting rods (if $\varepsilon = 0$) is spreading at the left to the right along the axis $x_1^{(j)}$ at *supervelocity* w greater than c:

$$s = l_0/\Delta t^{(j)} = c/\tanh\gamma_{ij} = c \cdot \coth\gamma_{ij} = c \cdot \cosh\upsilon_{ij} = c^2/v > c. \qquad (62A)$$

(See connections of these complementary hyperbolic angles γ and υ in (360), sect. 6.4.) During this accelerated movement the *coordinate supervelocity* decreases from ∞ to c (for the angle γ) and increases from c to ∞ (for the complementary angle υ). However, in the classic mechanics, the pairs of points meet simultaneously.

Note, that the set $\langle w \cdot \mathbf{e}_\alpha\rangle$ forms the hyperbolic cotangent vector space that is the cotangent models outside the trigonometric circle (the Cayley's oval) of radius 1 or c for supervelocity, where the motion angles γ is on a hyperboloid I (see Ch. 12, 6A, 7A).

In products (46A)–(48A) the hyperbolic deformational matrix is formally truncated, only three first rows are used (compare with rotational matrices in products (32A), (37A) in Ch. 3A, because the original objects (lineors) in forms (41A) are parallel to their proper Euclidean space $\langle\mathcal{E}^3\rangle^{(j)}$.

In the common pseudoplane of the hyperbolic rotation $roth\ \Gamma_{ij}$ in the base $\tilde{E}_i$ and the hyperbolic deformation $defh\ \Gamma_{ij}$ in the cross base $\tilde{E}_{ij}$ at Figure 1A, this problem is reduced to solving the *exterior right triangles*: either *pseudo-Euclidean* one ABC (sect. 6.4), where $l^{(i)}$ is similar to hypotenuse $AB = g$ and l_0, $\Delta ct^{(j)}$ are similar to legs a, b; or *quasi-Euclidean* one $A'B'D'$ (Figure 1A(2)), where otherwise $a = A'D'$ is similar to hypotenuse as l_0, $g = A'B'$ is similar to leg $l^{(i)}$ as contracted rod length, $b = B'D' = \Delta ct^{(j)}$; $i = 1$, $j = 2$). Then Lorentzian contraction is expressed formally in the *quasiplane* by the spherical rotation $rot\ \Phi(\Gamma_{ij})$ in (45A) in the universal base $\tilde{E}_i$, and hyperbolic cross projections are determined due to the Pythagorean theorem.

In a cross base $\tilde{E}_{ij}$, for two vectors (rods) applied in one world point M, there holds

$$\cos\beta_{12}^{(i,j)} = [\mathbf{a}_1^{(i,j)}]' \cdot \mathbf{a}_2^{(i,j)}/||\mathbf{a}_1^{(i,j)}|| \cdot ||\mathbf{a}_2^{(i,j)}|| = [\mathbf{e}_1^{(i,j)}]' \cdot \mathbf{e}_2^{(i,j)}, \quad (\beta_{12} \in [0;\pi]).$$

Here the algebraic formula for the cosine of the angle between two vectorial fixations in $\langle\mathcal{E}^3\rangle^{(i)}$ is given. Apply (54A) to this expression. The result is the trigonometric formula for the cosine of the angle between two moving vectors (rods) applied in M:

$$-1 \le \cos\beta_{12}^{(i)} = \frac{\cos\beta_{12}^{(j)} - \cos\varepsilon_1 \cdot \cos\varepsilon_2 \cdot \tanh^2\gamma}{\sqrt{1 - \cos^2\varepsilon_1 \cdot \tanh^2\gamma} \cdot \sqrt{1 - \cos^2\varepsilon_2 \cdot \tanh^2\gamma}} \le +1, \qquad (63A)$$

where $\beta_{12}^{(j)}$ and $\beta_{12}^{(i)}$ are the scalar angle between the vectors measured by Observers N_j and N_i. Two the initial vectors with the antivelocity vector form a triple in $\langle\mathcal{E}^3\rangle^{(j)}$. Due to the Hadamard Inequality, for their unit vectors Gram determinant, there holds

$$0 \le \det\{[\mathbf{e_1 e_2 e_3}]' \cdot [\mathbf{e_1 e_2 e_3}]\} = s_{123}^2 \le 1. \qquad (64A)$$

And from here the triple trigonometric inequality follows:

$$2\cos\alpha_{12}\cdot\cos\alpha_{13}\cdot\cos\alpha_{23} \le \cos^2\alpha_{12}+\cos^2\alpha_{13}+\cos^2\alpha_{23} \le 1+2\cos\alpha_{12}\cdot\cos\alpha_{13}\cdot\cos\alpha_{23}.$$

In our case, we have $\alpha_{13} = \varepsilon_1$, $\alpha_{23} = \varepsilon_2$, $\alpha_{12} = \beta_{12}$. These inequalities and condition $\tanh^2\gamma < 1$ infer (63A) *as inequality* too. If the initial angle between the vectors is $\beta_{12}^{(j)} = \pi/2 \to \cos\beta_{12}^{(j)} = 0$, then the new angle $\beta_{12}^{(i,j)}$ is either acute $(\cos\varepsilon_1 \cdot \cos\varepsilon_2 < 0)$, or obtuse $(\cos\varepsilon_1 \cdot \cos\varepsilon_2 > 0)$, or zero $(\cos\varepsilon_1 \cdot \cos\varepsilon_2 = 0)$. If $\beta_{12}^{(j)} = 0$, then $\varepsilon_1 = \varepsilon_2$ and $\beta_{12}^{(i,j)} = 0$. If both the vectors (and the angle between them) are orthogonal to the antivelocity vector, then the relativistic effect of the angle changing does not take place: $\cos\varepsilon_1 = \cos\varepsilon_2 = 0 \ \to\ \beta_{12}^{(i,j)} = \beta_{12}^{(j)}$. If one of these two vectors is collinear to the antivelocity vector, then $|\cos\beta_{12}|$ decreases, and the acute angle increases, the obtuse angle decreases ($\varepsilon_1 = 0 \to \beta_{12}^{(j)} = \varepsilon_2$):

$$0 < \cos\beta_{12}^{(i)} = \cos\beta_{12}^{(j)} \cdot \sqrt{\frac{1 - \tan^2\gamma}{1 - \cos^2\varepsilon_2 \cdot \tanh^2\gamma}} < \cos\beta_{12}^{(j)}. \qquad (65A)$$

Relativistic area of the parallelogram on two the vectors is

$$S_{12}^{(i,j)} = l_1^{(i,j)} \cdot l_2^{(i,j)} \cdot \sin\beta_{12}^{(i,j)} =$$

$$= \frac{S_{12}^{(j)}}{\sin\beta_{12}^{(j)}} \cdot \sqrt{\sin^2\beta_{12}^{(j)} - (\cos^2\varepsilon_1 + \cos^2\varepsilon_2 - 2\cos\beta_{12}^{(j)} \cdot \cos\varepsilon_1 \cdot \cos\varepsilon_2) \cdot \tanh^2\gamma}. \qquad (66A)$$

The diagonals of the moving parallelogram are subjected to Lorentzian contraction unless they are orthogonal to the velocity. In general, for the length of the diagonals, there holds:

$$[L^{(i,j)}]_{1,2}^2 = [L^{(j)}]_{1,2}^2 - [l_1^{(j)} \cdot \cos\varepsilon_1 \pm l_2^{(j)} \cdot \cos\varepsilon_2]^2 \cdot \tanh^2\gamma. \qquad (67A)$$

The volume of a parallelepiped (as well as of other body) decreases proportionally to the secant of the hyperbolic angle γ of motion - see in (40A). With the use of (54A), (64A) and (40A) the sine norm of a moving 3-dimensional lineor angle is evaluated:

$$s_{123}^{(i,j)} = \frac{s_{123}^{(j)} \cdot \mathrm{sech}\ \gamma}{\sqrt{1-\cos^2\varepsilon_1 \cdot \tanh^2\gamma} \cdot \sqrt{1-\cos^2\varepsilon_2 \cdot \tanh^2\gamma} \cdot \sqrt{1-\cos^2\varepsilon_3 \cdot \tanh^2\gamma} \cdot},$$

$$s_{123}^{(i,j)} \in (0;1). \qquad (68A)$$

Inequalities $0 < s_{123}^{(i,j)} < 1$ may be inferred by another way, with the use of formulae (63A) and (64A), because we have:

$$[s_{123}^{(i,j)}]^2 = 1 + 2 \cdot \cos\beta_{12}^{(i,j)} \cdot \cos\beta_{13}^{(i,j)} \cdot \cos\beta_{23}^{(i,j)} - \cos^2\beta_{12}^{(i,j)} - \cos^2\beta_{13}^{(i,j)} - \cos^2\beta_{23}^{(i,j)}.$$

The essential distinction between Lorentzian contraction of extent and Einsteinian dilation of time consists in the following. For multistep motions, the latter may be always expressed through multiplication of rotational matrices of all these particular motions with evaluating its summarized motive tensor angle (see in Chs. 5A and 7A). However, Lorentzian contraction, for multistep motions, is not expressed similarly through multiplication of all these particular deformations, because their hyperbolic tensor angles are not summable. But it may be expressed through the deformational matrix-function of the tensor angle in the rotational matrix-function obtained after multiplication of particular rotational matrices and following polar decomposition.

Moreover, due to (45A), the geometric result of the Lorentzian contraction is visually similar to the geometric object spherical rotation at the angle $\Phi(\Gamma)$ with the following spherical cosine projecting. Also, from the point of view of the tensor trigonometry, the equivalent spherical matrix $rot\ \Phi(\Gamma_{ij}) \equiv defh\ \Gamma_{ij}$ mathematically clear may interpret the relativistic effect "Terrell-Penrose visual rotation" under the Lorentzian contraction of moving geometric objects (of course, in the base $\tilde{E}_1$ of an immovable Observer).

We have an important peculiarity: the Lorentzian seeming contraction is a typical *artefact*, i. e., it is a really observational but seeming to N_1 space-like phenomenon evaluated in a certain universal base $\tilde{E}_1$. When the object returns to the rest state, its geometric sizes and angles are preserved. Internal mechanical stretches in a material object according only to any inertial movement are impossible.

Chapter 5A

Trigonometric models of two-, multistep, and integral collinear motions in STR and in hyperbolic geometries

Consider trigonometric interpretations of rectilinear physical movements summation. They are described mathematically (Ch. 2A) by hyperbolic rotational matrix-functions of compatible tensor angles (in their elementary form). According to **Rule 2** (sect. 5.7), *compatible rotational matrices commute, in their multiplications the tensor argument angles of motive type form an algebraic sum.* Hence, in this Chapter, we use *mainly* the scalar form for these motion angles and connected with them trigonometric functions and velocities. The latters may be subjected also to operations of integration (into some distances) and differentiation (into some accelerations), and what's more, these operations are realized inside of a certain pseudoplane of these compatible hyperbolic type motions! Some examples of similar physical movements are exposed at Figure 2A.

By this reason, the relativistic Einstein–Poincaré Law of two velocities summation (as well as hyperbolic tangents summation) for *collinear summands* has the following trigonometric interpretation as *compatible rotations in the hyperbolic angles* Γ_{jk}:

$$\left.\begin{aligned} & roth\ \Gamma_{13} = roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} = roth\ (\Gamma_{12} + \Gamma_{23}) \Rightarrow \\ & \Rightarrow \cos\alpha_{(13)} \cdot \gamma_{13} = \cos\alpha_{(12)} \cdot \gamma_{12} + \cos\alpha_{(23)} \cdot \gamma_{23}, \end{aligned}\right\} (\cos\alpha = \pm 1, \gamma > 0) \quad (69A)$$

$$\left.\begin{aligned} & \tanh\left[\cos\alpha_{(13)} \cdot \gamma_{13}\right] = \tanh\left[\cos\alpha_{(12)} \cdot \gamma_{12} + \cos\alpha_{(23)} \cdot \gamma_{23}\right] \Rightarrow \\ & \Rightarrow v_{13} = c \cdot \tanh\left[\operatorname{artanh} v_{12}/c + \operatorname{artanh} v_{23}/c\right] = \frac{v_{12} + v_{23}}{1 + v_{12}v_{23}/c^2}, \\ & v_{12}v_{23} > 0 \ \leftrightarrow \ |v_{13}| < |v_{12} + v_{23}|, \quad v_{12}v_{23} < 0 \ \leftrightarrow \ |v_{13}| > |v_{12} + v_{23}|. \end{aligned}\right\} \quad (70A)$$

Hyperbolic form of this law was first derived by Arnold Sommerfeld with geometric interpretation as if on a *sphere of imaginary radius* i [53, p. 111; 63], i. e., in fact (!) on a Minkowskian hyperboloid II (see sect. 12.1). This is based on the rule for summation of tangents of trigonometrically compatible hyperbolic angles. The relativistic law of summing *several collinear velocities* is expressed also in the simplest hyperbolic form:

$$\cos\alpha \cdot \gamma = \sum_{t=1}^{m} \cos\alpha_{(t)} \cdot \gamma_{(t)}, \quad (\cos\alpha = \pm 1, \gamma > 0) \qquad (71A)$$

$$v = c \cdot \tanh\,(\cos\alpha \cdot \gamma) = c \cdot \tanh \sum_{t=1}^{m} \operatorname{artanh} v_t/c. \qquad (72A)$$

The term "collinear" has here and further rather conventional character, it means merely that all these summarized particular velocities vectors $\mathbf{v}_t$ are directed in their common vectorial 3-dimensional Euclidean space coaxially with a set constant vector $\mathbf{e}_\alpha = \langle \cos\alpha_i \rangle = \mathbf{const}$ with $\cos\alpha_i$, $i = 1, 2, 3$. Hence, the particular velocity $\mathbf{v}_t$ can have only one of two values of vectors of directional cosines $\pm\mathbf{e}_\alpha$, i. e., in these contrary directions. In (69A)–(72A), this condition corresponds to values $\cos\alpha = \pm 1$.

An integral collinear motion as a curve world line in $\langle\mathcal{P}^{3+1}\rangle$ is projected hyperbolically into some Euclidean subspace $\langle\mathcal{E}^3\rangle$ as a rectilinear physical movement. More in details, such motion is realized in some only one pseudoplane $\langle\mathcal{P}^{1+1}\rangle$ with its specific directional vector $\mathbf{e}_\alpha$, but physically the motion is projected hyperbolically as a straight line into any its space axis, for example, $x^{(1)} = \chi$ in parallel to $\overrightarrow{ct}^{(1)}$. Hence, speaking strictly, "rectilinear movement" is a physical term, which has rather conventional character too in $\langle\mathcal{P}^{3+1}\rangle$. (In the Lagrangian space-time, a collinear motion is projected always into its Euclidean subspace as single one for all the bases in parallel to any $\overrightarrow{ct}$.)

Continuous summation of collinear motion angle differentials $d\gamma = d\gamma^{(m)}$ is accomplished with integrating either along instantaneous axis $x^{(1)}$ as differentials $d\gamma = dv^{(m)}/c$ of its inclination to the Euclidean subspace $\langle\mathcal{E}^3\rangle^{(1}$ or along instantaneous tangent to a world line as differentials $d\gamma$ of its inclination to the time-arrow $\overrightarrow{ct}^{(1)}$. Note, that these 1-st differentials $d\gamma$ and $dv^{(m)}$, as always, only are linear parts of curve increments $\Delta\gamma$ and $\Delta v^{(m)}$ (and in the current point M there holds: $v^{(m)} = 0$).

The space axis $x^{(1)}$ collinear with $\pm\mathbf{e}_\alpha$ and the time arrow $\overrightarrow{ct}^{(1)}$ determine the constant pseudoplane $\langle\mathcal{P}^{1+1}\rangle$ with this two-dimensional universal base. Such base $\tilde{E}_1$ corresponds to the rest state of inertial Observer N_1 of STR. In the base $\tilde{E}_1$, we have the concrete spherical-hyperbolic analogy (26A) between hyperbolic and spherical motion angles for very important applications. Further, we shall describe two-step, multistep and integral collinear motions mainly in the universal base $\tilde{E}_1$ – see at Figure 2A.

In the trigonometric version of STR, the *characteristic hyperbolic angle of motion* γ has relative nature as well as the time-arrow and the space. Here this angle is counted in the base $\tilde{E}_1$ off $\overrightarrow{ct}^{(1)}$ unless another condition is accepted. So, for a straight world line, the relative velocity between Observers N_1 and N_2 determines the hyperbolic tangent of the angle of motion γ_{ij} from two opposite points of view – Figure 2A(1):

$$\tanh\gamma_{12} = \frac{v_{12}}{c} = \frac{\Delta x^{(1)}}{\Delta(ct^{(1)})} = \frac{\Delta x^{(1)}\cdot \text{sech }\gamma_{12}}{\Delta(ct^{(1)})\cdot \text{sech }\gamma_{12}} = \frac{-\Delta x^{(2)}}{\Delta(ct^{(2)})} = -\tanh\gamma_{21}. \qquad (73A)$$

The same takes place also if a material object is moving rectilinearly with acceleration or deceleration along directions $\pm\mathbf{e}_\alpha$ with its current instantaneous base $\tilde{E}_m$. For each point M of its world line, the instantaneous pseudo-Cartesian base is associated with M, it takes into account translation of the coordinates origin into the point M:

$$\tilde{E}_m = roth\ \Gamma\cdot\tilde{E}_1 = F_1(\gamma,\mathbf{e}_\alpha)\cdot\tilde{E}_1. \qquad (74A)$$

The current tangent $\tanh\gamma$ determines the *coordinate velocity* of physical movement, it may be expressed by two ways: from the points of view of Observers N_1 and N_m:

$$\tanh\gamma = \frac{v}{c} = \frac{d\chi}{d(ct^{(1)})} = \frac{dx^{(1)}\cdot \text{sech }\gamma}{d(ct^{(1)})\cdot \text{sech }\gamma} = \frac{\mp dx^{(m)}}{d(c\tau)} = \mp\tanh(\mp\gamma). \qquad (75A)$$

These formulae correspond to the Lorentzian dilations of space and time in moving systems of reference with the coefficient sech $\gamma(v)$ in the base $\tilde{E}_1$ (see the end of Ch. 12). Further similar Greek notations $\chi = x^{(1)}$, $c\tau = ct^{(2)}$ stand for *the proper coordinates.*

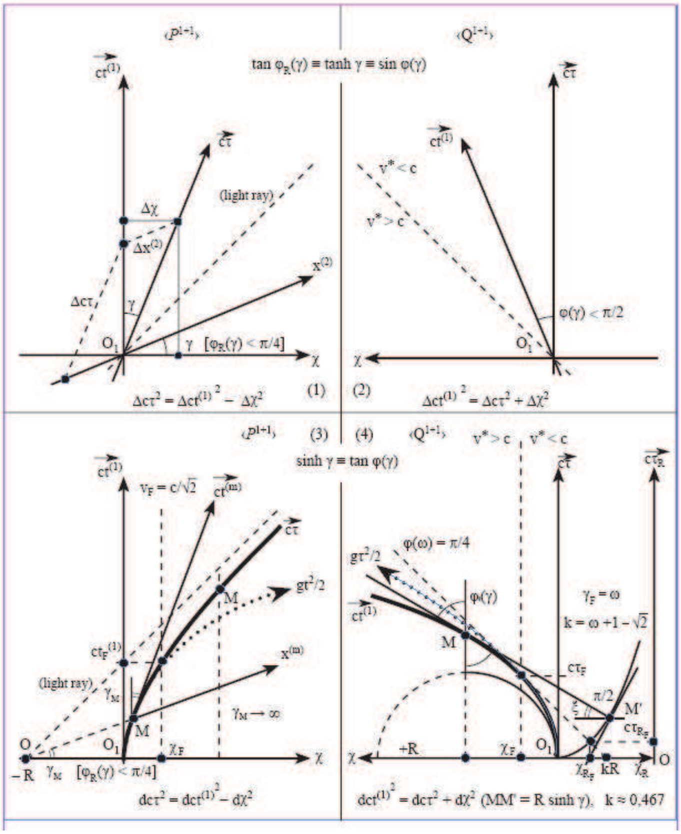

Figure 2A. The world lines of a material point M for simplest kinds of rectilinear physical movement: uniform one (1,2), uniformly accelerated one (3,4) in the universal, proper, and compressed bases. (Two dotted curves with their arrows are non-relativistic tangent kinematical parabolae.)

Here the proper (*true*) distance χ in $\langle\mathcal{E}^3\rangle^{(1)}$ is the segment of $x^{(1)}$ axis, it is immovable in the universal base $\tilde{E}_1$. The proper time differential $dc\tau$ here is the differential of pseudo-Euclidean length of a world line arc, i. e., along the world line. The proper time is counted with the clock of the moving object in the current subbase $\tilde{E}_m^{(3)}$. For the moving object, its curvilinear world line is identical to its proper-time-arrow $\overrightarrow{\int_0^t d(ct^{(m)})} \equiv \overrightarrow{c\tau}(\gamma)$, see Figure 2A(3). A pseudo-normal and a tangent to a curvilinear world line at point M form instantaneous directed axes $x^{(m)}$ and $\overrightarrow{ct}^{(m)}$ of the base $\tilde{E}_m$.

In (73A), (75A), the relative velocity v_{12} in $\tilde{E}_1$ of Observer N_2 with respect to N_1 is evaluated with the use of its coordinate time $t^{(1)}$ and its proper distance $x^{(1)} = \chi$. Similarly, the relative velocity v_{21} in $\tilde{E}_2$ of Observers N_1 with respect to N_2 is evaluated with the use of its decreased proper time $t^{(2)}$ ($dt^{(2)} = \text{sech}\ \gamma_{21}\ dt^{(1)}$) and its moving coordinate distance $x^{(1)}$ ($dx^{(1)} = \text{sech}\ \gamma_{21}\ dx^{(2)}$) – the latter is formally analogous to Einstein's dilation of time. Hence, the notion v is, in fact, the *coordinate* velocity.

The *proper velocity* of physical movement (39A) is defined with the use of *proper coordinates*, i. e., proper time $d(c\tau)$ in a moving Euclidean subspace in $\tilde{E}_m$ and immovable proper distance $d\chi = dx^{(1)}$ in $\tilde{E}_1$. It is expressed by the hyperbolic sine:

$$\frac{v^*}{c} = \frac{dx^{(1)}}{d(ct^{(m)})} = \frac{d\chi}{d(c\tau)} = \cosh\gamma \cdot \tanh\gamma = \sinh\gamma > \frac{v}{c}. \qquad (76A)$$

The proper velocity of a light is infinite as $d(c\tau) = 0$. Therefore the law of proper relativistic velocities summation for collinear summands has the following hyperbolic sine interpretation, though hyperbolic angles are summed as before, see in (70A):

$$\left.\begin{array}{l} v^*_{13} = c \cdot \sinh[\cos\alpha_{(13)} \cdot \gamma_{13}] = c \cdot \sinh[\cos\alpha_{(12)} \cdot \gamma_{12} + \cos\alpha_{(23)} \cdot \gamma_{23}] = \\ = c \cdot [\cos\alpha_{(12)} \cdot \sinh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\alpha_{(23)} \cdot \sinh\gamma_{23} \cdot \cosh\gamma_{12}], \Rightarrow \\ \Rightarrow v^*_{13} = v^*_{12} \cdot \sqrt{1 + (v^*_{23}/c)^2} + v^*_{23} \cdot \sqrt{1 + (v^*_{12}/c)^2}, \\ v_{12}v_{23} > 0 \;\leftrightarrow\; |v^*_{13}| > |v^*_{12} + v^*_{23}|. \end{array}\right\} \qquad (77A)$$

Furthermore, there holds: $v^* = v/\sqrt{1 - (v/c)^2} \;\to\; 1/c^2 = 1/v^2 - 1/(v^*)^2$. The latter is equivalent to the trigonometric identity: $1 = \coth^2\gamma - \mathrm{csch}^2\gamma$. It is invariant of cotangent-cosecant rotational matrix at complementary angle Υ! (Ch. 12). The directed cosines of vectors $\mathbf{v}^*$ and $\mathbf{sin}\ \gamma$ are equal to those of $\mathbf{v}$ and $\mathbf{tan}\ \gamma$, as they are obtained from the same vector differential $d\mathbf{x}$ in the numerator of their derivatives.

Let the frame of reference with Observer N_m moves also rectilinearly, but non-uniformly. Then N_m has the instantaneous coordinate velocity with respect to N_1 as

$$v^{(m)}_{21} = \frac{dx^{(m)}}{d\tau} = \frac{\mathrm{sech}\ \gamma \cdot dx^{(1)}}{\mathrm{sech}\ \gamma \cdot dt} = \frac{dx^{(1)}}{dt} = -v^{(m)}_{12}.$$

However, the infinitesimal instantaneous coordinate velocity-derivative of N_m in $\tilde{E}_m$, with respect to a certain previous current origine M of $\tilde{E}_m$ is expressed as $\frac{dx^{(m)}}{d\tau} = v^{(m)}$ (and exactly in M it is always zero, even for the non-uniform movement). This *inner velocity* $v^{(m)}$ has another sense. For the world trajectory passing through the point M, consider a neighborhood of M and introduce in it two hyperbolic angles: $\gamma^{(1)} = \gamma$ is a *general motion angle in* $\tilde{E}_1$, and $\gamma^{(m)} \to 0$ is a *additional infinitesimal motion angle in the base* $\tilde{E}_m$ determined by the *inner acceleration or deceleration of movement* in the neighborhood of M. For differentials of the two coordinate velocities with respect to $\tilde{E}_1$ and $\tilde{E}_m$ in the neighborhood of M, their trigonometric forms are expressed as

$$\left.\begin{array}{l} d\left(\dfrac{dx^{(1)}}{d(ct^{(1)})}\right) = d\left(\dfrac{d\chi}{d(ct^{(1)})}\right) = d\tanh\gamma = \mathrm{sech}^2\gamma\ d\gamma, \\ \\ d\left(\dfrac{dx^{(m)}}{d(ct^{(m)})}\right) = d\left(\dfrac{dx^{(m)}}{d(c\tau)}\right) = d\tanh\gamma^{(m)} = d\gamma^{(m)} = d\gamma, \end{array}\right\} \qquad (78A)$$

where $\gamma^{(m)} \to 0$ is counted in the base $\tilde{E}_m$ from the current point M, but the angle γ is counted in the base $\tilde{E}_1$ from the origin O along the same world line.

The angle γ is counted from the axes $x^{(1)}$ and $\overrightarrow{ct}^{(1)}$ of the base $\tilde{E}_1$ up to the axes $x^{(m)}$ and $\overrightarrow{c\tau}$ of the base $\tilde{E}_m$ applied to the point M. So, for a curve world line segment, the infinitesimal angle $\gamma^{(m)}$ is counted in the current point M from the time-arrow $\overrightarrow{c\tau}$ (the tangent) as a *time-like angle* or from the axis $x^{(m)}$ (the pseudo-normal) as a *space-like angle* in two opposite directions to the light cone between them. There holds $\gamma^{(m)} \to 0$ in a neighborhood of the point M as $v_M^{(m)} = 0$. (For a straight world line segment, angles $d\gamma$ and $\gamma^{(m)}$ are zero.) For collinear motion in its pseudoplane, $d\gamma$ is expressed also in the same instantaneous base $\tilde{E}_m$, i. e., $\gamma^{(m)} = d\gamma^{(m)} = d\gamma$. At the point M, the *inner 3-acceleration in* $\tilde{E}_m$ (but it is expressed as *4-acceleration in* $\tilde{E}_1$) is

$$\frac{d^2x^{(m)}}{d\tau^2} = \frac{dv^{(m)}}{d\tau} = c \cdot \frac{d(\tanh\ \gamma^{(m)})}{d\tau} = c \cdot \frac{d(\tanh\ d\gamma)}{d\tau} = c \cdot \frac{d\gamma}{d\tau} = g^{(m)}(\tau). \qquad (79A)$$

From here, for collinear motions, *the fundamental trigonometric formulae* follow as

$$\boxed{d^2\mathbf{x}_M^{(m)} = d\gamma \cdot d(c\tau) = dv^{(m)} \cdot d\tau,\ dv^{(m)} = c\ d\gamma,\ dx_M^{(m)} = 0}\ ,\ dx^{(m)} = v\ d\tau; \quad (80A)$$

in $\langle \mathcal{P}^{3+1} \rangle$: $d^2\mathbf{x}^{(m)} = d^2x^{(m)} \cdot \mathbf{e}_\alpha = d\gamma \cdot d(c\tau) \cdot \mathbf{e}_\alpha,\ d\mathbf{x}_M^{(m)} = \mathbf{0},\ \mathbf{e}_\alpha = \pm\,\mathbf{const},\ d(c\tau) \neq 0.$

$d(c\tau) = d\lambda$ is 1-st differential of the pseudo-Euclidean length of a world line segment; $d\gamma$ is space-like or time like. It is counted from M along the current $x^{(m)}$ or tangent. Formulae (80A) connect three differential parameters of curvilinear collinear motion. So, we obtain the *inner velocity and acceleration* as $v^{(m)} = c \cdot \gamma^{(m)}$ and $g^{(m)} = dv^{(m)}/d\tau$.

We use the trigonometric opportunities in the Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$ for clear descriptions of relativistic motions, in particular, collinear ones, with their kinematics and dynamics in uninertial (or accelerated) frames of reference, but from the point of view of any inertial (Galilean) frame of reference $\tilde{E}_i$. That is why in STR the base $\tilde{E}_m$ may be considered in $\tilde{E}_i$ as instantaneously inertial [53]. At a moment of the time τ, an *inner 3-force* F acted on $\mathbf{M}$, with caused by it the inner 3-acceleration $g^{(m)}$ and 3-velocity $v^{(m)} = g^{(m)} d\tau$, are directed in $\tilde{E}_m$ along the $x^{(m)}$-axis. Hence, in the base $\tilde{E}_m$ *these three instantaneous characteristics are always collinear ones.* According to the 2-nd Newton's Law of mechanics and relation (79A), and with the instantaneous radius of the hyperbolic pseudo-curvature $\overline{\overline{R}} = 1/\overline{\overline{K}}$ along a world line, there holds:

$$\mathbf{g}(\tau) = \frac{F(\tau) \cdot \mathbf{e}_\alpha}{m_0} = \frac{d^2x^{(m)} \cdot \mathbf{e}_\alpha}{d\tau^2} = c^2 \frac{d\gamma}{d(c\tau)} \cdot \mathbf{e}_\alpha,\ \ d\gamma = \overline{\overline{K}}\ d(c\tau)\ \Rightarrow\ g = c^2/\overline{\overline{R}}. \quad (81A)$$

$|\mathbf{F}|_P = m_0 \cdot |\mathbf{g}|_P$ is the same for Observers in inertial bases. (If $\mathbf{F}$ is a force of inertia, then $|\mathbf{F}|_P$ is the number showed at the scale of a dynamometer in $\tilde{E}_m^{(3)}$.) The rest *own mass* $m_0 \neq 0$ of a material point (object) M does not depend on the own frame of reference. The *absolute value of inner acceleration* determined by (79A) and (81A) is *an invariant* (strongly at constant temperature $m_0 = \text{const}$). In $\tilde{E}_m$, it does not depend on γ (or the velocity of movement) contrary to corresponding *relative characteristics.* Due to this, exactly $\mathbf{g}(\tau)$ is considered in STR as the basic inner 3- or 4-acceleration.

If acceleration $\mathbf{g}$ is *tangent* to velocity $\mathbf{v}$ or $\mathbf{v}^*$, then the world line, with $\mathbf{v}$ and $\mathbf{g}$, stays in the same pseudoplane (the motion is *coplanar* one). The constant and pure tangent to $\mathbf{v}$ acceleration $\mathbf{g}$ determines rectilinear uniformly accelerated or decelerated physical movement. The absolute motion is described in a certain pseudoplane along a time-like hyperbola with the radius of pseudo-curvature $\overline{\overline{R}} = d\lambda/d\gamma = 1/\overline{\overline{K}}$ (see it in sect. 6.4). For such *hyperbolic motion*, formula (79A) is modified as follows

$$\frac{d^2x^{(m)}}{[d(c\tau)]^2} = \frac{dv^{(m)}}{c^2 d\tau}) = g^{(m)}/c^2 = \frac{d\gamma}{d(c\tau)} = \frac{d\gamma}{d\lambda} = \overline{\overline{K}} = \text{const} \;\Rightarrow\; d^2\mathbf{x}^{(m)} = \overline{\overline{R}}\,(d\gamma)^2,$$

where $d\gamma \neq 0$, $d(c\tau) = d\lambda = \overline{\overline{R}}\,d\gamma$ is the hyperbolic arc with its radius-vector of pseudo-normal radiated out of the hyperbola center O. The kinematical hyperbola, for example, is mapped on a Minkowskian hyperboloid I and is going out its point C_1 (see sect. 12.1 at Figure 4). The simplest kind of relativistic accelerated movement was first analyzed by H. Minkowski [53, p. 111]; then M. Born and A. Sommerfeld [64; 63]. There are two more types of tangent acceleration, in addition to (79A). The *proper 3-acceleration* in $\tilde{E} = (\chi, \overrightarrow{c\tau})$, with taking into account (76A) and (80A), is

$$\overline{\overline{g^*}}(\tau) = \frac{d^2\chi}{d\tau^2} = \frac{dv^*}{d\tau} = c\cdot\frac{d\sinh\gamma}{d\tau} = c\cdot\cosh\gamma\cdot\frac{d\gamma}{d\tau} = \cosh\gamma\cdot g(\tau) > g(\tau). \qquad (82A)$$

It is greater than inner acceleration in (79A), as the differentials $d^2x^{(m)}$ is decreased proper differential $d^2\chi$ due to relativistic dilation as result of rotation of the axis $x^{(m)}$. Contrary, the *coordinate 3-acceleration* in $\tilde{E}_1$ due to (78A) is very less than inner one:

$$\overline{\overline{g^{(1)}}}(t^{(1)}) = \frac{dv}{dt^{(1)}} = \frac{d^2\chi}{(dt^{(1)})^2} = c\cdot\frac{d\tanh\gamma}{dt^{(1)}} = c\cdot\text{sech}^2\gamma\cdot\frac{d\gamma}{dt^{(1)}} = c\cdot\text{sech}^3\gamma\cdot\frac{d\gamma}{d\tau} \;\Rightarrow$$

$$\Rightarrow\; \boxed{\overline{\overline{g^{(1)}}}(t^{(1)}) = g[\tau(t)^{(1)}]/\cosh^3\gamma} \;\Rightarrow\; \{\overline{\overline{g}}[\tau(t)^{(1)}] \ll g[\tau(t)^{(1)}] < \overline{\overline{g^*}}[\tau(t)^{(1)}]\}. \qquad (83A)$$

The last formula for tangential acceleration $\overline{\overline{g}}^{(1)}(t)$ (here in clear trigonometric form) is well-known in STR. In kinematics, the parameters $ct^{(1)}$ and $c\tau$ are used as arguments of motion functions. The parameters are synchronous in the universal base $\tilde{E}_1$ if they are fixed with clocks of N_1 and N_m simultaneously. Simultaneity is defined (Ch. 4A) by the differential and integral forms derived from projecting time in parallel to $\langle\mathcal{E}^3\rangle^{(1)}$:

$$d(c\tau) = \text{sech}\,\gamma\; d(ct^{(1)}) < d(ct^{(1)}), \qquad c\tau = \int_0^{ct^{(1)}} \text{sech}\,\gamma\; d(ct^{(1)}) < ct^{(1)}; \qquad (84A)$$

$$d(ct^{(1)}) = \cosh\gamma\; d(c\tau) > d(c\tau), \qquad ct^{(1)} = \int_0^{c\tau} \cosh\gamma\; d(c\tau) > c\tau. \qquad (85A)$$

They are obtained with cut parallel to the axis $x^{(1)} = \chi$. Here $c\tau$ is, according to (84A), the pseudo-Euclidean length of a world line counted from the base $\tilde{E}_1$ origin.

If motion is *integral* (and the vectors **v** and **g** in the velocity 3D-space are collinear as before), then the angle γ and the velocity v and v^* vary continuously. In particular, for the hyperbolic motion (as uniformly accelerated or decelerated one) determined above, there holds $g(\tau) = g = \text{const}$. Then, due to (79A) and (84A), we have

$$\left.\begin{aligned} d\gamma = g\cdot d\tau/c \;\Rightarrow\; d(c\tau) = \overline{\overline{R}}\, d\gamma, \\ \gamma = g\cdot \tau/c \;\Rightarrow\; c\tau = \overline{\overline{R}}\cdot\gamma, \end{aligned}\right\} \quad (g=\text{const}); \tag{86A}$$

$$\left.\begin{aligned} d\sinh\gamma = g\cdot dt^{(1)}/c \;\Rightarrow\; \overline{\overline{R}}\cdot d\sinh\gamma = d(ct^{(1)}), \\ \sinh\gamma = g\cdot t^{(1)}/c \;\Rightarrow\; ct^{(1)} = \overline{\overline{R}}\cdot\sinh\gamma. \end{aligned}\right\} \quad (g=\{\text{const}). \tag{87A}$$

Trigonometric parameters in (86A) and (87A) are instantaneous, and from there we have the relation used for its synchronized consideration: $t^{(1)}/\tau = \sinh\gamma/\gamma$! They allow one to express simultaneously the velocity and distance functions for hyperbolic (uniformly accelerated) motion in terms of two-types time arguments:

$$\left.\begin{aligned} d(c\tau) = \frac{d(ct^{(1)})}{\cosh\gamma} = \frac{d(ct^{(1)})}{\sqrt{1+\left[\overline{\overline{g}}t^{(1)}/c\right]^2}} = \frac{d(ct^{(1)})}{\sqrt{1+\left[ct^{(1)}/\overline{\overline{R}}\right]^2}}, \\ c\tau = \overline{\overline{R}}\;\cdot\gamma = (c^2/g)\cdot\text{arsinh}\,[g\cdot t^{(1)}/c] = \overline{\overline{R}}\cdot\text{arsinh}\,\left[ct^{(1)}/\overline{\overline{R}}\right]; \end{aligned}\right\} \tag{88A}$$

$$\left.\begin{aligned} d(ct^{(1)} = \cosh\gamma\cdot d(c\tau) = \cosh(g\cdot\tau/c)\; d(c\tau) = \cosh(c\tau/\overline{\overline{R}})\; d(c\tau), \\ ct^{(1)} = (c^2/g)\cdot\sinh\gamma = (c^2/g)\cdot\sinh(g\cdot\tau/c) = \overline{\overline{R}}\cdot\sinh(c\tau/\overline{\overline{R}}). \end{aligned}\right\} \tag{89A}$$

In the hyperbolic motion, from (86A) and (87A), very useful relation $t^{(1)}/\tau = \sinh\gamma/\gamma$ is acted. For this motion as the physical movement, coordinate and proper velocities (see above) are functions in coordinate time (they are expressed synchronically in $\tilde{E}_1$, in terms of proper time too):

$$v = v_t(t^{(1)}) = c\cdot\tanh\gamma = \frac{g\cdot t^{(1)}}{\sqrt{1+\left[g\cdot t^{(1)}/c\right]^2}} \equiv$$

$$\equiv v_\tau(\tau) = c\cdot\tanh(g\cdot\tau/c) < g\cdot\tau < g\cdot t^{(1)}, \tag{90A}$$

$$v^* = v^*_\tau(\tau) = c\cdot\sinh\gamma = c\cdot\sinh(g\cdot\tau/c) \equiv v^*_t(t^{(1)}) = g\cdot t^{(1)} > g\cdot\tau. \tag{91A}$$

These inequalities may be also interpreted by the trigonometric way as analogs of $\tanh\gamma < \gamma < \sinh\gamma < \cosh\gamma$.

The proper distance as a function in time $t^{(1)}$ counted with the clock of N_1 is

$$\chi = \chi_t(t^{(1)}) = \int\limits_0^{t^{(1)}} v_t(t^{(1)})dt^{(1)} = \overline{\overline{R}}\cdot(\cosh\gamma - 1) = \overline{\overline{R}}\cdot\left(\sqrt{1+\left[ct^{(1)}/\overline{\overline{R}}\right]^2} - 1\right). \tag{92A}$$

Functional relation χ and $ct^{(1)}$ may be expressed in the parametric and invariant forms

$$\left.\begin{aligned} ct^{(1)} &= \overline{\overline{R}} \cdot \sinh\gamma, \\ \chi + \overline{\overline{R}} &= \overline{\overline{R}} \cdot \cosh\gamma, \end{aligned}\right\} \Rightarrow (\chi + \overline{\overline{R}})^2 - (ct^{(1)})^2 = \overline{\overline{R}}^2. \tag{93A}$$

This corresponds generally to equation of a *kinematical hyperbola* in a pseudoplane and its *sine-cosine time-like invariant* $\sinh^2\gamma - |\mathbf{cosh}\gamma|^2 = i^2$ describing movement in the base $\tilde{E}_1 = \{\chi, \overrightarrow{ct^{(1)}}\}$. Three more forms for the invariant $\overline{\overline{R}}$ (with $\chi, ct, c\tau$) are

$$\overline{\overline{R}} = \frac{\chi}{\cosh\gamma - 1} = \frac{ct^{(1)}}{\sinh\gamma} = \frac{c\tau}{\gamma} = \text{const}, \tag{94A}$$

it may be derived from (92A) as well as from (93A) taking into account (86A), (87A).

In Minkowskian geometry, this equation determines a circle of radius $\overline{\overline{R}} = c^2/g$ in $\langle \mathcal{P}^{3+1} \rangle$, its affine interpretation is a hyperbola. The trajectory has constant *hyperbolic curvature* $\overline{\overline{K}} = 1/\overline{\overline{R}}$. Thus we deal with *hyperbolic movement* in STR, it is the simplest type of collinear integral movement. A kinematical hyperbola is the intermediate form between the Newtonian kinematical parabola in $t^{(1)}$ and an isotropic straight line of the light ray $\chi = ct^{(1)} - \overline{\overline{R}}$ going out of the point O, see Figure 2A(3):

$$ct^{(1)} - \overline{\overline{R}} \ < \ \chi = \chi_t(t^{(1)}) \ < \ \overline{\overline{g}} \cdot (t^{(1)})^2/2. \tag{95A}$$

The inequality is interpreted with (87A, 92A) as $\sinh\gamma - 1 < \cosh\gamma - 1 < (\sinh^2\gamma)/2$.

The proper distance measured with the clock of N_m and velocity v^*_τ is the *catenary*:

$$\left.\begin{aligned} &\chi = \chi_\tau(\tau) = \int\limits_0^\tau v^*_\tau(\tau)d\tau = c\int\limits_0^\tau \sinh\gamma(\tau)d\tau = \overline{\overline{R}}\int\limits_0^\gamma \sinh\gamma\ d\gamma = \\ &= \overline{\overline{R}} \cdot [\cosh(c\tau/\overline{\overline{R}}) - 1] = \overline{\overline{R}} \cdot (\cosh\gamma - 1) \equiv \overline{\overline{R}} \cdot [\sec\varphi(\gamma) - 1]. \end{aligned}\right\} \tag{96A}$$

Due to (94A) one may infer, that $R = 1$ reduces (93A) and (96A) to these unique trigonometric objects as unity hyperbola and unity catenary. All hyperbolae (93A) and all catenaries (96A) are homothetic to these unity objects with coefficient R, however the first in pseudo-Euclidean metric, the second in Euclidean metric, i. e., in their basis spaces. This relate also to hyperboloids I, II and catenoids I, II – see more in Ch. 6A.

Formula (96A) is the equation of a *hyperbolic cosine curve* (catenary) in the both *proper quasi-Cartesian coordinates* of space and time $\tilde{E} = (\chi, \overrightarrow{c\tau})$, see Figure 2A(4). The straight-line axis $\overrightarrow{c\tau}$ is obtained from the hyperbolic world line $\overrightarrow{c\tau}(\gamma)$ in $\tilde{E}_1$ with its rectification by spherical orthogonalization with respect to the *proper Euclidean subspace* $\langle \mathcal{E}^3 \rangle \equiv \langle \mathcal{E}^3 \rangle^{(1)}$, i. e., to the axis $\chi = x^{(1)}$. It is formally realized with transformation of instantaneous motion angles with the use of concrete spherical–hyperbolic sine–tangent analogy (sect. 6.2): $\sin\varphi(\gamma) \equiv \tanh\gamma = v/c$, $\tan\varphi(\gamma) \equiv \sinh\gamma = v^*/c$. This means that, in the special quasi-Cartesian base $\tilde{E} = (\chi, \overrightarrow{c\tau})$, the spherical tangent of the world line inclination angle and the hyperbolic sine are equivalent in $\tilde{E}$ and $\tilde{E}_1$. Then (86A) and (87A) give differentials and lengths of the world line in $\tilde{E}_1$ and $\tilde{E}$.

This can be interpreted as passage into the *uninertial time-like quasi-Euclidean space-time*, which is direct spherically orthogonal sum of the proper Euclidean space $\langle\mathcal{E}^3\rangle^{(1)}$ and the *rectified proper-time-arrow* $\overrightarrow{c\tau}$, obtained with the sine-tangent analogy for slopes of any world lines at their transitions from the universal base $\tilde{E}_1 = \{\chi, \overrightarrow{ct}\}$ with inertial N_1 into the *quasi-Cartesian base* $\tilde{E} = \{\chi, \overrightarrow{c\tau}\}$ with uninertial N_m

$$\langle\mathcal{Q}^{3+1}\rangle^{\updownarrow} \equiv \langle\mathcal{E}^3\rangle \boxplus \overrightarrow{c\tau}, \quad \text{where } \langle\mathcal{E}^3\rangle \equiv \text{CONST}, \ \overrightarrow{c\tau} \equiv \text{Const.} \tag{97A}$$

and where *rot* Θ are admitted. The analogy for slopes of world lines in $\tilde{E}_1$ and $\tilde{E}$ is

$$d\chi = c \cdot \tanh\gamma \ d(ct) = c \cdot \sinh\gamma \ d(c\tau) \equiv \ c \cdot \tan\varphi \ d(c\tau) \rightarrow \sinh\gamma \equiv \tan\varphi \ \leftrightarrow$$

$$\leftrightarrow \ d\chi/d(ct) = \tanh\gamma, \ d\chi/d(c\tau) = \tan\varphi(\gamma) \leftrightarrow \ d\varphi = \text{sech}\gamma \ d\gamma, d\gamma = \sec\varphi \ d\varphi.$$

In these two instantaneous bases, we have the pseudo-Euclidean measure in $\langle\mathcal{P}^{3+1}\rangle$ and the Euclidean measure in $\langle\mathcal{Q}^{3+1}\rangle^{\updownarrow}$ *with hyperbolic and spherical motion angles*

$$[d(c\tau)]^2 = [d(ct(\gamma))]^2 - [d\chi(\gamma)]^2 \rightarrow [d(ct)]^2 = \{d(c\tau[\varphi(\gamma)])\}^2 + \{d\chi[\varphi(\gamma)]\}^2.$$

The principal motion angles Γ and $\Phi(\Gamma)$, with respect to $\tilde{E}_1$ and $\tilde{E}$, are connected through both velocities as $\tanh\gamma = \sin\varphi(\gamma) = v/c \leftrightarrow \tan\varphi(\gamma) = \sinh\gamma = v^*/c$ taking into account (76A). The *Special quasi-Euclidean space* has the same *reflector tensor* $\{I^{\pm}\}$ with admissible orthospherical transformations *rot* Θ and an exchange of $\overrightarrow{c\tau}$ and $\overrightarrow{ct^{(1)}}$. The principal rotations *roth* Γ and *rot* $\Phi(\Gamma)$ in both these bases are connected by the same analogy as indicated above – see more in sect. 6.4 and Ch. 6A. The Euclidean length of the original world line $\overrightarrow{c\tau}$ as the new ordinate $\overrightarrow{c\tau}$ in the base $\tilde{E}$ corresponds to the proper time of N_m; the Euclidean length of this curvilinear in general world line $\overrightarrow{ct^{(1)}}(\varphi)$ in this base corresponds to the coordinate time of N_1.

In addition to (93A), we obtain in $\langle\mathcal{Q}^{3+1}\rangle^{\updownarrow}$, due to (94A), the equation for a catenary, *invariant only to Lorentzian transformations* of its pro-hyperbola with same *rot* Θ:

$$\left.\begin{aligned} &\left[\frac{\overline{\overline{R}}^2}{\overline{\overline{R}} + \chi}\right]^2 + \left[\frac{\tanh\gamma(\varphi)}{\gamma(\varphi)} \cdot c\tau(\varphi)\right]^2 = \overline{\overline{R}}^2 = \\ &= \overline{\overline{R}}^2 \cdot [\text{sech}^2\gamma(\varphi) + \tanh^2\gamma(\varphi)] = \overline{\overline{R}}^2 \cdot (\cos^2\varphi + \sin^2\varphi), \end{aligned}\right\} (c\tau = \overline{\overline{R}}\gamma). \tag{98A}$$

Uniform rectilinear movement is described by a straight line with visual inclination $\varphi = \varphi(\gamma) = \text{const}$ identical to the angle in an usual quasi-Euclidean space. Uniformly accelerated movement is described by hyperbolic cosine curve catenary (96A).

The *focal hyperbolic angle* of inclination for these space-like hyperbola and catenary (see at Figure 2A, (3) and (4)) is $\gamma_F = \omega = \text{arsinh } 1 \approx 0.881$ defined by the concrete sine-tangent analogy. It is the especial hyperbolic angle introduced in sect. 6.4 for applications in different geometries with principal hyperbolic angles, in particular, as the hyperbolic analog of spherical angle and number $\pi/4$! We used it already some times and shall use further for more descriptive trigonometric considerations.

In the base $\tilde{E}$, the proper distance for the catenary $\chi = \overline{\overline{R}} \cdot (\cosh\gamma - 1)$ tends to parabola $f(c\tau) = g\tau^2/2 = \overline{\overline{R}}\gamma^2/2$ from above (at $\tau \to \infty$) due to $(\cosh\gamma - 1) \sim \gamma^2/2$. The catenary lies under the kinematical parabola in τ and under the catenary focal tangent to it (with inclination spherical angle $\pi/4$), but it lies above the tangent circle (informatively up to $\chi = \overline{\overline{R}}$) in the special quasiplane $\langle \mathcal{Q}^{1+1} \rangle^{\updownarrow}$, see at Figure 2A (4):

$$g\tau^2/2 < \chi, \quad c\tau - k\overline{\overline{R}} \ \leq \chi, \quad \chi = \chi_\tau(\tau) < R - \sqrt{(\overline{\overline{R}})^2 - (c\tau)^2} \ \text{ if } \ c\tau \leq |\overline{\overline{R}}|.$$

These inequalities are interpreted as follows: $\gamma^2/2 < \cosh\gamma - 1 < 1 - \sqrt{1-\gamma^2}$ $(\gamma \leq 1)$.

In a pseudo-Cartesian base and in quasi-Cartesian one, both these world lines of hyperbolic motion lie at different sides of the two kinematical parabolae, see at Figure 2A (3) and (4). If the angle of motion γ is equal to $\gamma_F = \omega$ (and $\varphi(\gamma_F) = \pi/4$), then the coordinate velocity v achieves value $v_F = c \cdot \tanh\ \omega = c/\sqrt{2}$ (for the hyperbola), and the proper velocity v^* achieves value $v^*_F = c \cdot \sinh\ \omega = c$ (for the catenary). Furthermore, $v^* > c$ if $\gamma > \omega$ and $\varphi(\gamma) > \pi/4$. Proper velocity of light is infinite. The maximal proper velocity for material objects is $v^* \to \infty$! (This is velocity of austronauts by their a clock – see further.) The coordinates of the focal point F in these bases are expressed in terms of the hyperbolic characteristic radius $\overline{\overline{R}} = g/c^2$:

$$\chi_F = (\sqrt{2} - 1)\overline{\overline{R}} \approx 0.41\overline{\overline{R}}; \ \ ct_F^{(1)} = \overline{\overline{R}}, \ \ c\tau_F = \omega\overline{\overline{R}} \approx 0.881\overline{\overline{R}}, \ \text{(but } c\tau = \overline{\overline{R}} \text{ at } \gamma = 1);$$

$$kR = c\tau_F - \chi_F \ \Rightarrow \ k = \omega + 1 - \sqrt{2} \approx 0.467, \ \text{as } \gamma = \omega \text{ and } \varphi(\omega) = \pi/4 \text{ at } F.$$

Hyperbolic motion has the *inner constant parameters*: *pseudo-curvature* $\overline{\overline{K}}_R = 1/\overline{\overline{R}}$ and hyperbolic angular velocity as its main kinematical characteristic (in rad/sec):

$$\eta_K = \frac{d\gamma}{d\tau} = c/\overline{\overline{R}} = c\overline{\overline{K_R}} = \frac{g}{c}. \qquad (99A)$$

It expresses also the hyperbolic rotation of tangent i with pseudonormal p at moving instantaneous point M for collinear motion in its pseudoplane. The curve is hyperbola at hyperbolic motion, then its pseudonormal is radiated from center O (Figure 2A(3)).

The classical *principle of correspondence* in its trigonometric interpretation means that the kinematical hyperbola, catenary, and parabola have the same tangent circle of radius R at their zero point O_1, see Figure 2A(3,4). This is equivalent to the fact that these three curves have at point O_1 the same derivatives of the 1-st (they are zero) and the 2-nd order. Consequently, the two kinematical parabolae of t and of τ (first is classical one) approximating the hyperbola and catenary in a neighborhood of O_1 (in non-relativistic region) has the same " characteristic radius" (i. e., these hyperbola and catenary with the two approximating parabolae have the same radius of curvature) in the own different coordinates, this follows from the approximating relations:

$| \ (g \cdot \tau^2)/2 < \chi = (c^2/g) \cdot \{\cosh[g \cdot \tau(t)/c] - 1\} < (g \cdot t^2)/2 \ | \Rightarrow (g \cdot t^2)/2,$ if $v/c \to 0$.

Another simplest uniform *pseudoscrewed motion* will be considered in Ch. 10A.

* * *

Trigonometric representations are valid for dynamical relativistic characteristics too. For rectilinear physical progressive movement of mass M, we define scalar, vector, and tensor trigonometric expressions of the characteristics from the point of view of inertial Observer N_1 in the original base $\tilde{E}_1$ and in $\tilde{E}_m$. Differential relations of simultaneity (84A) and (85A) will be used. The moving material body is reduced to its barycenter as a material point M. According to the 2-nd Newtonian Law, in $\langle \mathcal{P}^{3+1} \rangle$ there holds

$$F = F_\tau(\tau) = m_0 g(\tau) = m_0 \cdot \frac{dv^{(m)}}{d\tau} = \frac{d[m_0 v^{(m)}]}{d\tau} = \frac{dp^{(m)}}{d\tau} = m_0 c \cdot \frac{g(\tau)}{c} = m_0 c \cdot \frac{d\gamma}{d\tau} \equiv$$

$$\equiv m_0 c \cdot \frac{\cosh\gamma \; d\gamma}{dt^{(1)}} = \frac{d(m_0 c \cdot \sinh\gamma)}{dt^{(1)}} = \frac{d[(\cosh\gamma \cdot m_0) \cdot (\tanh\gamma \cdot c)]}{dt^{(1)}} =$$

$$= \frac{d(m_0 v^*)}{dt^{(1)}} = \frac{d(mv)}{dt^{(1)}} = \frac{dp^{(1)}}{dt^{(1)}} = \frac{dp^{(m)}}{d\tau} = F_t(t^{(1)}).$$

(In general, the inner acceleration g and proportional to it differential $d\gamma$ may be decomposed into parallel and normal projections to velocity v, see Chs. 7A and 10A).

Formulae of the first row hold only in an instantaneous pseudo-Cartesian base where $m_0 = \text{const}$ is the *own mass*. Hence, the form of the 2-nd Newtonian Law is covariant! Then the same inner force F in $\tilde{E}_1$ and $\tilde{E}_m$ is determined with the use of simultaneity at the world point of mass M in both these bases of $\langle \mathcal{P}^{3+1} \rangle$. (The force of inertia $|\mathbf{F}|$ is the number showed as if at the scale of a dynamometer in $\tilde{E}_m^{(3)}$.) Capacity of the inner force action, according to Newtonian mechanics, is represented in the base $\tilde{E}_1$ as

$$N = F \cdot v = m_0 c^2 \cdot \frac{\cosh\gamma \; d\gamma}{dt^{(1)}} \cdot \tanh\gamma = \frac{d(\cosh\gamma \cdot m_0 c^2)}{dt^{(1)}} = \frac{d(mc^2)}{dt^{(1)}} = \frac{dE}{dt^{(1)}}.$$

First, both these STR equations were obtained in *physical forms* by H. Poincaré [47].

These expressions allow to evaluate instantaneous dynamical characteristics in $\tilde{E}_m$ and $\tilde{E}_1$: the *own 4-momentum* $\mathbf{P}_0 = m_0\mathbf{c}$, the *total momentum* $P = mc = \cosh\gamma \cdot P_0$, the *real 3-momentum* $\mathbf{p} = m\mathbf{v} = m_0\mathbf{v}^* = \mathbf{sinh}\ \gamma \cdot m_0 c = \mathbf{sinh}\ \gamma \cdot P_0$; the *own energy* $E_0 = m_0 c^2$, the *total energy* $E = mc^2 = \cosh\gamma \cdot E_0$, *the real kinetic energy* $\Delta E = E - E_0 = (\cosh\gamma - 1) \cdot E_0 \approx (\tanh^2\gamma/2) \cdot E_0 = m_0 v^2/2$, the useful sine energy part $e = pc = mvc = m_0 v^* c = \sinh\gamma \cdot E_0$. The time-like total momentum is increased as a *cosine* orthoprojection onto the time-arrow $\overrightarrow{ct^{(1)}}$, the space-like real momentum is a *sine* orthoprojection into the Euclidean subspace $\langle \mathcal{E}^3 \rangle^{(1)}$ (from a world line).

For the total momentum we obtain $P = mc = \sqrt{P_0^2 + p^2} = \sqrt{(m_0 c)^2 + (mv)^2} \approx$

$$\approx P_0 + m_0 (v^*)^2/(2c) \approx P_0 + m_0 v^2/(2c) \;\Rightarrow\; (mc)^2 = (m_0 c)^2 + (mv)^2.$$

However P is non-invariant in $\langle \mathcal{P}^{3+1} \rangle$ and depends on velocity. Invariant is P_0 given by the pseudo-Euclidean *Absolute Pythagorean Theorem* for 3 momenta (see in Ch. 10A), here for the same right triangle of 3 momenta with hypotenuse $\mathbf{P}_0 = m_0\mathbf{c}$ in $\langle \mathcal{P}^{3+1} \rangle$:

$$P_0 \cdot \mathbf{i} = P \cdot \mathbf{i}_1 + p \cdot \mathbf{j} \Rightarrow (iP_0)^2 = (iP)^2 + p^2 - \text{for tensor } I^{\pm} \text{ in correct form (17A).}$$

From here Poincaré–Einstein relativistic formula for mass-energy follows [83], [50]:

$$E = Pc = mc^2 = \sqrt{E_0^2 + e^2} = \sqrt{E_0^2 + (pc)^2} \approx E_0 + m_0(v^*)^2/2 \approx E_0 + m_0 v^2/2.$$

(With such mechanical way, relation $E = mc^2$ was inferred by G. Lewis in 1908 [68].) The former approximate values in these formulae for m, P and E are upper bounds for the characteristics, second ones are lower bounds, this follows from inequalities: $1+\sinh^2(\gamma/2) > \cosh\gamma > 1+\tanh^2(\gamma/2)$. Last expression is *cosine* time-like Hamilton function of γ as $E = \sqrt{E_0^2 + (pc)^2} = \sqrt{E_0^2 + (E_0 \cdot ||\mathbf{sinh}\ \gamma||)^2} = \cosh\gamma \cdot E_0$. Both these *pseudo-Euclidean invariants* in $\langle \mathcal{P}^{3+1} \rangle$ are $P_0 = m_0 c > 0$, $E_0 = +\sqrt{E^2 - (pc)^2}$. **In addition**, we express trigonometrically the phase velocity of the de Broglie wave as $w_B = E/p = \coth\gamma \cdot c = c^2/v$ and its real velocity as $v_B = dE/dp = \tanh\gamma \cdot c = v$.

Trigonometrically, the total and real momenta as defining dynamical characteristics may be represented in the space-time $\langle \mathcal{P}^{3+1} \rangle$ as scalar cosine and 3×1-vectorial sine projections of the united in STR 4×1-momentum $\mathbf{P}_0$ of a material point M:

$$\mathbf{P}_0 = P_0 \cdot \mathbf{i}_\alpha = m_0 \cdot \mathbf{c}_\alpha = P_0 \cdot \begin{bmatrix} \mathbf{sinh}\ \gamma \\ \cosh\gamma \end{bmatrix} = P_0 \cdot \begin{bmatrix} \sinh\ \gamma \cdot \mathbf{e}_\alpha \\ \cosh\gamma \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ E/c \end{bmatrix}.$$

It is preserved under $\mathbf{F} = \mathbf{0} \leftrightarrow \mathbf{P}_0 = \mathbf{Const}$. The scalar value $P_0 = m_0 c = E_0/c$ is pseudo-Euclidean invariant for the body or material point M. As vectorial differential characteristic, it has the 1-st order of differentiation along a world line. In Chapter 10A, we consider trigonometrically characteristics of absolute movement of M along its world line in $\langle \mathcal{P}^{3+1} \rangle$, with respect to the base $\tilde{E}_1$, up to the superior 5-th order.

These trigonometric forms of the dynamical characteristics are obtained from Laws of the Newtonian mechanics and the relativistic Law of summing physical velocities for collinear two-step motions (v or γ in $\tilde{E}_1$ and dv or $d\gamma$ in $\tilde{E}_m$). The hyperbolic angles of motion are bivalent 4×4-tensors Γ and $d\Gamma$ in $\tilde{E}_1$ and $\tilde{E}_m$. The former is a main argument of the *hyperbolic tensor of motion* acting in space-time $\langle \mathcal{P}^{3+1} \rangle$ – see about it also in Chs. 6. It is a *pseudobiorthogonal tensor.* In the original base $\tilde{E}_1$, its definition and canonical forms due to (324), (348) and (362), (363) are following:

$$\{roth\ (\pm\Gamma)\}_{(3+1)\times(3+1)} = \cosh\Gamma \pm \sinh\Gamma = F(\gamma, \mathbf{e}_\alpha)$$

$$\left|\begin{array}{c|c} \cosh\gamma \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & \pm\sinh\gamma \cdot \mathbf{e}_\alpha \\ \hline \pm\sinh\gamma \cdot \mathbf{e}'_\alpha & \cosh\gamma \end{array}\right| =$$

$$\langle roth\ \Gamma \rangle : \quad roth\Gamma \cdot I^\pm \cdot roth\Gamma = I^\pm \quad (\mathbf{e}_\alpha \mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha \mathbf{e}'_\alpha}) \qquad (100A)$$

$$= \left|\begin{array}{c|c} I_{3\times3} + (\cosh\gamma - 1) \cdot \mathbf{e}_\alpha \mathbf{e}'_\alpha & \pm\sinh\gamma \cdot \mathbf{e}_\alpha \\ \hline \pm\sinh\gamma \cdot \mathbf{e}'_\alpha & \cosh\gamma \end{array}\right|.$$

It is splitted projectively in 3×3-tensor orthoprojection into $\langle \mathcal{E}^3 \rangle^{(1)}$, scalar cosine orthoprojection onto $\overrightarrow{ct^{(1)}}$ and two mutually transposed sine vector oblique projections.

Suppose that a material object M is moving progressively with respect to $\tilde{E}_1$ in $\langle \mathcal{P}^{3+1} \rangle$ at instantaneous velocity $\mathbf{v} = v \cdot \mathbf{e}_\alpha = c \cdot \mathbf{tanh}\ \gamma = c \cdot \tanh\gamma \cdot \mathbf{e}_\alpha$ or proper velocity $\mathbf{v}^* = v^* \cdot \mathbf{e}_\alpha = c \cdot \mathbf{sinh}\ \gamma = c \cdot \sinh\gamma \cdot \mathbf{e}_\alpha$ in the subspace $\langle \mathcal{E}^3 \rangle^{(1)}$. On its arbitrary $4D$-world line in the same base $\tilde{E}_1$, we obtain some more general kinematical parameters: $V^* = c{\cdot}\sinh\Gamma$ as a tensor of proper velocity and $\mathcal{T}_V = c{\cdot}roth\ \Gamma$ as a tensor of absolute 4×4-velocity. For its right column as the 4×1-velocity $\mathbf{c} = c\mathbf{i}$ of Poincaré the pseudo-Euclidean module is ic. (Recall, that at $\gamma = \omega$ we have $v^* = c,\ v = c/\sqrt{2}$.)

The dynamical tensor characteristics are proportional to the tensor of motion (100A) *as their dimensionless trigonometric prototype*, with using two coefficients: c and m_0. Mainly, these following instantaneous dynamical tensors of momentum–energy $\mathcal{T}_P$ and of energy–momentum $\mathcal{T}_E$ are defined in the original base $\tilde{E}_1$ as:

$$\mathcal{T}_P = P_0 \cdot roth\ \Gamma = m_0 c \cdot roth\ \Gamma, \quad \mathcal{T}_E = P_0 c \cdot roth\ \Gamma = E_0 \cdot roth\ \Gamma = m_0 c^2 \cdot roth\ \Gamma.$$

If we suppose, that c = const, then $\mathcal{T}_E \sim \mathcal{T}_P$). Of course, all these three tensors are compatible with metrical reflector tensor of the Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$. Moreover, they are pseudo-Euclidean orthogonal and preserve their symmetric form under orthospherical transformation of $\tilde{E}_1$, i. e., in $\langle \tilde{E}_{1u} \rangle$. Non-symmetrical tensors after two-step or multistep motions may be represented in their polar form (19A) – see initially in sect. 11.3, and for the $(3+1) \times (3+1)$-tensors further in Ch. 7A. For example, consider the tensor $\mathcal{T}_P$. Its canonical tensor form is preserved under $\mathbf{F} = \mathbf{0} \leftrightarrow \mathcal{T}_P = \mathrm{CONST}$. In the base $\tilde{E}_1$, from (100A) it has this physical form:

$$\mathcal{T}_P = \begin{array}{|c|c|} \hline P \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' + P_0 \cdot \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' & \mathbf{p} \\ \hline \mathbf{p}' & E/c \\ \hline \end{array} = \begin{array}{|c|c|} \hline mc \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' + m_0 c \cdot \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' & m\mathbf{v} \\ \hline m\mathbf{v}' & mc \\ \hline \end{array}. \qquad (101A)$$

The $(3+1) \times (3+1)$-tensor is splitted projectively in the 3×3-tensor orthoprojection $\{[\cosh\ \gamma{\cdot}\overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}'] {\cdot} P_0\}$ into $\langle \mathcal{E}^3 \rangle^{(1)}$, the scalar cosine projection $P = P_0{\cdot}\cosh\gamma$ onto the time-arrow $\overrightarrow{ct}^{(1)}$, and two mutually transposed sine 3×1- and 1×3-vectorial oblique projections $\mathbf{p} = P_0 \cdot \sinh\gamma \cdot \mathbf{e}_\alpha = m_0 \mathbf{v}^* = m\mathbf{v}$ and $\mathbf{p}'$. In all admissible pseudo-Cartesian bases, the values $P_0 = m_0 c$ and $E_0 = m_0 c^2$ for a massive material point are the pseudo-Euclidean scalar invariants, but $\mathbf{P}_0 = m_0 \mathbf{c}$ and right column $\mathbf{P}_0$ in (101A) are such *geometric invariants* in space-time $\langle \mathcal{P}^{3+1} \rangle$ similar to a world line.

In its turn, the Lorentzian contraction of moving objects extent in the direction of movement, fixed also by Observer in the universal base $\tilde{E}_1$, has coordinate nature. It is described in 3-dimensional variant by the trigonometric $(3+1) \times (3+1)$-tensor of hyperbolic deformation (Ch. 4A). Due to Lorentzian *seeming* decreasing of moving body volume, *its coordinate density* seems to increase. There is no pressing force acting on the body in the direction of movement. Inner physical force is absolute characteristic in (81A), its scalar value is the same in all inertial bases, not only in $\tilde{E}_1$.

In this trigonometric interpretation of STR, all the relativistic transformations of physical values may be determined more clearly and briefly with the use of these trigonometric tensors and further operations of mathematical analysis over them.

Note (!), that in our applications of tensor trigonometry to the relativistic physics, we rested for the Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$ its *negative signature* $q = 1$, i. e., with the *real-valued* Euclidean space $\langle \mathcal{E}^3 \rangle$ and the *imaginary* time-arrow $\overrightarrow{ict}$, as this was introduced by Poincaré in 1905 [47], and respectively in trigonometric tensor of motion (100A) and proportional to it physical tensor of momentum–energy. However, some physicists use the space-time with $p = 1, q = 3$ from the imaginary anti-Euclidean space $\langle i\mathcal{E}^3 \rangle$ and the real-valued time-arrow $\overrightarrow{ct}$ (?), contrary to (17A), though our geometric $3D$-space of Nature is the real-valued notion!

* * *

We use (79A), (81A) and (86A) for deducing the *relativistic analog* of Ziolkovsky formula, in particular, for a photon rocket moving due to reactive force of the light [66].

$$F = m_0(\tau) \cdot g(\tau) = u \cdot \frac{dm_0(\tau)}{d\tau} \;\Rightarrow\; u \cdot \frac{dm_0(\tau)}{m_0(\tau)} = g(\tau) d\tau = c\, d\gamma(\tau) \Rightarrow$$

$$\Rightarrow m_0(\tau) = m_0 \exp[-(c/u) \cdot \gamma(\tau)] = m_0 \exp\{-(c/u) \cdot \operatorname{arsinh}\, [v^*(\tau)/c]\},$$

where m_0 and m are the initial and current mass of the rocket in the base $\tilde{E}_m$, and u is the fuel outflow velocity, $\gamma(\tau) = \operatorname{arsinh}\, [v^*(\tau)/c]$. We deal with the hyperbolic motion! For a hypothetical photon rocket (as theoretically ideal variant), there holds $u = c$, and

$$m_0(\tau) = m_0 \exp[-\gamma(\tau)] = m_0 \exp\{-\operatorname{arsinh}\, [v^*(\tau)/c]\} = m_0 \exp\{-\operatorname{artanh}\, [v(t)/c]\}.$$

Compare the values of the own mass in terms of the coordinate and proper velocities of the rocket obtained by the Ziolkovsky formula and its relativistic variant above:

$$m_0 \exp(-v^*/u) < m_0 \exp[-\operatorname{arsinh}\, (v^*/c)] = m_0 \exp[-\operatorname{artanh}\, (v/c)] < m_0 \exp(-v/u),$$

and this is equivalent to the trigonometric inequalities $\sinh \gamma > \gamma > \tanh \gamma$.

Consider trigonometric computations for data of the reverse hyperbolic movement of a rocket, see at Figure 3A. This example illustrates, in particular, *the twins paradox*. Similar examples were first analyzed by P. Langevin [65]. For the *free flight*, we have:

$$\chi = L/2 = R \cdot (\cosh \gamma_{max} - 1), \quad \cosh \gamma_{max} = \chi/R + 1, \quad (R = c^2/g);$$

$$\tau = 4(c/g)\gamma_{max}, \qquad t^{(1)} = 4(c/g) \sinh \gamma_{max};$$

$$v_{max} = c \cdot \tanh \gamma_{max}, \qquad v^*_{max} = c \cdot \sinh \gamma_{max};$$

$$m_0(\tau)/m_0 = \exp[4(-c/u)\gamma_{max}].$$

Suppose that a hypothetical photon rocket flies to the nearest star Proxima Centauri and returns to the Earth. Then the *ideal parameters* (by taken time) of the flight are:

- the fuel outflow velocity $u = c$ for a photon rocket (as the theoretical maximum),
- the constant inner acceleration $g = 10$ m/sec^2, (as on the Earth),
- the one-way distance $L = 2\chi \approx 40.3 \cdot 10^{15} m \approx 4.26$ light years.

Computations give the following results:

$$\chi \approx 20.15 \cdot 10^{15} \text{ m}, \quad R \approx 9 \cdot 10^{15} \text{ m}, \quad t_F \approx 305 \text{ days};$$

$$\cosh \gamma_{max} \approx 3.239, \quad \sinh \gamma_{max} \approx 3.081, \quad \tanh \gamma_{max} \approx 0.951, \quad \gamma_{max} \approx 1.8437;$$

$$v_{max} \approx 0.951c, \qquad v^*_{max} \approx 3.061c;$$

$$2L = 4\chi \approx 8.52 \text{ light years}, \quad t^{(1)} \approx 3.70 \cdot 10^8 \text{ sec} \approx 11.7 \text{ years},$$

$$\tau \approx 2.21 \cdot 10^8 \text{ sec} \approx 7{,}01 \text{ years} < 2L \approx 8{,}52 \text{ light years (!)}$$

This is the concrete example for trigonometric interpretation of *the twins paradox*: we have for the 1-st twin-astronaut $\tau \approx 7$ years and for the 2-nd twin on the Earth $t^{(1)} \approx 11.7$ years. Time on the Earth of light spreading there and back with velocity c ($2L \approx 8.52$ light years) is greater than proper time of the flight for this twin-astronaut! Relative decreasing of the own mass due to only expenditure of fuel, according to the relativistic formula, is $m_0(\tau)/m_0 = \exp(-4\gamma_{max}) \approx 1/1600$!!!

A photon rocket with *terrestrial acceleration* reaches the proper velocity c for period less than one year, and further the velocity increases up to $3c$, but at the end of the trip the own mass of the rocket becomes insignificantly small ($m_0/1600$). That is why, such cosmic flights to stars with return of astronauts onto Earth by STR laws are impossible for contemporary men (no for robots) as well as empty projects based on GTR (throuh "wormholes-tunnels" in the Universe etc.)!

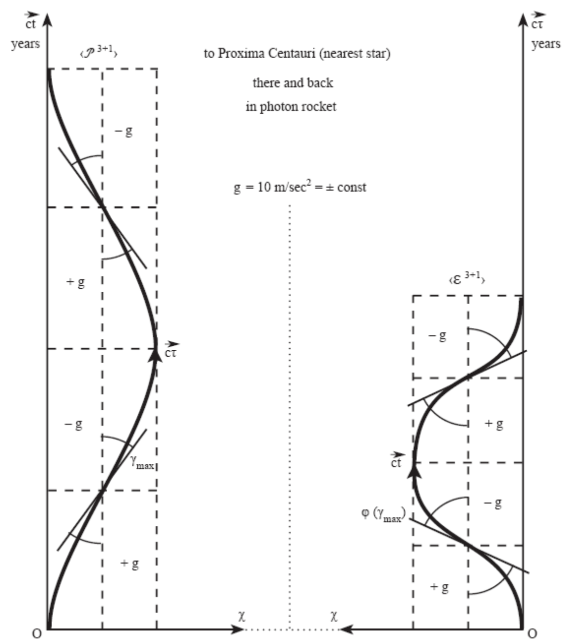

Figure 3A. Reverse hyperbolic movement of a material body in pseudo-Cartesian (at the left) and quasi-Cartesian (at the right) coordinates under constant proper force causing proper acceleration.

However, the paradoxical inequality $\tau \approx 7,01$ years $< 2L \approx 8,52$ light years (gotten due to the concrete initial parameters of the flight) shows, that astronauts during such reverse cosmic flight as if outstrip the light!!! Indeed, a radio-signal sent by the astronauts at the moment of their departure from the Earth to the Star theoretically after its reflection of the Star must return to the Earth in $2L = 4\chi \approx 8,52$ light years. But the astronauts return onto the Earth in $\tau \approx 7$ years $< 2L$ by their *same clock*! This unusual paradox of STR, may be interpreted as follows.

In the instantaneous space $\langle\mathcal{E}^3\rangle^{(m)}$ connected with the rocket and in the space $\langle\mathcal{E}^3\rangle^{(1)}$, light spreads at usual coordinate velocity $c = dx^{(m)}/d\tau = d\chi/d(ct^{(1)})$. However, from the point of view of the astronauts by their clock in the rocket, relative of them velocity of light in $\langle\mathcal{E}^3\rangle^{(1)}$ is $d\chi/d\tau = dx^{(1)}/d(ct^{(m)}) = \cosh\gamma\cdot c > v^* = \sinh\gamma\cdot c$, i. e., *the astronauts do not outstrip the light in* $\langle\mathcal{E}^3\rangle^{(1)}$! (It is caused by the reason, that the space $\langle\mathcal{E}^3\rangle^{(m)}$ and time $\overrightarrow{ct^{(m)}}$ with respect to ones in the base $\tilde{E}_1$ are rotated at the hyperbolic angle $\gamma = \text{arsinh}\ (v^*/c) = \text{artanh}\ (v/c)$ with dilation of time and space in the rocket (Ch. 3A). Consequently, the radio-signal returns to people of the Earth in $t^{(1)} = 2L \approx 8.52$ years, they will meet the astronauts on the Earth in $t^{(1)} \approx 11.7$ years. This paradox is interpreted also by tensor trigonometry. In general, similar kinematic effects of STR, with real difference of time in different frames of reference, are possible only under action of two the great basis Principles of Nature. They are the Principle of Relativity by Poincaré and the Mach Principle (see in sect. 12.3 and in Ch. 9A).

* * *

In conclusion, define in parallel instantaneous parameters of distortions inside of $\langle\mathcal{P}^{1+1}\rangle \subset \langle\mathcal{P}^{3+1}\rangle$ and $\langle\mathcal{Q}^{1+1}\rangle^{\updownarrow} \subset \langle\mathcal{Q}^{3+1}\rangle^{\updownarrow}$ for a world line of a nonuniform rectilinear movement. Due to this simplest nonuniformity, the lines are distorted, but stay in the same pseudo- or quasiplane. At a point M of a certain world line, this distortion is determined up to 2-nd order by parameters of instantaneous specific tangent curves to it: a tangent hyperbola and a tangent circle in $\langle\mathcal{P}^{1+1}\rangle$ (a tangent catenary and a tangent circle in $\langle\mathcal{Q}^{1+1}\rangle^{\updownarrow}$). Hence, these tangent curves may be called *identical* to the world line in a neighborhood of the point M of the 2-nd order, with the same derivatives up to 2-nd order as if with real hyperbolic and visual spherical curvatures. We take advantage of the fact, that *all world lines are regular curves*, i. e., any inner physical acceleration cannot be infinite! In a pseudoplane $\langle\mathcal{P}^{1+1}\rangle$, the radius-vector of hyperbolic curvature of the tangent hyperbola and the world line at point M is directed along the pseudonormal vector $\overline{\overline{\mathbf{p}}}$ (along the $x^{(m)}$-axis) out the center O of the tangent hyperbola (see at Figure 2A(3)). In a quasiplane $\langle\mathcal{Q}^{1+1}\rangle^{\updownarrow}$, the radius-vector of spherical curvature of the tangent catenary and the world line at M is directed along the quasinormal vector $\overline{\overline{\mathbf{q}}}$ (along the $x^{(m)}$-axis) to the center O of the tangent circle (Figure 2A(4)). The tangent vector $\mathbf{i}$ is pseudoorthogonal to $\overline{\overline{\mathbf{p}}}$ or quasiorthogonal to $\overline{\overline{\mathbf{q}}}$. All three these vectors are unity in pseudo-Euclidean or Euclidean metrics.

Any nonuniform rectilinear physical movement of a particle M has in the base $\tilde{E}_1$ its characteristic motion angle γ with the constant vector of directional cosines. Consider its world line in $\langle\mathcal{P}^{3+1}\rangle$ and fix coordinates of the world line points with respect to $\tilde{E}_1$. A pseudoplane containing the curvilinear world line $\overrightarrow{c\tau}(\gamma)$ contains the time-arrow $\overrightarrow{ct^{(1)}}$ and the space axis $x^{(1)} = \chi$ of this universal base $\tilde{E}_1$. The axes $\overrightarrow{ct^{(1)}}$ and χ, the tangent $\mathbf{i}$ and the pseudonormal vector $\overline{\overline{\mathbf{p}}}$ have the same vector of directional cosines $\mathbf{e}_\alpha = \mathbf{const}$. In the base $\tilde{E}_1$, the proper axis χ forms the space-like angle γ with $\overline{\overline{\mathbf{p}}}$, the time-arrow $\overrightarrow{ct}$ forms such time-like angle with $\mathbf{i}$.

As a result of differentiation along the world lines, the curvilinear lines in $\langle\mathcal{P}^{1+1}\rangle$ and in $\langle\mathcal{Q}^{1+1}\rangle^{\updownarrow}$ have at the point M the following hyperbolic and spherical curvatures:

$$1/\overline{\overline{R}} = \frac{d\gamma}{d\lambda},\ d\lambda = d(c\tau);\quad 1/\overline{\overline{r}} = \frac{d\varphi}{dl},\ dl = d(ct)\ \rightarrow\ 1/\overline{\overline{r}} = \operatorname{sech}^2\gamma\cdot(1/\overline{\overline{R}}),\quad (102A)$$

as $d(ct) = \cosh\gamma\ d(c\tau)$, $d\gamma(\varphi) = \cosh\gamma\ d\varphi$ – see in Ch. 6 (sect 6.2 and 6.4). Then basic parameters of distortion for the tangent hyperbola to the world line $\chi(ct)$ are evaluated with respect to the universal base $\tilde{E}_1$ in $\langle\mathcal{P}^{1+1}\rangle$ as follows ($\mathbf{e}_\alpha = \mathbf{const}$):

$$\left.\begin{array}{l} 1/\overline{\overline{R}} = \dfrac{|d\gamma|}{d(c\tau)} = \dfrac{d\ \operatorname{artanh}\left|\dfrac{d\chi}{d(ct)}\right|}{\sqrt{d(ct)^2 - d\chi^2}} = \dfrac{\left|\dfrac{d^2\chi}{d(ct)^2}\right|}{\left[1-\left(\dfrac{d\chi}{d(ct)}\right)^2\right]^{3/2}} = \left|\dfrac{d^2x^m}{d(c\tau)^2}\right|, \\ \text{In } C_R: \chi_c = \chi - \cosh\gamma\cdot\mathbf{e}_\alpha\cdot\overline{\overline{R}},\ ct_c = ct - \sinh\gamma\cdot\overline{\overline{R}};\ \tanh\ \gamma = \dfrac{d\chi}{d(ct)} \\ \rightarrow\ d^2x^{(m)} = d\gamma\cdot d(c\tau) - \text{see in (80A) !} \end{array}\right\}\quad (103A)$$

After transformation $\gamma\rightarrow\varphi(\gamma)$ of the universal base $\tilde{E}_1$, we obtain the following. In the new base $\tilde{E} = \{\chi, \overrightarrow{ct}\}$ of the quasiplane, the rectilinear time-arrow $\overrightarrow{ct}$ forms the angle φ with the tangent $\mathbf{i}$, the proper axis χ forms the angle φ with the quasinormal $\overline{\overline{\mathbf{q}}}$. Basic parameters of distortion for the tangent catenary and circle to the world line $\chi(c\tau)$ are evaluated with respect to the universal base $\tilde{E}$ in $\langle\mathcal{Q}^{1+1}\rangle^{\updownarrow}$ as follows:

$$\left.\begin{array}{l} 1/\overline{\overline{R}} = \cosh^2\gamma\cdot(1/\overline{\overline{r}}) = \cosh^2\gamma\cdot\dfrac{d\varphi}{d(ct)} = \cosh^2\gamma\cdot\dfrac{d\ \arctan\left|\dfrac{d\chi}{d(c\tau)}\right|}{\sqrt{d(c\tau)^2 + d\chi^2}} = \\ = \dfrac{\left|\dfrac{d^2\chi}{d(c\tau)^2}\right|}{\sqrt{1+\left(\dfrac{d\chi}{d(c\tau)}\right)^2}},\quad 1/\overline{\overline{r}} = \dfrac{d\varphi}{d(ct)} = \dfrac{\left|\dfrac{d^2\chi}{d(c\tau)^2}\right|}{\left[1+\left(\dfrac{d\chi}{d(c\tau)}\right)^2\right]^{3/2}} = \left|\dfrac{d^2x^m}{d(ct)^2}\right|, \\ \text{In } C_R:\ \chi_c = \chi - \cosh\gamma\cdot\mathbf{e}_\alpha\cdot\overline{\overline{R}},\quad c\tau_c = c\tau - \gamma\cdot\overline{\overline{R}}; \\ \text{In } C_r:\ \chi_c\ = \chi + \cos\varphi\cdot\mathbf{e}_\alpha\cdot\overline{\overline{r}},\quad c\tau_r = c\tau - \sin\varphi\cdot\overline{\overline{r}}; \\ \sinh\gamma = \dfrac{d\chi}{d(c\tau)} \equiv \tan\varphi(\gamma)\ \rightarrow\ d^2x^{(m)} = d\varphi\cdot d(ct) - \text{see in (80A) !} \end{array}\right\}\quad (104A)$$

If angles γ and φ are independent, then the similar connection of three differentials is mapped according to *abstract* spherical–hyperbolic analogy (322): $\varphi\leftrightarrow -i\varphi\leftrightarrow\gamma$.

Similarly, in the enveloping space $\langle\mathcal{P}^{n+1}\rangle$ or $\langle\mathcal{Q}^{n+1}\rangle$ for n-dimensional hyperbolic or spherical non-Euclidean geometry, with respect to the current base $\tilde{E}_m^{2\times 2} = \{\mathbf{i}, \overline{\overline{\mathbf{p}}}\}$ or $\tilde{E}_m^{2\times 2} = \{\mathbf{i}, \overline{\overline{\mathbf{q}}}\}$ *in the osculating* pseudoplane $\langle\mathcal{P}^{1+1}\rangle$ or quasiplane $\langle\mathcal{Q}^{1+1}\rangle$, the instantaneous parameters of tangent hyperbolae or tangent circles to curves may be evaluated at its current point M analogously with formulae (102A)–(104A).

Chapter 6A

Isomorphic mapping of a pseudo-Euclidean space into time- and space-like quasi-Euclidean ones, Beltrami pseudosphere

Space itself, without matter moving including material fields, has no physical sense. It (as well as its geometry) is an abstract mathematical model, used for adequate and convenient description, according to Poincaré, of general laws of matter movement in coordinate forms. In Ch. 5A, with this approach, we introduced the uninertial *time-like Special quasi-Euclidean space* $\langle \mathcal{Q}^{3+1} \rangle^{\updownarrow}$ (97A), with respect to the base $\tilde{E}_1$ of $\langle \mathcal{P}^{3+1} \rangle$. Its Euclidean subspace $\langle \mathcal{E}^3 \rangle^{(1)}$ is the same as original one in the base $\tilde{E}_1$. In it, ortho-spherical rotations are *rot* Θ. The world line as proper time-arrow $\overrightarrow{c\tau}(\gamma)$ is transformed in a new base $\tilde{E} = \{\chi, \overrightarrow{c\tau}\}$ of the uninertial in general space-time $\langle \mathcal{Q}^{3+1} \rangle^{\updownarrow}$ into a new rectilinear (rectified) ordinate axis, as it is permanently orthogonalized, with respect to $\langle \mathcal{E}^3 \rangle^{(1)} \equiv$ CONST. As a consequence of this fact, for each variant of the world line $\overrightarrow{c\tau}(\gamma)$ of N_m in the base $\tilde{E}_1$ of $\langle \mathcal{P}^{3+1} \rangle$, in the base $\tilde{E}$ of $\langle \mathcal{Q}^{3+1} \rangle^{\updownarrow}$ there exists a map of the original rectilinear time-arrow $\overrightarrow{ct}$ into the new world line $\overrightarrow{ct}(\varphi)$ but of N_1, see Figure 2A(1)–(4). Coordinates of points of this new world line in the new base $\tilde{E} = \{\chi, \overrightarrow{c\tau}\}$ fix proper time $c\tau$ and proper distance χ for Observer N_m. Synchronism of events for N_1 and N_m is parallelism to the common subspace $\langle \mathcal{E}^3 \rangle^{(1)}$ (see this in Ch. 4A).

The *time-like* $\langle \mathcal{Q}^{3+1} \rangle^{\updownarrow}$ is synthesized from the external conic cavity with an exchange of $\overrightarrow{ct}$ and $\overrightarrow{c\tau}$. The additional *space-like* $\langle \mathcal{Q}^{3+1} \rangle^{\leftrightarrow}$ is synthesized from the internal conic cavity with an exchange of $\langle \mathcal{E}^3 \rangle^{(1)}$ and $\langle \mathcal{E}^3 \rangle^{(2)}$. Further we are interested in similar transformations of $\langle \mathcal{P}^{3+1} \rangle$ into the Special spaces with hyperboloids I and II. Note, that *each motion has its invariant in all these three spaces vector of directional cosines* $\mathbf{e}_\alpha$!

In 1-st variant, the hyperboloid I as a locus of time-like curves as hyperbolae $\pm\overrightarrow{c\tau}(\gamma)$ expressed in the base $\tilde{E}_1$ is transformed into a cylinder expressed in the *new* base $\tilde{E}$; its generating lines are these rectified hyperbolae as *new time axes* $\pm\overrightarrow{c\tau}$. A similar circular set of axes $\overrightarrow{ct}$ expressed in the base $\tilde{E}_1$ is transformed with the direction outside the central axis $\overrightarrow{ct}$ into a catenoid I as a locus in $\langle \mathcal{Q}^{3+1} \rangle^{\updownarrow}$ of *new time-like curves as catenaries* $\pm\overrightarrow{ct}(\varphi)$ expressed in the base $\tilde{E}$ (at $ct \leftrightarrow c\tau$). The two cavities of the internal light cone with the space-like hyperboloid II are concentrated *into the new centralized proper time axis* $\overrightarrow{c\tau}$. A catenoid I is a *minimal hypersurface* $F_I(R\varphi)$ in the quasi-Euclidean space $\langle \mathcal{Q}^{3+1} \rangle^{\updownarrow}$. It has cylindrical topology (as a hyperboloid I) and is obtained with revolving one time-like catenary $\pm\overrightarrow{ct}(\varphi)$ around the new time axis as ordinate $\overrightarrow{c\tau}$. The Euclidean length of the world line $\overrightarrow{ct}(\varphi)$ in $\langle \mathcal{Q}^{3+1} \rangle^{\updownarrow}$ is the coordinate time ct of Observer N_1 due to (98A). The proper time $c\tau$ is measured *by Euclidean way* along the ordinate $\overrightarrow{c\tau}$. Transformation $\langle \mathcal{P}^{3+1} \rangle \to \langle \mathcal{Q}^{3+1} \rangle^{\updownarrow}$, as it can be seen for transformation of hyperbolae into catenaries, consists in replacing pseudo-Euclidean measure for time by Euclidean one in the space-time $\langle \mathcal{Q}^{3+1} \rangle^{\updownarrow}$ and only with respect to any universal base. The catenoid I is the result of *dilating* the hyperboloid I time axis, with the current local coefficient $k = d\tau/dt = \mathrm{sech}\gamma$ due to (38A), Ch. 3A.

In 2-nd variant, we change in axes and curves $\chi \to ct = \text{const}$ and $c\tau \to \lambda$ in (96A) for realizing the space-like hyperbolae and catenaries! Then by analogy the two-sheet hyperboloid II as a twain locus of space-like curves as hyperbolae $\lambda(\gamma)$ expressed in the base $\tilde{E}_1$ is transformed into a twain circular set of these rectified hyperbolae λ expressed in the *new* base $\tilde{E}$ and radiated from their two centers O_{II} (Figure 4) as *new space axes* λ in the Euclidean subspace $\langle \mathcal{E}^3 \rangle^{(2)}$. A similar twain circular set of axes χ expressed in the base $\tilde{E}_1$ is transformed with the direction to the ordinate axis $\pm\overrightarrow{ct}$ into a two-sheet catenoid II as a twain locus in space-like $\langle \mathcal{Q}^{n+1} \rangle^{\leftrightarrow}$ of *new space-like curves as catenaries* $\chi(\varphi)$ expressed in the base $\tilde{E}$ (at $\chi \leftrightarrow \lambda$). The single cavity of the external light cone with the time-like hyperboloid I is concentrated into the Euclidean subspace $\langle \mathcal{E}^3 \rangle^{(2)}$. A catenoid II is a two-sheet *sag hypersurface* $F_{II}(R\varphi)$ in the quasi-Euclidean space $\langle \mathcal{Q}^{2+1} \rangle^{\leftrightarrow}$. Its two sheets have also Euclidean topology (as two sheets of a hyperboloid II) and are obtained with revolving two space-like catenaries $\chi(\varphi)$ around the ordinate axis $\overrightarrow{ct}$. The catenoid II is the result of *dilating* the two-sheet hyperboloid II circular space axes, with local coefficient $k = d\lambda/d\chi = \text{sech}\gamma$.

There exists a one-to-one correspondence between points of the hyperboloid I (II) in $\langle \mathcal{P}^{n+1} \rangle$ and the catenoid I (II) in $\langle \mathcal{Q}^{n+1} \rangle^{\updownarrow}$ ($\langle \mathcal{Q}^{n+1} \rangle^{\leftrightarrow}$). The hyperboloid I in $\langle \mathcal{P}^{2+1} \rangle$ *may be mapped isometrically* on a certain hypersurface in the special enveloping quasi-Euclidean space. In order to do it, perform special compressing the catenoid I along χ and $c\tau$ with the following transformation of the space in the space-like $\langle \mathcal{Q}_R^{2+1} \rangle^{\updownarrow}$.

The involute of catenary $\overrightarrow{ct}(c\tau)$ *(*$c\tau = R\gamma(\varphi)$*) is a tractrix* $\overrightarrow{l_R}(c\tau_R)$. The Euclidean length of a catenary is equal to $ct = R\sinh\gamma$ till the point M, see (87A); the length is the same here for the tangent to the catenary at M (it is rectified $\overrightarrow{ct}$). This tangent MM' is the normal radius of curvature of the tractrix, see Figure 2A(4). This normal or tangent is the vector-distance $\overrightarrow{ct}$ between these curves as the same world line in different bases. Revolving a double time-like catenary around the axis $\overrightarrow{c\tau_R}$ produces a catenoid I. Revolving together with a double space-like tractrix produces *a tractricoid* I inside this catenoid I. A double tractrix is a *continuous complete line* with its especial quasi-Cartesian base $\tilde{E}_R = \{\chi_R, \overrightarrow{c\tau_R}\}$ in the *Especial quasi-Euclidean space* $\langle \mathcal{Q}_R^{2+1} \rangle^{\updownarrow}$.

These hypersurfaces of revolution: a hyperboloid I, a cylinder, a catenoid I, and a tractricoid I – are homeomorphic and isomorphic, all they have the determining parameter R. Among them the hyperboloid I and the tractricoid I as *the pseudosphere of Beltrami* have the equal and constant negative Gaussian (if $n = 2$) curvature. Generally, for curvilinear surfaces, their isomorphism (or homeomorphism) and their equal and constant Gaussian curvature are sufficient for their isometry on the whole, what is in accordance and inferred with the classic Minding Theorem [40; 16, p. 533]. Thus, in addition to results obtained in sect. 12.1, we conclude the following:

Main Inference. *The cylindrical non-Euclidean geometry on a hyperboloid I in* $\langle \mathcal{P}^{2+1} \rangle$ *is isometric to the geometry on a real-valued Beltrami pseudosphere in* $\langle \mathcal{Q}_R^{2+1} \rangle^{\updownarrow}$ *with the same parameter* R *and Gaussian curvature* $K_G = -1/R^2 = -1/(R_1 R_2) = \text{const}$.

The analogous but n-dimensional isometry, with the constant negative curvature, is valid for the n-dimensional hyperboloid I and hypertractricoid in their enveloped spaces $\langle\mathcal{P}^{n+1}\rangle$ and $\langle\mathcal{Q}_R^{n+1}\rangle^{\updownarrow}$ as isometry of their n-dimensional hyperbolic geometry with concomitant $(n-1)$-dimensional orthospherical geometry. Of course, this geometry is invariant to this orthospherical rotations. The central circular zone as *equator* of the hyperboloid I and of the double hyperpseudosphere (at $\gamma = 0$) corresponds to the conventional infinite border of the whole cylindrical tangent projective space (Ch. 12). Figures cannot pass through the equator of a pseudosphere under regular motions, but they pass it as broken under 180°, but its metric and topological properties are preserved. However, figures on the hyperboloid I pass freely through this its equator (without the broken) as also conventional border in its cylindrical tangent projective model. The infinitely far border for its flat cotangent model is an infinite ring. Further we explain how the tractrix coordinates are connected in consequent bases under transformations a hyperbola → a catenary → a tractrix.

The time asymptotic ordinate axis $\overrightarrow{c\tau_R}$ of generating tractrices for the pseudosphere is the axis of its proper revolution. It is parallel to $\overrightarrow{c\tau}$, see at Figure 2A(4). At beginning, the tractrix is interpreted in its trigonometric *compressed coordinates* $\tilde{E}_R = \{\chi_R, \overrightarrow{c\tau_R}\}$ in $\langle\mathcal{Q}_R^{3+1}\rangle^{\updownarrow}$ with respect to the bases $\tilde{E}_1$ and $\tilde{E}$. The tractrix abscissa axes χ_R have a space interval on χ from the value 0 up to R between O_1 and O, i. e., $|\chi_R| \le R$ in $\tilde{E}_R = \{\chi_R, \overrightarrow{c\tau_R}\}$. Thus the abscissa axes have the vector of directional cosines $\mathbf{e}_\alpha$ as for the hyperbola and the catenary. The bases $\tilde{E}$ and $\tilde{E}_R$ have the common center O_1, it is the zero point of these connected catenary $\overrightarrow{ct}(c\tau)$ and tractrix $\overrightarrow{l_R}(c\tau_R)$. The point O_1 is a cusp for the complete tractrix, hence it belongs to the curve. It is the mapping of a zero point C_I of the initial hyperbola $\overrightarrow{c\tau}$ in $\tilde{E}_1$ of the space-time $\langle\mathcal{P}^{3+1}\rangle$ (Figure 4). Under STR $c\tau > 0$ and, in the upper and lower parts of the complete tractrix, we have $v > 0$ and $v < 0$, g=const>0, $d\gamma > 0$; at the point O_1 $v = 0, \chi = \chi_R = 0, c\tau = c\tau_R = 0$. Taking into account (86A), (87A), (94A), in $\langle\mathcal{Q}_R^{3+1}\rangle^{\updownarrow}$ the tractrix *radius of curvature* is $ct = R\cdot\sinh\gamma$, $c\tau = R\gamma$ and its *compressed two coordinates* are bonded with ones of these hyperbola and catenary in $\tilde{E}_1$ and $\tilde{E}$ as

$$\left.\begin{array}{l} \chi_R = \sin\varphi(\gamma)ct - \chi \equiv \tanh\gamma\cdot ct - \chi = \text{sech}\ \gamma\cdot\chi = k_1\cdot\chi\ , \\ c\tau_R = c\tau - \cos\varphi(\gamma)ct \equiv c\tau - \text{sech}\ \gamma\cdot ct = (1-\tanh\gamma/\gamma)c\tau = k_2\cdot c\tau. \end{array}\right\} \quad (105A)$$

Here ($\gamma = 0 \to \chi_R = 0,\ c\tau_R = 0$.) The coefficients of compression monotonically change from 1 to 0 (k_1) and from 0 to 1 (k_2) as the world point M is moving from the point O_1 to O. They influence on coordinates mappings $\chi \to \chi_R$, $c\tau \to c\tau_R$ and transform the original curves into the *complete continuous tractrix* $\chi_R(c\tau_R)$. Due to formulae (86A) and (87A) for hyperbolic motion, two equations (105A) may be also represented in the pure trigonometric form (with its constant parameter R) as the functions of the principal angle γ or with the use of sine-tangent analogy (sect. 6.2) as the functions of the angle $\varphi(\gamma)$. Further we reduce relations (105A) to equations in the base $\tilde{E}_R = \{\chi_R, \overrightarrow{c\tau_R}\}$ in $\langle\mathcal{Q}_R^{3+1}\rangle^{\updownarrow}$*in the canonical parametric and explicit forms.*

Note, that similar hyperbolic equations for a tractrix were be given, for example, in the directory [The CRC Concise Encyclopedia of Mathematics by Eric W. Weisstein. – A CRC Press Company. – Boca Raton – London – New York – Washington: 2003, p. 349] till issue of my book [17] in 2004. But in it and later by other authors such equations were interpreted notrightly – in a usual Euclidean space with Cartesian bases.

$$\left.\begin{array}{l}\chi_R = R\cdot s = R(1-\text{sech}\ \gamma) = R(1-\text{sech}\ c\tau/R) = f(R\gamma),\\ c\tau_R = R\cdot h = R(\gamma-\tanh\gamma) = R(c\tau/R-\tanh\ c\tau/R),\end{array}\right\}\boxed{\mathcal{L}_R = x^{(m)}}. \quad (106A)$$

Corollary 1. Condition $R = 1$ *come to unity tractrix as unique trigonometric object. All tractrices $\chi_R(c\tau_R)$ are homothetic to each other with coefficient R (off unity one) as well as such homothetic curves as equilateral hyperbolae, catenaries, circles, etc.* The length of a tractrix arc is expressed by Pythagorean Theorem in $\langle\mathcal{Q}_R^{3+1}\rangle^{\updownarrow}$ with to its radius of curvature $R_t = ct = R\cdot\sinh\gamma = R\cdot\tan\varphi(\gamma)$, or by the way $x^{(m)}$ in (80A):

$$(d\mathcal{L}_R)^2 == (d\chi_R)^2+(dc\tau_R)^2 = (R\tanh\gamma\, d\gamma)^2 \equiv (R\tan\varphi\, d\varphi)^2 = (R_t d\varphi)^2, (\varphi = \varphi(\gamma)),$$

$$d\mathcal{L}_R = R_t d\varphi = R\tan\varphi\ d\varphi \equiv R\tanh\gamma\ d\gamma \equiv dx^{(m)} = d\chi/\cosh\gamma, (d\varphi = \text{sech}\ \gamma\ d\gamma),$$

$$\mathcal{L}_R = R\cdot\mathcal{L} = R\cdot\ln\cosh\gamma \equiv R\cdot\ln\sec\ \varphi \equiv x^{(m)} < c\tau = R\gamma < ct = R\cdot\sinh\gamma.$$

If $\gamma\to 0$, then $\mathcal{L}_R\to gt^2/2$, where $g = F/m_0 = c^2/R$ is the inner acceleration in $\tilde{E}_m$.

$$\frac{d(c\tau_R)}{d\chi_R} = \frac{dh}{ds} = \sinh\gamma \equiv \tan\varphi(\gamma) \Leftrightarrow \frac{d\chi_R}{d(c\tau_R)} = \sinh\ \upsilon = \text{csch}\gamma \equiv \tan\xi(\upsilon) = \cot\varphi(\gamma).$$

The velocity in the base $\tilde{E}_R = \{\chi_R, \overrightarrow{c\tau_R}\}$, according to (106A), is formally related to the *proper supervelocity* (the *coordinate supervelocity* s was defined in (62A), Ch. 4A:

$$s^* = \frac{d\chi_R}{d\tau_R} = c\cdot\text{csch}\ \gamma = \frac{c^2}{\text{v}^*},\quad s = \frac{c^2}{\text{v}} \to s^2-(s^*)^2 = c^2 \leftrightarrow \coth^2\gamma - \text{csch}^2\gamma = 1.$$

It is the cotangent-cosecant invariant with respect to $\{I^{\pm}\}$ of complementary rotation $\overline{rot}\ \Gamma$ *at the angle* Υ (Chs. 6, 12, 7A)! The supervelocity s^* decreases from ∞ up to 0. If it is expressed through the angle υ, then $s^* = c\cdot\sinh\upsilon$ increases from 0 up to ∞!

The explicit and parametric equations of a unity tractrix four branches (as for four branches of quadrohyperbola – Ch. 6) in the base $\tilde{E}_R = \{\chi_R, \overrightarrow{c\tau_R}\}$ are the following.

- direct hyperbolic equations of unity tractrix with the different arguments ($\gamma > 0$):

$$\left.\begin{array}{r}\pm h = \pm h(\gamma) = \gamma - \tanh\gamma = \pm h(z) = \text{arsech}(z) - \sqrt{1-z^2} \equiv\\ \equiv \pm h[\gamma(\varphi)] = h'(\varphi) = \gamma(\varphi) - \sin\varphi = \text{artanh}(\sin\varphi) - \sin\varphi,\end{array}\right\} \quad (107A)$$

where $z = 1-d = \text{sech}\ \gamma$, see (106A), $0 < z \le 1$; $z_R = R\cdot z$ is *radius of revolving!*

$$\mathcal{L} = \ln\cosh\gamma = -\ln\text{sech}\gamma \equiv -\ln\cos\varphi(\gamma);\quad \text{if } \gamma\to 0, \text{ then } \mathcal{L}\to\gamma^2/2.$$

- parametric hyperbolic equations of a unity tractrix with the parameter γ or $\varphi(\gamma)$:

$$\left.\begin{array}{r}z = 1-d = \text{sech}\gamma \equiv \cos\varphi,\\ \pm h = \gamma - \tanh\gamma \equiv \gamma(\varphi) - \sin\varphi,\end{array}\right\}\ 0\le\varphi(\gamma)\le\pi/2. \quad (108A)$$

A tractrix was applied first by Ferdinand Minding in 1838 [40] as a generating line for constructing a pseudosphere as the surface with constant negative Gaussian curvature. (In addition, one may use so called s- and u-shape tractrices as pure *regular curves* !)

Compare them with parametric equations of spherical unity cycloid:

$$\left.\begin{aligned} z &= \cos\varphi, \\ \pm h &= (\varphi - \sin\varphi), \end{aligned}\right\} \quad z_R = R\cdot z = f(R\varphi), \quad h_R = R\cdot h;$$

$$\mathcal{L}_R = R\cdot\mathcal{L} = 4R[1-\cos(\varphi/2)].$$

Corollary 2. *A tractrix is the hyperbolic analog of a spherical cycloid with one cycle. All cycloids $f(R\varphi)$ are homothetic. If $R = 1$ the cycloid is unique trigonometric object.*

At the focus of the tractrix, related to $\gamma_F = \omega = \text{arsinh}\ 1 \approx 0.881$, we have

$$z_F = \sqrt{2}/2 \approx 0.707,\ h_F = \omega - \sqrt{2}/2 \approx 0.174,\ \mathcal{L}_F = \ln 2/2;\ \left(\frac{ds}{dh}\right)_F = 1,\ w_F^* = c.$$

In addition, at Figure 2(4) we have $k = d_F - h_F = 1 - \text{sech}\ \gamma_F + \gamma_F - \tanh\gamma_F \approx 0.467$. From (106A), (105A) and (87A), the useful limit formulae may be easily inferred:

$$\lim_{\gamma\to\infty}\chi_R = \lim_{\gamma\to\infty}(c\tau - c\tau_R) = R,\ \lim_{\gamma\to\infty}(\mathcal{L}_R - c\tau_R) = R(1-\ln 2),\ \text{where } c\tau > \mathcal{L}_R > c\tau_R,$$

This tractrix in process of uniformly accelerated movement, due to description in $\tilde{E}_R = \{\chi_R, \overrightarrow{c\tau_R}\}$, asymptotically tends to the axis $\overrightarrow{c\tau_R}$. If F is the focus of a hyperbola or a tractrix, then $c\tau_{R(F)} + \chi_{R(F)} = kR = c\tau_F - \chi_F$, as in F these catenary and tractrix have $\varphi(\omega) = \pi/4$, it can be seen at Figure 2A(4). With the use of (360) and (106A), in addition to (93A) and (98A), we obtain in $\langle\mathcal{Q}_R^{2+1}\rangle^\updownarrow$ the equation for a tractrix, *invariant only to Lorentzian transformations* of its pro-hyperbola with same *rot* Θ:

$$(R-\chi_R)^2 + (R\gamma - c\tau_R)^2 = R^2 = R^2\cdot(\text{sech}^2\gamma + \tanh^2\gamma) = R^2\cdot(\tanh^2\upsilon + \text{sech}^2\upsilon) \equiv$$
$$\equiv R^2\cdot[\cos^2\varphi(\gamma) + \sin^2\varphi(\gamma)] = R^2\cdot[\sin^2\xi(\upsilon) + \cos^2\xi(\upsilon)], \quad (|\chi_R| \le R).$$

The secondary revolving orthospherical rotations *rot* Θ (in $\langle\mathcal{E}^n\rangle$ are common in the chain: hyperboloid I – catenoid I – tractroid I (Figure 2A)! In the local bases $\tilde{E}_m$ from sine-tangent analogy along these identical world lines, there hold: $d\varphi = \text{sech}\ \gamma\ d\gamma \leftrightarrow d\gamma = \sec\varphi\ d\varphi \leftrightarrow d\xi = \text{sech}\ \upsilon\ d\upsilon \leftrightarrow d\upsilon = \sec\xi\ d\xi$. The quasi-Euclidean space $\langle\mathcal{Q}_R^{n+1}\rangle^\updownarrow$ has admissible *rot* Θ and not-admissible *rot* $\Phi(\Gamma)$, with the same unity metric tensor.

The Beltrami pseudosphere (tractricoid I) with parameter R is obtained by revolving its *generating tractrix* with local radius $z_R = R\cdot\text{sech}\gamma$ around its asymptotic axis $\overrightarrow{c\tau_R}$. All pseudospheres are homothetic with coefficient R (off unity one) in the enveloping Especial quasi-Euclidean space $\langle\mathcal{Q}_R^{2+1}\rangle^\updownarrow$ (as hyperboloids I, II, catenoids I, II, and so one). The two principal radii of curvature, along the generating tractrix R_1 and normally to it R_2 (Figure 2A(4)), are expressed trigonometrically as follows

$$\left.\begin{aligned} R_1 &= -R_t = -R\cdot\sinh\gamma \equiv -R\cdot\tan\varphi(\gamma) = -R\cdot\cot\ \xi < 0, \\ R_2 &= R/\sinh\gamma = z_R/\tanh\gamma \equiv R/\tan\varphi(\gamma) = z_R/\sin\varphi = z_R/\cos\ \xi. \end{aligned}\right\} \quad (109A)$$

In focus F we have $-R_{1F} = R_{2F} = R$, $\varphi(\omega) = \pi/4$. Radius R_t along the tractrix $\mathcal{L}_R = u_R$ is a length of the normal till the catenary, and $z_R = R\cdot\text{sech}\gamma = R/\cosh\gamma$ is radius of orthogonal arc $d\alpha_R = Rd\alpha = z_R d\varphi = Rd\varphi/\cosh\gamma$ and radius of revolution.

Here $\xi(\upsilon) = \pi/2 - \varphi(\gamma)$ is the complementary angle to φ between the normal and the vector-radius of revolving $\mathbf{z}_R$, in accordance with the Meusnier Theorem [16, p. 526]. Besides, $\varphi(\gamma)$, $\xi(\upsilon)$ are *covariant and contravariant (Lobachevskian) parallel angles* of a spherical nature (see sect. 6.4 and in the end of Ch. 1A), and also in the quasi-Euclidean geometry! The Gaussian curvature $K_G = 1/(R_1R_2) = -1/R^2 = \text{const}$ determines locally according to the Beltrami Theorem [41] on the pseudosphere the metric of the Lobachevsky–Bolyai geometry. Since a pseudosphere is determined in the enveloping quasi-Euclidean space $\langle \mathcal{Q}^{2+1}\rangle^{\updownarrow}$ (produced from $\langle \mathcal{P}^{2+1}\rangle$), we have for it the Euclidean measure dl and two Lambert angular ones – along tractrices $\varphi(\gamma)$ and orthospherical arcs α ("parallel circles"). Multiplying by $\sqrt{-K} = 1/R = \text{const}$ gives the measures in the Euclidean metric. Only one tractrix at each point may be independent as geodesic in admissible angular sector. Tractrices from different points are parallel lines in term of Lobachevsky, since all they converge uniformly as asymptotes.

Let's express the all pseudosphere (at $n = 2$) in the $3D$ especial quasi-Cartesian coordinates $\{x, y, z\}$ from (106A), but now as a surface of revolution in them. Thus, below we translate the two-steps motion on the pseudosphere in its polar coordinates with the use of the angle α of orthospherical rotation and with orthogonal projecting under the principal angle $\varphi(\gamma)$. Simultaneity, we calculate easily the 1-st two-steps metrical normal form on the pseudosphere in these two consequent variants $d\lambda_R \equiv dl_R$:

$$d(s_R/R)^2 = \tanh^2\gamma\, d\gamma^2 + \text{sech}^2\gamma\, d\gamma^2 \equiv d\mathcal{L}^2(\gamma) + d\varphi(\gamma)^2 = d\mathcal{L}^2(\gamma) + \cosh^2\gamma\, d\alpha^2 \equiv d\mathcal{L}^2(\varphi) + \sec^2\varphi(\gamma)\, d\alpha^2.$$

Note, the hyperbolic variant is an original prototype of ds_R. On an n-pseudosphere, we can represent two- and multi-steps motions only as isomorphic maps of the original sequential motions on a hyperboloid I in its pseudo-Cartesian bases of the pseudo-Euclidean space $\langle \mathcal{P}^{n+1}\rangle$ into isomorphic motions on this pseudosphere in its especial quasi-Cartesian coordinates of the especial quasi-Euclidean space $\langle \mathcal{Q}^{n+1}\rangle^{\updownarrow}$. It is caused by the fact, that the last space has the orthospherical group $\langle rot\ \Theta\rangle$, the set $\langle rot\ \Phi(\Gamma)\rangle$ with $\langle rot\ \Theta\rangle$ is not a group here with the same unity reflector tensor $I^{\pm}$. So, if we rotate this especial base of the pseudosphere at any principal spherical angle by $rot\ \Phi(\Gamma)$, then, in this new base, it is no longer a surface of constant curvature! For clarity, we must divide such hypersurfaces of constant curvature as *perfect at $R = const$* and *imperfect*.

With this trigonometric approach, we obtain easyly these 1-st two-steps metrical forms (at $n = 2$): in $3D$ quasi-Cartesian base $\tilde{E}_1$ for a hyperspheroid ($\varphi_0 = 0$); in $3D$ pseudo-Cartesian base $\tilde{E}_1$ for a hyperboloid II ($\gamma_0 = 0$) and for a hyperboloid I ($\gamma_0 = 0$). See descriptively these three main geometric objects (at $R = 1$) of the quasi-Euclidean and pseudo-Euclidean tensor trigonometries in their sections at Figure-4, in Ch. 12.

$$[d(l_R/R)]^2 = d\varphi_p^2 = d\varphi_i^2 + \sin^2\varphi_i\, d\alpha^2 = \left(\overline{\overline{d\varphi_p}}\right)_Q^2 + \left(\overset{\perp}{d\varphi_p}\right)_E^2 = d\xi_i^2 + \cos^2\xi_i\, d\alpha^2 \text{ – hyperspheroid (199A).}$$

$$[d(\lambda_R/R)]^2 = d\gamma_p^2 = d\gamma_i^2 + \sinh^2\gamma_i\, d\alpha^2 = \left(\overline{\overline{d\gamma_p}}\right)_P^2 + \left(\overset{\perp}{d\gamma_p}\right)_E^2 \text{ – two-sheets hyperboloid II (146A).}$$

$$\mp[d(\lambda_R/R)]^2 = \mp d\gamma_p^2 = d\gamma_i^2 - \cosh^2\gamma_i\, d\alpha^2 = \left(\overline{\overline{d\gamma_p}}\right)_P^2 - \left(\overset{\perp}{d\gamma_p}\right)_E^2 \text{ – one-sheet hyperboloid I (149A).}$$

A pseudosphere corresponds by isometry to hyperboloid I, a hyperspheroid by analogy (322) to hyperbolid II with $I^{\mp}$. Differentials along hyperbolae are accompanied by $d\alpha$ with projecting under the principal angle. These orthospherical parts are situated in the Euclidean subspace normally (Euclidean or pseudo-Euclidean) to first principal ones. They are caused by rotation of the directional vector $\mathbf{e}_\alpha$ of the principal motion (see in Chs. 7A, 8A and in details in Ch. 10A). The relations are represented here and generally in Ch. 10A by the *Absolute non-Euclidean Pythagorean Theorems* in scalar, vector and tensor forms, with the enveloping quasi-Euclidean or pseudo-Euclidean spaces, or only in scalar forms for internal non-Euclidean geometries.

Chapter 7A

Trigonometric models of two-step, multistep, and integral non-collinear motions in STR and in hyperbolic geometries

We continue studying both two-step and general multistep principal motions $\langle roth\ \Gamma\rangle$. They are analyzed with more wide using the tensor trigonometry in two directions:
1) The relativistic motions-rotations in $\langle \mathcal{P}^{2+1}\rangle \equiv \langle \mathcal{E}^2\rangle \boxtimes \overrightarrow{y} \equiv \mathrm{CONST}$, what correspond to the physical plane movements in STR with coplanar vectors of directional cosines; and on the embedding into $\langle \mathcal{P}^{2+1}\rangle$ $2D$ Minkowskian hyperboloid II ($n = 2$) or, that is equivalent, on the Lobachevsky–Bolyai hyperbolic plane at $R = \mathrm{const}$.
2) The geometric motions-rotations in $\langle \mathcal{P}^{n+1}\rangle \equiv \langle \mathcal{E}^n\rangle \boxtimes \overrightarrow{y} \equiv \mathrm{CONST}$, what correspond at $n = 3$ to the physical space movements in STR; and on the nD Minkowskian hyperboloids ($n \geq 3$) or, that is equivalent, in the Lobachevsky–Bolyai hyperbolic spaces at $R = \mathrm{const}$. See before in Chs. 11, 12, 2A and 5A.

These operations are admitted in $\langle \mathcal{P}^{n+1}\rangle \equiv \langle \mathcal{E}^n\rangle \boxtimes \overrightarrow{y} \equiv \mathrm{CONST}$ with right bases:
1) rotations of the two types, as principal hyperbolic $roth\ \Gamma$ and orthospherical $rot\ \Theta$;
2) parallel translations preserving the space structure with reflector tensor $I^{\pm}$.

Hyperbolic and orthospherical rotations have their real canonical forms in $\tilde{E}_1 = \{I\}$. That is why, in polar and summing formulae they are given *initially* in $\tilde{E}_1$, but really they may be translated from $\tilde{E}_1$ into the bases of action $\tilde{E}_k$, due to the Rule of multistep transformations (Ch. 11). Hyperbolic tensor of motion (100A), on the basis of its protensor (324), is defined due to relations (348) by $\langle roth\ \Gamma\rangle : \ roth\Gamma \cdot I^{\pm} \cdot roth\Gamma = I^{\pm}$. The orthospherical tensor has in $\langle \mathcal{P}^{3+1}\rangle$ and $\langle \mathcal{Q}^{3+1}\rangle$ canonical form (497). Their structures in $\langle \mathcal{P}^{n+1}\rangle$ or space-time $\langle \mathcal{P}^{3+1}\rangle \equiv \langle \mathcal{E}^3 \boxtimes \overrightarrow{ct}\,\rangle$ correspond to the metric reflector tensor:

$$\{roth\Gamma\}_{4\times4} = \cosh\Gamma + \sinh\Gamma \qquad\qquad rot\ \Theta \qquad\qquad I^{\pm}$$

$$\begin{array}{|c|c|}\hline \cosh\gamma_i \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & \sinh\gamma_i \cdot \mathbf{e}_\alpha \\ \hline \sinh\gamma_i \cdot \mathbf{e}'_\alpha & \cosh\gamma_i \\ \hline \end{array} \cdots\cdots \begin{array}{|c|c|}\hline \{rot\ \Theta\}_{3\times3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \\ \hline \end{array} \cdots\cdots \begin{array}{|c|c|}\hline I_{3\times3} & \mathbf{0} \\ \hline \mathbf{0}' & -1 \\ \hline \end{array} \ (\overleftarrow{\mathbf{e}_\alpha \mathbf{e}'_\alpha} = \mathbf{e}_\alpha \mathbf{e}'_\alpha). \quad (110A)$$

The orthospherical rotation-motion in the angle Θ as a rule is secondary for principal angle. The *hyperbolic motion tensor* $roth\ \Gamma$ with $\mathbf{e}_\alpha$ in $\tilde{E}_1$ as well as in another universal base $\tilde{E}_{1u} = rot\ \Theta \cdot \tilde{E}_1$ (with $rot'\Theta_{3\times3} \cdot \mathbf{e}_\alpha$) has canonical form (362) – see in Ch. 6. The time-arrows $\overrightarrow{ct}^{(k)}$ are used as the frame axes for counting the hyperbolic angle γ. At first, we consider two-step hyperbolic rotations realized as if in $\langle \mathcal{P}^{2+1}\rangle \equiv \mathrm{CONST}$ – see above, in order to infer the general law of summing two-step motions (velocities) in tensor, vectorial, and scalar forms. The new pseudo-Cartesian base can be represented in $\tilde{E}_1 = \{I\}$ by two ways: with ordering (485) of matrices and in the polar forms (491)

$$\tilde{E}_3 = roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot \tilde{E}_1 = (roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot roth^{-1}\Gamma_{12})_{\tilde{E}_2} \cdot roth\ \Gamma_{12} \cdot \tilde{E}_1 =$$

$$= roth\ \Gamma_{13} \cdot rot\ \Theta_{13} \cdot \tilde{E}_1 = (roth\ \Gamma_{13} \cdot rot\ \Theta_{13} \cdot roth^{-1}\Gamma_{13})_{\tilde{E}_{1h}} \cdot roth\ \Gamma_{13} \cdot \tilde{E}_1 = \quad (111A)$$

$$= rot\ \Theta_{13} \cdot roth\ \overset{\angle}{\Gamma}_{13} \cdot \tilde{E}_1 = (rot\ \Theta_{13} \cdot roth\ \overset{\angle}{\Gamma}_{13} \cdot rot'\Theta_{13})_{\tilde{E}_{1u}} \cdot rot\ \Theta_{13} \cdot \tilde{E}_1 = T_{13} \cdot \tilde{E}_1.$$

First pairs of matrices in each three rows of (111A) are given initially in the base $\tilde{E}_1 = \{I\}$ in their canonical forms. Further, the second matrix from these pairs is being translated in each row in the indicated base of its real action $\tilde{E}_k$. This relates to the two-step motion in the first row and to both these polar representations of the summary matrix T_{13} in the second and third rows with right and contrary ordering hyperbolic and orthospherical rotations, according to general formulae (485)–(488) and (491) from Ch. 11. For example, in the first variant of polar decomposition, $rot\ \Theta_{13}$ has the center of its application in the final point of the rotation $roth\ \Gamma_{13}$.

Corollary. *Generally, two-step noncollinear hyperbolic motions* $roth\ \Gamma_{ij}$ *in* $\langle \mathcal{P}^{n+1} \rangle$ *or on the hyperboloid II can be represented as hyperbolic one and orthospherical rotation.* Hyperbolic rotations are executed in $\langle \mathcal{P}^{n+1} \rangle$ relatively of the frame axis $\overrightarrow{ct}$. Orthospherical rotations are executed in $\langle \mathcal{E}^3 \rangle$ *for an object or a base* around the axis $\overrightarrow{\mathbf{e}_N}$.

In accordance with (352), the bases $\langle \tilde{E}_{1u} \rangle = \langle rot\ \Theta \cdot \tilde{E}_1 \rangle$ are universal too (in STR, they are called the rest bases). Due to (111A), there holds

$$roth\ \overset{\angle}{\Gamma}_{13} = rot\ (-\Theta_{13}) \cdot roth\ \Gamma_{13} \cdot rot\ \Theta_{13} = rot'\Theta_{13} \cdot roth\ \Gamma_{13} \cdot rot\ \Theta_{13}. \quad (112A)$$

For $\overset{\angle}{\Gamma}_{13}$, the vector of directional cosines in (363) is shifted with respect to that of Γ_{13} to backwards at Θ_{13}. Moreover, in $\langle \mathcal{P}^{3+1} \rangle$, for hyperbolic two-step rotations, the angle of secondary orthospherical shifting is realized contrary to the direction of summing principal angles, i. e., $\theta_{13} < 0$ due to (499), see in sect 12.2 and below:

$$\mathbf{e}_{\overset{\angle}{\sigma}} = \{rot\ (-\Theta_{13})\}_{3\times 3} \cdot \mathbf{e}_\sigma \text{ (under rule } \varepsilon > 0 \to \theta_{13} < 0\,!) \ \Rightarrow\ \cos\theta_{13} = \mathbf{e}'_{\overset{\angle}{\sigma}} \cdot \mathbf{e}_\sigma. \quad (113A)$$

In accordance with (474), (475) and by (111A) and (325), there holds

$$roth\ \Gamma_{13} = \sqrt{TT'} = \sqrt{roth\ \Gamma_{12} \cdot roth\ (2\Gamma_{23}) \cdot roth\ \Gamma_{12}} = \sqrt{roth\ (2\Gamma_{13})}, \quad (114A)$$

$$rot\ \Theta_{13} = roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot roth\ \overset{\angle}{\Gamma}_{31} = roth\ \Gamma_{31} \cdot roth\ \Gamma_{12} \cdot roth\ \Gamma_{23}. \quad (115A)$$

Formula (115A) represents $rot\ \Theta_{13}$ as the angular defect Θ_{13} of the closed cycle of motions $roth\ \Gamma_{ij}$ in the hyperbolic triangle 123. It is executed from the first point 1 to the final point 3 in the bases of particular rotations along of the triangle legs! If rotations $roth\ \Gamma_{ij}$ are collinear, then the triangle degenerates into the segment γ_{13}.

Further, we shall often use the operation of permutation of particular motions with change of their order into contrary one (for some more simple calculations). In the original universal base $\tilde{E}_1 = \{I\}$, permutation in (111A) of two motions (velocities) leads to a new pseudo-Cartesian base $\tilde{E}'_3 = \{T'\}$:

$$\tilde{E}'_3 = roth\ \Gamma_{23} \cdot roth\ \Gamma_{12} \cdot \tilde{E}_1 = T' \cdot \tilde{E}_1 =$$

$$= roth\ \overset{\angle}{\Gamma}_{13} \cdot rot\ (-\Theta_{13}) \cdot \tilde{E}_1 = rot\ (-\Theta_{13}) \cdot roth\ \Gamma_{13} \cdot \tilde{E}_1. \quad (116A)$$

Thus there are two points of view at matrix (112A): as in (111A) and as in (116A)!

In addition to (114A) and (115A), if matrices in $\tilde{E}_1$ are ordered inversely, then

$$roth\ \overset{\angle}{\Gamma}_{13} = \sqrt{T'T} = \sqrt{roth\ \Gamma_{23} \cdot roth\ (2\Gamma_{12}) \cdot roth\ \Gamma_{23}} = \sqrt{roth\ (2\ \overset{\angle}{\Gamma}_{13})}, \quad (117A)$$

$$rot\ (-\Theta_{13}) = roth\ \overset{\angle}{\Gamma}_{13} \cdot roth\ \Gamma_{32} \cdot\ roth\ \Gamma_{21} = roth\ \Gamma_{32} \cdot roth\ \Gamma_{21} \cdot\ roth\ \Gamma_{13}. \quad (118A)$$

Formula (118A) represents $rot\ (-\Theta)$ for the inverse closed cycle (115A) of $roth\ \Gamma_{ij}$.

In STR, the angle Θ_{13} of the orthospherical shift has the pure relativistic nature. In fact, with respect to the original base $\tilde{E}_1$, we have the following relativistic effect: a nonpoint object is seen by Observer N_1 spherically turned backwards in the plane of noncollinear velocities $\mathbf{v}_{12}$, $\mathbf{v}_{23}$, i. e., contrary to direction of their summation into $\mathbf{v}_{13}$. Similar effect of orthospherical shift of the final subbase $\tilde{E}^{(3)}$ with its nonpoint objects takes place in external hyperbolic and spherical geometries as additional result for noncollinear summing principal motions. It is connected (see further) with deviation in these geometries of angles sum in a figure constructed from geodesic lines.

The orthospherical scalar angular shift θ was discussed first by É. Borel in 1913 [54], as consequence of Lorentzian transformations non-commutativity. In 1926 L. Thomas gave his famous relativistic interpretation [70] of the experimental correcting coefficient 1/2 to the electron spin precession based on this orthospherical shifting. This scientific event was one of first convincing and obvious affirmations of STR, as the experimental coefficient 1/2 had no other interpretation! Later in 1928 the *Thomas precession* got more general and final interpretation in the relativistic–invariant (in the Minkowskian space-time) quantum wave equation of Paul Dirac [55].

The angles Γ_{13} and $\overset{\angle}{\Gamma}_{13}$ differ only by their vectors of directional cosines. Due to (491) or (112A), the scalar summary hyperbolic angle does not depend on ordering of summands (direct or inverse). The case when the directional cosines of motions are either equal or additively opposite to each other, corresponds to collinear motions.

Let $\mathbf{e}_\alpha = \{\cos\alpha_k,\ k = 1, 2, 3\}$ be the vector of directional cosines for Γ_{12}, $\mathbf{sinh}\ \gamma_{12}$, $\mathbf{tanh}\ \boldsymbol{\gamma_{12}}$, and $\mathbf{v}_{12}$ in the Cartesian subbase $\tilde{E}_1^{(3)}$; $\mathbf{e}_\beta = \{\cos\beta_k,\ k = 1, 2, 3\}$ be the vector of directional cosines for Γ_{23}, $\mathbf{sinh}\ \gamma_{23}$, $\mathbf{tanh}\ \boldsymbol{\gamma_{23}}$, and $\mathbf{v}_{23}$ in the Cartesian subbase $\tilde{E}_2^{(3)}$. Define *the conditional characteristic*, the angle ε between $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ as if they are in the same subspace $\langle\mathcal{E}^3\rangle$ by the following formal value of its cosine:

$$\cos\varepsilon = \begin{bmatrix} \cos\beta_1 \\ \cos\beta_2 \\ \cos\beta_3 \end{bmatrix}' \cdot \begin{bmatrix} \cos\alpha_1 \\ \cos\alpha_2 \\ \cos\alpha_3 \end{bmatrix} = \mathbf{e}'_\beta \mathbf{e}_\alpha = \mathbf{e}'_\alpha \mathbf{e}_\beta, \qquad 0 \le |\varepsilon| \le \pi \qquad (119A).$$

Here $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = \cos^2\beta_1 + \cos^2\beta_2 + \cos^2\beta_3 = 1$. If the partial cosines are pairly equal, then $\cos\varepsilon = 1$. If they are pairly additively opposite, then $\cos\varepsilon = -1$. Thus, in these cases, $\mathbf{v}_{12}$ and $\mathbf{v}_{23}$ are conventionally collinear, with the same or opposite directions. If $\cos\varepsilon = 0$, then $\mathbf{v}_{12}$ and $\mathbf{v}_{23}$ are conventionally orthogonal. In general, they form the conventional angle ε (as $\mathbf{v}_{12}$ and $\mathbf{v}_{23}$ is in different Euclidean spaces).

We have *invariant* $R = 1$ for any types of rotations related to the hyperboloid radius.

Further, evaluate the final hyperbolic matrix $roth\ \Gamma_{13}$ with the use of (114A), in that number, the eigen angle γ_{13} in the base $\tilde{E}_1$ and directional cosines $\cos\sigma_k$, $k = 1, 2, 3$, of $roth\ \Gamma_{13}$ in the Cartesian subbase $\tilde{E}_1^{(3)}$. For two-step motions in the inverse order, the scalar angle of summary motion $roth\ \overset{\angle}{\Gamma}_{13}$ is the same γ_{13} according to (112A). The directional cosines of $roth\ \overset{\angle}{\Gamma}_{13}$ are $\cos\overset{\angle}{\sigma}_k$, $k = 1, 2, 3$. By (113A), we obtain

$$\cos\theta_{13} = \begin{bmatrix} \cos\overset{\angle}{\sigma}_1 \\ \cos\overset{\angle}{\sigma}_2 \\ \cos\overset{\angle}{\sigma}_3 \end{bmatrix}' \cdot \begin{bmatrix} \cos\sigma_1 \\ \cos\sigma_2 \\ \cos\sigma_3 \end{bmatrix} = \mathbf{e}'_{\overset{\angle}{\sigma}} \cdot \mathbf{e}_\sigma = \cos\theta_{13} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\angle}{\sigma}}, \qquad (120A),$$

where $\sin\theta_{13} < 0$, if $\varepsilon > 0$. Recall (sect. 12.2), that in $\langle \mathcal{P}^{3+1} \rangle$ (and in $\langle \mathcal{Q}^{3+1} \rangle$) this follows from a Rule *for the sign of* θ_{13} *in result of summation of non-collinear principal motions* γ_{12} and γ_{23}, where ε is the external angle between them. The angles θ and ε act in the same Euclidean plane $\langle \mathcal{E}^2 \rangle \equiv \langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle$ orthogonal to the frame axis $\mathbf{r}_N$ as (499) in sect. 12.2 of the orthospherical rotation in $\langle \mathcal{E}^3 \rangle$ ($\mp\theta \leftrightarrow \pm\varepsilon$). For θ, the Euclidean Lambert measure is used, as these angles take place in Euclidean subspaces!

Note for further trigonometric transformations, that the two variants (direct and inverse) of two-step motion are connected by substitution of partial angles:

$$\gamma_{12} \leftrightarrow \gamma_{23}, \qquad \alpha_k \leftrightarrow \beta_k, \quad (\text{but } \gamma_{13} = \text{const}). \qquad (121A)$$

In (111A), block-to-block multiplication of matrices with structure (363) is unwieldy. It may merely illustrate further the General Law of summing principal motions. We use for two-step motions more simple way. At first evaluate matrix product in (114A)

$$B = \{roth\ \Gamma_{12} \cdot roth\ (2\Gamma_{23})\} = \{b_{ij}\}.$$

For tensor trigonometric analysis of two-step hyperbolic motions and plane relativistic movements in STR it is enough to use 3×3-matrices. For generality we use 4×4- and $(n+1)\times(n+1)$-matrices! Only fourth row of B is used in further computations. The matrices $roth\ \Gamma$ may be used in any of canonical forms (362), (363). Then we obtain:

$$b_{41} = [\sinh\gamma_{12} \cdot \cosh(2\gamma_{23}) \cdot \cos\varepsilon + \cosh\gamma_{12} \cdot \sinh(2\gamma_{23})] \cdot \cos\beta_1 + \\ + \sinh\gamma_{12} \cdot (\cos\alpha_1 - \cos\varepsilon \cdot \cos\beta_1),$$

$$b_{42} = [\sinh\gamma_{12} \cdot \cosh(2\gamma_{23}) \cdot \cos\varepsilon + \cosh\gamma_{12} \cdot \sinh(2\gamma_{23})] \cdot \cos\beta_2 + \\ + \sinh\gamma_{12} \cdot (\cos\alpha_2 - \cos\varepsilon \cdot \cos\beta_2),$$

$$b_{43} = [\sinh\gamma_{12} \cdot \cosh(2\gamma_{23}) \cdot \cos\varepsilon + \cosh\gamma_{12} \cdot \sinh(2\gamma_{23})] \cdot \cos\beta_3 + \\ + \sinh\gamma_{12} \cdot (\cos\alpha_3 - \cos\varepsilon \cdot \cos\beta_3),$$

$$b_{44} = \sinh\gamma_{12} \cdot \sinh(2\gamma_{23}) \cdot \cos\varepsilon + \cosh\gamma_{12} \cdot \cosh(2\gamma_{23}).$$

At the beginning, we evaluate the diagonal corner element s_{44} of the symmetric matrix $S = roth^2\ \Gamma_{13} = roth\ (2\Gamma_{13})$ multiplying the 4-th row of B and the 4-th column of $roth\ \Gamma_{12}$:

$$s_{44} = \cosh(2\gamma_{13}) = 2\cosh^2\gamma_{13} - 1 =$$
$$= \cosh(2\gamma_{12})\cdot\cosh(2\gamma_{23}) + \cos\varepsilon\cdot\sinh(2\gamma_{12})\cdot\sinh(2\gamma_{23}) - 2\sin^2\varepsilon\cdot\sinh^2\gamma_{12}\cdot\sinh^2\gamma_{23} =$$
$$= 2(\cosh\gamma_{12}\cdot\cosh\gamma_{23} + \cos\varepsilon\cdot\sinh\gamma_{12}\cdot\sinh\gamma_{23})^2 - 1.$$

From here we obtain directly the first metric *commutative scalar* formula of the non-Euclidean Lobachevsky–Bolyai geometry for the cosine of the summary angle γ_{13}:

$$\cosh\gamma_{13} = \cosh\gamma_{12}\cdot\cosh\gamma_{23} + \cos\varepsilon\cdot\sinh\gamma_{12}\cdot\sinh\gamma_{23} =$$
$$= \cosh\gamma_{12}\cdot\cosh\gamma_{23} - \cos(\pi-\varepsilon)\cdot\sinh\gamma_{12}\cdot\sinh\gamma_{23} =$$
$$= \cosh\gamma_{12}\cdot\cosh\gamma_{23} - \cos A_{123}\cdot\sinh\gamma_{12}\cdot\sinh\gamma_{23}. \qquad (122A)$$

In non-Euclidean geometry, $A_{123} = \pi - \varepsilon$ is the *internal angle* of the triangle between legs 12 and 23; and, in STR, ε is the *external* angle adjacent to A_{123}. For any relativistic physical movements in $\langle\mathcal{P}^{3+1}\rangle$, there holds $\gamma_{ij} > 0$, because the angles correspond to inequality $\Delta ct > 0$ (for motions to future). In particular, these hyperbolic angles are represented by segments on the unity Minkowskian hyperboloid II, and its tangents lead to velocities. For the motions angles and the segments, the lengths by Lambert measure γ follow to the *Rule of a parallelogram* similar to one in Euclidean geometry:

$$|\gamma_{12} - \gamma_{23}| \le \gamma_{13} \le \gamma_{12} + \gamma_{23}, \quad (\varepsilon \in [0;\pi]). \qquad (123A)$$

For the angles of motion or their trigonometric projections in Euclidean subspaces, their directional cosines range is $[-1;+1]$. Due to unequalities $\gamma > 0$ and (123A), distance in hyperbolic geometry by the measure γ is a norm. (For spherical arcs in any subspace $\langle\mathcal{E}^3\rangle \subset \langle\mathcal{P}^{3+1}\rangle$, corresponding measure of length stays spherical as $r\varphi$.)

Note, that *scalar cosine* formula of the type (122A) was historically first one for describing legs of pseudo-spherical triangles as if on the hypothetic sphere of imaginary radius iR based on the analogy with spherical triangle on the real sphere of radius R. Franz Taurinus was the first geometer (1825), who considered with results so called "logarithmic–spherical" non-Euclidean geometry with relation between three legs of a hypothetic triangle in relation (122A) [36]. Hence, by the fact, (122A) would be called cosine formula of Taurinus! Here it expresses the leg a_{13} through the legs a_{12} and a_{23}.

Due to (122A) and the following scalar trigonometric formulae, *the scalar value of summary motion γ_{13} or summary velocity v_{13} does not depend on the summands ordering.* However, due to (111A), the *complete law of summation* of two or more particular motions or velocities for nonpoint vector and lineor objects or for an initial base must include the secondary orthospherical rotation too. Only the vectorial and more tensor similar trigonometric formulae may give the complete law of summation in its general form. This in details will be discussed later.

The scalar sine is evaluated trigonometrically with (122A), in that number, in two commutative variants as the *ums of squares* provided that $\gamma_{12} \leftrightarrow \gamma_{23}$:

$$\sinh^2\gamma_{13} = \sinh^2\gamma_{12} + \sinh^2\gamma_{23} + \sinh^2\gamma_{12}\cdot\sinh^2\gamma_{23}\cdot(1+\cos^2\varepsilon) +$$
$$+2\sinh\gamma_{12}\cdot\cosh\gamma_{12}\cdot\sinh\gamma_{23}\cdot\cosh\gamma_{23}\cdot\cos\varepsilon =$$
$$= (\sinh\gamma_{12}\cosh\gamma_{23} + \cos\varepsilon\cdot\sinh\gamma_{23}\cdot\cosh\gamma_{12})^2 + (\sin\varepsilon\cdot\sinh\gamma_{23})^2 =$$
$$= (\sinh\gamma_{23}\cosh\gamma_{12} + \cos\varepsilon\cdot\sinh\gamma_{12}\cdot\cosh\gamma_{23})^2 + (\sin\varepsilon\cdot\sinh\gamma_{12})^2. \qquad (124A)$$

The scalar tangent is evaluated trigonometrically with the use of (122A) and (124A) also commutatively as the *sums of squares* provided that $\gamma_{12} \leftrightarrow \gamma_{23}$:

$$\tanh^2\gamma_{13} =$$
$$= \left[\frac{\tanh\gamma_{12} + \cos\varepsilon\cdot\tanh\gamma_{23}}{1+\cos\varepsilon\cdot\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\right]^2 + \left[\frac{\sin\varepsilon\cdot\tanh\gamma_{23}\cdot\mathrm{sech}\,\gamma_{12}}{1+\cos\varepsilon\cdot\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\right]^2 =$$
$$= \left[\frac{\tanh\gamma_{23} + \cos\varepsilon\cdot\tanh\gamma_{12}}{1+\cos\varepsilon\cdot\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\right]^2 + \left[\frac{\sin\varepsilon\cdot\tanh\gamma_{12}\cdot\mathrm{sech}\,\gamma_{23}}{1+\cos\varepsilon\cdot\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\right]^2. \qquad (125A)$$

From (125A), with $\tanh\gamma_{12} = v_{12}/c$, $\tanh\gamma_{23} = v_{23}/c$, $\tanh\gamma_{13} = v_{13}/c$ and after reducing the Poincaré–Einstein relativistic Law of two coordinate velocities summation follows, for example, in [53, p. 34]. Below it is given in the trigonometric *tangent form*

$$\tanh\gamma_{13} = \frac{\sqrt{\tanh^2\gamma_{12} + \tanh^2\gamma_{23} + 2\cos\varepsilon\cdot\tanh\gamma_{12}\cdot\tanh\gamma_{23} - \sin^2\varepsilon\cdot\tanh^2\gamma_{12}\cdot\tanh^2\gamma_{23}}}{1+\cos\varepsilon\cdot\tanh\gamma_{12}\cdot\tanh\gamma_{23}}. \qquad (126A)$$

One more scalar commutative variant of two velocities summation is expressed from (122A) in terms of *relativistic factors* [53, p. 35], or in the trigonometric *secant form*

$$\mathrm{sech}\gamma_{13} = \sqrt{1-\tanh^2\gamma_{13}} = \frac{\mathrm{sech}\gamma_{12}\cdot\mathrm{sech}\gamma_{23}}{1+\cos\varepsilon\cdot\tanh\gamma_{12}\cdot\tanh\gamma_{23}}. \qquad (127A)$$

Formulae (122A), (124A) and (125A) give the scalar interpretations for the sum of two hyperbolic segments if $n \geq 2$ in terms of different functions. The case of $\cos\varepsilon = 0$ corresponds to sum of orthogonal segments. The case of $\cos\varepsilon = \pm 1$ gives the additive rules (69A)-(72A). Formulae for cosecant and cotangent are obtained with inversions of sine and tangent. In (124A) and (125A) we see the sine and tangent *Big Pythagorean Theorems* in $\tilde{E}_1^{(3)}$, they will be discussed and interpreted visually later. If $\cos\varepsilon = 0$, then for two conventionally orthogonal hyperbolic segments we have the sine and tangent *Small Pythagorean Theorems.* In the base $\tilde{E}_1 = \{I\}$ there hold:

$$\cosh\gamma_{13} = \cosh\gamma_{12}\cdot\cosh\gamma_{23} \;\Leftrightarrow\; \mathrm{sech}\,\gamma_{13} = \mathrm{sech}\gamma_{12}\cdot\mathrm{sech}\gamma_{23}, \qquad (128A)$$

$$\sinh^2\gamma_{13} = \sinh^2\gamma_{12} + (\sinh\gamma_{23}\cdot\cosh\gamma_{12})^2 = \sinh^2\gamma_{23} + (\sinh\gamma_{12}\cdot\cosh\gamma_{23})^2. \quad (129A)$$

$$\tanh^2\gamma_{13} = \tanh^2\gamma_{12} + (\tanh\gamma_{23}\cdot\mathrm{sech}\gamma_{12})^2 = \tanh^2\gamma_{23} + (\tanh\gamma_{12}\cdot\mathrm{sech}\gamma_{23})^2. \quad (130A)$$

In 3-dimensional Euclidean space, not more than three vectors can be conventionally orthogonal. Perform sequentially two operations of three conventionally orthogonal segments summing, we obtain three-step scalar commutative trigonometric formulae:

$$\cosh\gamma_{14} = \cosh\gamma_{12}\cdot\cosh\gamma_{23}\cdot\cosh\gamma_{34} \Leftrightarrow \mathrm{sech}\gamma_{14} = \mathrm{sech}\gamma_{12}\cdot sech\gamma_{23}\cdot\mathrm{sech}\gamma_{34}. \quad (131A)$$

$$\sinh^2\gamma_{14} = \sinh^2\gamma_{12} + \sinh^2\gamma_{23} + \sinh^2\gamma_{34} +$$
$$+\sinh^2\gamma_{12}\cdot\sinh^2\gamma_{23} + \sinh^2\gamma_{12}\cdot\sinh^2\gamma_{34} + \sinh^2\gamma_{23}\cdot\sinh^2\gamma_{34} +$$
$$+\sinh^2\gamma_{12}\cdot\sinh^2\gamma_{23}\cdot\sinh^2\gamma_{34} =$$
$$= \sinh^2\gamma_{12} + (\sinh\gamma_{23}\cdot\cosh\gamma_{12})^2 + (\sinh\gamma_{34}\cdot\cosh\gamma_{12}\cdot\cosh\gamma_{23})^2. \quad (132A)$$

$$\tanh^2\gamma_{14} = \tanh^2\gamma_{12} + \tanh^2\gamma_{23} + \tanh^2\gamma_{34} -$$
$$-(\tanh^2\gamma_{12}\cdot\tanh^2\gamma_{23} + \tanh^2\gamma_{12}\cdot\tanh^2\gamma_{34} + \tanh^2\gamma_{23}\cdot\tanh^2\gamma_{34}) +$$
$$+\tanh^2\gamma_{12}\cdot\tanh^2\gamma_{23}\cdot\tanh^2\gamma_{34} =$$
$$= \tanh^2\gamma_{12} + (\tanh\gamma_{23}\cdot\mathrm{sech}\ \gamma_{12})^2 + (\tanh\gamma_{34}\cdot\mathrm{sech}\ \gamma_{12}\cdot\mathrm{sech}\ \gamma_{23})^2. \quad (133A)$$

Scalar formulae of types (132A)-(133A) for the trigonometric function of the summary angle in $\langle\mathcal{P}^{n+1}\rangle$ may be always represented in the quadric form as sum of n quadrates by $n!$ identical variants. (We give only one last example in (132A) and in (133A) in the direct order of the motions from the six variants.) If in these summation formulae at least one of the particular angles is infinite ($\gamma_{ij} = \infty$, $\tanh\gamma_{ij} = 1$ or $v_{ij} = c$), then the final angle is infinite too. This corresponds to Einstein's Velocity Postulate (15A).

As generalization of summation multiplicative cosine variants (128A) and (131A) for a lot of conventionally orthogonal hyperbolic segments $\gamma_{(k)}$ in the n-dimensional Lobachevsky–Bolyai space or on **a hyperboloid II** of two coupled sheets in $\langle\mathcal{P}^{n+1}\rangle$ (it is seeming as *two symmetric cups* – see at Figure 4, sect. 12.1), this simplest multiplicatively commutative scalar cosine formula is realized in the base $\tilde{E}_1 = \{I\}$ too:

$$\cosh\gamma = \prod_{k=1}^{t}\cosh\gamma_{(k)}, \quad \gamma = \mathrm{arcosh}\left(\prod_{k=1}^{t}\cosh\gamma_{(k)}\right); \quad \varepsilon_{(k)} = \pm\pi/2 \text{ (on } \overrightarrow{y} \text{ and } \overrightarrow{y}^{(k)}).$$

The final *scalar* distance γ does not depend on ordering conventionally orthogonal particular angles. Furthermore, if all t orthogonal segments are infinitesimal, i. e., $\gamma_{(k)} \to 0$, then the special *Infinitesimal Pythagorean Theorem* holds for such (now non-conventionally) spherically orthogonal infinitesimal hyperbolic segments with Lambert measure too, with the non-relativistic law of summation for such velocities. Indeed, take into account the decomposition $\boxed{\cosh\gamma \to 1 + \gamma^2/2 + \cdots}$, and realize it as substitution in the cosine formulae. Then we obtain for any $t \leq n$ of independent infinitesimal orthogonal partial segments or infinitesimal principal angles the result:

$$\lim_{\gamma_{(k)} \to 0}\gamma = \sqrt{\sum_{k=1}^{t}\gamma^2_{(k)}}, \qquad (\varepsilon_{ij} = \pm\pi/2) \text{ (on the axes } \overrightarrow{x} \text{ and orthogonal } \overrightarrow{x}^{(k)}).$$

Now, instead of infinitesimal angles, we introduce the n partial space-like mutually orthogonal linear differentials $d\gamma_{(k)}$, where $k = 1, 2, ..., n$, of the total one $d\gamma$ (applied also at the point M of the unity hyperboloid II). All they are situated in the tangent n-dimensional *Euclidean hyperspace* $\mathcal{E}^n_M$, by its topological nature (see above), and its slope is only in the external cavity of the isotropic cone. It can be seen from this formula that cosine mapping approximates well to the Euclidean metric of the hyperboloid II with $d\gamma \to 0$. Moreover, these linear differentials, applied at M, are situated between this hyperboloid and these cosine orthoprojections $\cosh d\gamma$, according to the differential formula $\boxed{\cosh d\gamma = 1 + (d\gamma)^2/2}$ with exactness up to 2-nd extent in $d\gamma$. (Note, that this hyperbolic cosine projection is space-like one unlike the hyperbolic time-like sine projection − see such sine-cosine decompositions of the 1-st differential $d\gamma$ on the hyperboloid II in (162A), (163A).) Further realize substitutions by the expressions in the multiplicative cosine formula and get the *orthogonal commutative decomposition* of $d\gamma$ into the n partial differentials $d\gamma_{(k)}$ as maps of geometric summation of hyperbolic angles increments on the unity hyperboloid II:

$$(d\gamma)^2 = \sum_{k=1}^{n} [d\gamma_{(k)}]^2, \qquad (\varepsilon_{ij} = \pm\pi/2).$$

The two-step Riemannian 1-st metrical normal form on a hyperboloid II is

$$(d\lambda_R/R)^2 = d\gamma_p^2 = d\gamma_i^2 + \sinh^2\gamma_i \; d\alpha^2 = \left(\overline{\overline{d\gamma_p}}\right)^2_P + \left(\overset{\perp}{d\gamma_p}\right)^2_E \quad (n \geq 2).$$

It is inferred in $\langle\mathcal{P}^{3+1}\rangle$ from (162A), (163A), generally from the *Absolute Pythagorean theorem* in Ch. 10A; the motion $d\gamma_i$ along the basic geodesic is accompanied by $d\vartheta_\alpha$. See in scalar forms in the end of Ch. 6A, in tensor and vector $4D$-forms in Ch. 10A.

A hyperboloid I of one sheet is seeming as an *hourglass* (see at Figure 4, sect. 12.1). By this reason, so, on its upper part, the tangent hyperplane to this n-dimensional hyperboloid of radius $R = 1$ at a point M is the *pseudo-Euclidean subspace* $\mathcal{P}^{(m+1)}_M$, where $m + 1 = n$, pseudonormal to its space-like radius OM in $\langle\mathcal{P}^{n+1}\rangle$. In the hyperplane, we have one time-like differential $d\gamma$ under inclination $\varphi_R(\gamma) > \pi/4$ as orthoprojection of the increment along the geodesic hyperbola in this angular sector, and the m partial space-like differentials $d\alpha_{(k)}$ (besides the direction of OM) under inclinations $\varphi_R(\gamma_{(k)}) < \pi/4$ (single at $n = 2$) as the orthoprojections of the increments along the m geodesic ellipsoidal arcs in this angular sector. They form the $m + 1$ pseudo-Euclidean quadric. These time-like and space-like differentials are divided by isotropic lines of the cone as a map of the horocyrcles arcs, when the angle is $\pi/4$. The two-step pseudo-Riemannian 1-st metrical normal form on a hyperboloid I is

$$\mp(d\lambda_R/R)^2 = \mp d\gamma_p^2 = d\gamma_i^2 - \cosh^2\gamma_i \; d\alpha^2 = \left(\overline{\overline{d\gamma_p}}\right)^2_P - \left(\overset{\perp}{d\gamma_p}\right)^2_E \quad (n \geq 2).$$

See in scalar forms in the end of Ch. 6A, in tensor and vector $4D$-forms in Ch. 10A.

The directional cosines of final rotation $roth\Gamma_{13} = \sqrt{S}$ in (114A), and those of the vectors $\mathbf{sinh}\gamma_{13}$, $\mathbf{tanh}\gamma_{13}$, and $\mathbf{v}_{13}$ in the Cartesian subbase $\tilde{E}_1^{(3)}$ we evaluate with the use of tensor trigonometry. Take advantage of their equality for matrices $roth\ \Gamma_{13}$ and $roth\ (2\Gamma_{13})$. (Recall, that we use the arithmetic, as trigonometric, square root $\sqrt{S}$, because in it the angle Γ_{13} is bisected, see in sect. 6.3!) Compute the three remained non-diagonal $(4,k)$-th elements of the 4-th row of the matrix $S = \{s_{ij}\}$. Thus we need to multiply the 4-th row of $B = \{b_{ij}\}$ and the k-th column of $roth\ \Gamma_{12}$, $k = 1, 2, 3$:

$$s_{4k} = s_{k4} = \sinh(2\gamma_{13}) \cdot \cos\sigma_k = 2\cosh\gamma_{13} \cdot \sinh\gamma_{13} \cdot \cos\sigma_k =$$
$$= 2\cosh\gamma_{13} \cdot [(\sinh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{23} \cdot \cosh\gamma_{12}) \cdot \cos\alpha_k + \sinh\gamma_{23} \cdot (cos\beta_k - \cos\varepsilon \cdot \cos\alpha_k)]. \quad (134A)$$

This allows to infer all *vectorial* trigonometric formulae for two-step motions in the hyperbolic non-Euclidean geometry. The vectorial formulae with their directional cosines illustrate also any isometric to them hyperbolic motions on the hyperboloid II. They depend on ordering of the two summands γ_{12} and γ_{23}. So, *vectorial sines* in these contrary variants of ordering two motions in the subbase $\tilde{E}_1^{(3)}$ are the following:

$$\left.\begin{array}{l}
\mathbf{sinh}\gamma_{13} = \sinh\gamma_{13} \cdot \mathbf{e}_\sigma = \\
= (\sinh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{23} \cdot \cosh\gamma_{12}) \cdot \mathbf{e}_\alpha + \\
+ \sin\varepsilon \cdot \sinh\gamma_{23} \cdot \mathbf{e}_\nu = \\
= [\sinh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{23} \cdot (\cosh\gamma_{12} - 1)] \cdot \mathbf{e}_\alpha + \\
+ \sinh\gamma_{23} \cdot \mathbf{e}_\beta; \\
\mathbf{sinh}\gamma_{13} = \sinh\gamma_{13} \cdot \mathbf{e}_{\underset{\sigma}{\angle}} = \\
= (\sinh\gamma_{23} \cdot \cosh\gamma_{12} + \cos\varepsilon \cdot \sinh\gamma_{12} \cdot \cosh\gamma_{23}) \cdot \mathbf{e}_\beta + \\
+ \sin\varepsilon \cdot \sinh\gamma_{12} \cdot \mathbf{e}_{\underset{\nu}{\angle}} = \\
= [\sinh\gamma_{23} \cdot \cosh\gamma_{12} + \cos\varepsilon \cdot \sinh\gamma_{12} \cdot (\cosh\gamma_{23} - 1)] \cdot \mathbf{e}_\beta + \\
+ \sinh\gamma_{12} \cdot \mathbf{e}_\alpha;
\end{array}\right\} \quad (135A)$$

$$\sinh\gamma_{13} \cdot \cos\sigma_k = (\sinh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{23} \cdot \cosh\gamma_{12}) \cdot \cos\alpha_k +$$
$$+ \sinh\gamma_{23} \cdot (\cos\beta_k - \cos\varepsilon \cdot \cos\alpha_k), \quad k = 1, 2, 3; \text{ (for direct order).}$$

Here $\mathbf{e}_\sigma = \{\cos\sigma_k\}$ is the *unity vector of directional cosines for the summary hyperbolic motion* γ_{13} in structure (363) with ordering γ_{12}, γ_{23}, and

$$\mathbf{e}_\nu = \left\{\frac{\cos\beta_k - \cos\varepsilon \cdot \cos\alpha_k}{\sin\varepsilon}\right\}_{k=1,2,3} = \frac{\mathbf{e}_\beta - \cos\varepsilon \cdot \mathbf{e}_\alpha}{\sin\varepsilon} = \frac{\overrightarrow{\mathbf{e}_\alpha \mathbf{e}'_\alpha} \cdot \mathbf{e}_\beta}{||\overrightarrow{\mathbf{e}_\alpha \mathbf{e}'_\alpha} \cdot \mathbf{e}_\beta||} \quad (136A)$$

is the *unity vector of conventionally orthogonal increment of full motion with respect to* $\mathbf{e}_\alpha$, i. e., *to the vector of the first motion as tangential one.*

The vector $\mathbf{e}_\nu$ (and $\mathbf{e}_{\underset{\nu}{\angle}}$ for inversely ordered summary motions at $\mathbf{e}_\alpha \leftrightarrow \mathbf{e}_\beta$) is used in biorthogonal decompositions of principal motion increment into tangential and normal parts, for physical velocities, inner accelerations, curvature etc.. It is evaluated from biorthogonal representation of the 2-nd vector in the sum:

$$\mathbf{e}_\beta = \cos\varepsilon \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \mathbf{e}_\nu, \quad \mathbf{e}'_\nu \cdot \mathbf{e}_\alpha = 0, \quad \mathbf{e}'_\nu \cdot \mathbf{e}_\beta = \sin\varepsilon \ \ (\varepsilon \in [0; \pi]). \quad (137A).$$

From vectorial formulae (135A) and scalar formula (122A) similar vector relations for tangents in ordering γ_{12}, γ_{23} (and vice versa for γ_{23}, γ_{12} – see in (135A)) are inferred:

$$\mathbf{tanh}\gamma_{13} = \tanh\gamma_{13}\cdot\mathbf{e}_\sigma = \frac{\mathbf{sinh}\ \gamma_{13}}{\cosh\gamma_{13}} = \tag{138A}$$

$$= \frac{\tanh\gamma_{12} + \cos\varepsilon\cdot\tanh\gamma_{23}}{1+\cos\varepsilon\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\cdot\mathbf{e}_\alpha + \frac{\sin\varepsilon\cdot\tanh\gamma_{23}\cdot\text{sech}\ \gamma_{12}}{1+\cos\varepsilon\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\cdot\mathbf{e}_\nu =$$

$$= \frac{\tanh\gamma_{12} + \cos\varepsilon\cdot\tanh\gamma_{23}\cdot(1-\text{sech}\ \gamma_{12})}{1+\cos\varepsilon\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\cdot\mathbf{e}_\alpha + \frac{\tanh\gamma_{23}\cdot\text{sech}\ \gamma_{12}}{1+\cos\varepsilon\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\cdot\mathbf{e}_\beta.$$

Sine and tangent formulae, in squared and vectorial variants (124A), (135A) and (125A), (138A), have in $\tilde{E}_1^{(3)}$ the following interpretation. The second segment γ_{23} on a hyperboloid II is decomposed into a pair of segments such that their projections into $\langle\mathcal{E}^3\rangle^{(1)}$ are directed along $\mathbf{e}_\alpha$ and $\mathbf{e}_\nu$. We obtain these big and small hyperbolic right triangles on a hyperboloid II $\boxed{\gamma_{13} = (\gamma_{12} + \overset{=}{\gamma}_{23}) \boxplus \overset{\perp}{\gamma}_{23}}$ and $\boxed{\gamma_{23} = \overset{=}{\gamma}_{23} \boxplus \overset{\perp}{\gamma}_{23}}$ with such spherically orthogonal sums and corresponding to them sine or tangent right triangles in the Euclidean subspace $\langle\mathcal{E}^3\rangle^{(1)}$. (*Segments γ are 4-dimensional, but their space projections are 3-dimensional*!) For beginning, perform the hyperbolic sine projecting γ_{13} and γ_{23} (in its spherically orthogonal decomposition) into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(1)}$. The result is these two orthogonalized projections of γ_{23} and γ_{13} into $\langle\mathcal{E}^3\rangle^{(1)}$:

$$\mathbf{sinh}\ \gamma_{23} = \overset{=}{\mathbf{sinh}}\ \gamma_{23} + \overset{\perp}{\mathbf{sinh}}\ \gamma_{23} \ \rightarrow\ \mathbf{sinh}\gamma_{13} = (\mathbf{sinh}\ \gamma_{12} + \overset{=}{\mathbf{sinh}}\ \gamma_{23}) + \overset{\perp}{\mathbf{sinh}}\ \gamma_{23}.$$

Both these relations are compatible. So, as results, we obtain the Big Pythagorean Theorem in its squared variant corresponding to (124A), and, as a consequence, the Small Pythagorean Theorem for the second segment in $\tilde{E}_1^{(3)}$, with trivial case (129A):

$$\boxed{\sinh^2\gamma_{13} = \sinh^2(\gamma_{12} + \overset{=}{\gamma}_{23}) + \sinh^2\overset{\perp}{\gamma}_{23}}\ ,\quad \boxed{\sinh^2\gamma_{23} = \sinh^2\overset{=}{\gamma}_{23} + \sinh^2\overset{\perp}{\gamma}_{23}}\ .$$

In these formulae, $\sinh\overset{=}{\gamma}_{13} = \cos\varepsilon\cdot\sinh\gamma_{13}$, $\sinh\overset{\perp}{\gamma}_{23} = \sinh\overset{\perp}{\gamma}_{13} = \sin\varepsilon\cdot\sinh\gamma_{13}$. Their *cosines*, are, due to (122A), the scalar projections into $\overrightarrow{ct}$ parallel to $\langle\mathcal{E}^3\rangle$.

Tangent formulae (125A) and (138A) are interpreted by analogous way, but with the use of tangent cross projecting. The angle γ_{23} is decomposed as before and then all these parallel and normal components are subjected to *cross projecting* (see in Ch. 4A) into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(2)}$. It should be taken into account by correction with additional coefficient sechγ_{12} (only by formal analogy with Lorentzian contraction). Their *tangent summation*, with these analogous Big and Small Pythagorean Theorems (125A) and (138A), are identical to *tangent model* – see further at Figure 4A. It is also equivalent to well-known *geometric summation* in Klein's homogeneous coordinates.

Furthermore, this important property of the summations into **sinh** γ_{13}, **tanh** γ_{13} unites to a certain extent the Euclidean geometry with non-Euclidean *hyperbolic* and *spherical* geometries! In Chapter 8A, analogous results will be represented in $\langle\mathcal{Q}^{2+1}\rangle$.

Distinction is the following. In Euclidean geometry the particular vectors $\mathbf{a}_{12} = a_{12}\cdot\mathbf{e}_\alpha$ and $\mathbf{a}_{23} = a_{23}\cdot\mathbf{e}_\beta$ are summarized commutatively, i. e., in their direct and inverse orders with the same result $\mathbf{a}_{13} = a_{13}\cdot\mathbf{e}_\sigma$. Geometrically, two variants of the biorthogonal non-Euclidean summation (direct and inverse) are noncommutative from the different sign of the angle of the secondary orthospherical rotation $(\mp\theta_{13})$ after summing. The Big Pythagorean Theorem is valid for two variants of their orthoprojections: onto $\mathbf{e}_\alpha$ and $\mathbf{e}_\nu$, as well as onto $\mathbf{e}_\beta$ and $\overset{\angle}{\mathbf{e}}_\nu$. Although, in both the cases, modules of hypotenuses are equal, but the directional summary vectors $\mathbf{a}_{13}$ are distinct by the orthospherical rotation as in (120A). Thus, formulae (124A), (135A) and (125A), (138A), can be represented in the two biorthogonal forms with decompositions either of projection γ_{12} with respect to $\mathbf{e}_\alpha$ or of projection γ_{23} with respect to $\mathbf{e}_\beta$. We obtain the following.
Theorem. *Sum of two motions can be represented in the Special biorthogonal form, commutative in Euclidean geometry and noncommutative in non-Euclidean geometry.* (All of they are quasi- and pseudo-Euclidean, spherical and hyperbolic non-Euclidean!) So, in pseudo-Euclidean and hyperbolic non-Euclidean geometries, the theorem is valid in $\langle\mathcal{E}^3\rangle$ from the center at the start point of summation, with correction of next vector. This geometric theorem allowed formally H. Poincaré (into three projections) and A. Einstein to infer the relativistic Law of summing two noncollinear velocities in vector and scalar forms without loss of generality under preliminary conditions: $\{\cos\alpha_1 = 1, \cos\alpha_2 = \cos\alpha_3 = 0\} \to \cos\varepsilon = \cos\beta_1$. Orthogonal projections of the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ along the axis x_1 as well as the axes x_2 and x_3 were considered independent and then were summed by these famous authors of STR just due to this Theorem.

In vector formula (138A), put $\tanh\gamma_{12}\cdot\cos\alpha_1 = \pm v/c \approx 10^{-4}$, $\cos\alpha_1 = \pm 1$, Then $\cos\varepsilon = \pm\cos\beta_1$, see (119A), and $\tanh\gamma_{23} = c/c = 1$, that is why $\tanh\gamma_{13} = 1$ too. Here $v \approx 30$ km/sec is the orbital velocity of the Earth moving around the Sun. Hence,

$$\mathbf{tanh}\ \gamma_{13} = \mathbf{e}_\sigma = \frac{[\tanh\gamma_{12} \pm \cos\beta_1\cdot(1 - \mathrm{sech}\ \gamma_{12})]\cdot\mathbf{e}_\alpha + \mathrm{sech}\ \gamma_{12}\cdot\mathbf{e}_\beta}{1 \pm \cos\beta_1\cdot\tanh\gamma_{12}} =$$

$$= \frac{1}{1 \pm \cos\beta_1\cdot\tanh\gamma_{12}}\cdot\begin{bmatrix} \pm\tanh\gamma_{12} + \cos\beta_1 \\ \mathrm{sech}\ \gamma_{12}\cdot\cos\beta_2 \\ \mathrm{sech}\ \gamma_{12}\cdot\cos\beta_3 \end{bmatrix} = \begin{bmatrix} \cos\sigma_1 \\ \cos\sigma_2 \\ \cos\sigma_3 \end{bmatrix}, \ (\tanh\gamma_{13} = 1),$$

where $\beta_1, \beta_2, \beta_3$ and $\sigma_1, \sigma_2, \sigma_3$ are the true and seemed angles, under which the Star is observed. From this, the complete list of *relativistic formulae for aberration* follows:

$$\tan\beta_1' = \frac{\cos\sigma_2}{\cos\sigma_1} = \frac{\mathrm{sech}\ \gamma_{12}\cdot\cos\beta_2}{\pm\tanh\gamma_{12} + \cos\beta_1}, \qquad \tan\beta_2' = \frac{\cos\sigma_3}{\cos\sigma_1} = \frac{\mathrm{sech}\ \gamma_{12}\cdot\cos\beta_3}{\pm\tanh\gamma_{12} + \cos\beta_1},$$

$$\cos\delta^\pm = (\mathbf{e}_\sigma^+)'\cdot\mathbf{e}_\sigma^- = \frac{\mathrm{sech}^2\gamma_{12} - \sin^2\beta_1\cdot\tanh^2\gamma_{12}}{1 - \cos^2\beta_1\cdot\tanh^2\gamma_{12}}, \quad R_a = \frac{\delta^\pm}{2} \ \ (\text{as how } \gamma_{12} \pm \gamma_{23}).$$

If the Star is observed in the simplest variant under $\beta_1 = \pi/2$, then for maximal δ^m:

$$\cos\delta^m = \mathrm{sech}^2\gamma_{12} - \tanh^2\gamma_{12} \equiv \cos 2\varphi(\gamma_{12}), \ \sin\delta^m = 2\tanh\gamma_{12}\cdot\mathrm{sech}\gamma_{12} \equiv \sin 2\varphi(\gamma_{12}).$$

If $\beta_3 = \pi/2$, then $\cos\beta_2 = \sin\beta_1$. In this special case, we obtain *trigonometric variant* of Einstein's formula for the orthogonally observed aberration [53, p. 36–39]:

$$\tan\beta_1' = \frac{\sin\beta_1 \cdot \text{sech}\ \gamma_{12}}{\cos\beta_1 \pm \tanh\gamma_{12}}$$

$$\left(\sin\beta_1' = \frac{\sin\beta_1 \cdot \text{sech}\ \gamma_{12}}{1 \pm \cos\beta_1 \cdot \tanh\gamma_{12}}, \qquad \cos\beta_1' = \frac{\cos\beta_1 \pm \tanh\gamma_{12}}{1 \pm \cos\beta_1 \cdot \tanh\gamma_{12}}\right).$$

For the orthogonally observed aberration, we have the simplest formulae:

$$\sigma_1 = \beta_1' = \pi/2 - \sigma_2, \quad \cos\sigma_2 = \sin\sigma_1 = \sin\beta_1' \quad \sigma_3 = \beta_3 = \pi/2.$$

Then either $\beta_1' < \beta_1$ (if the sign $+$ is chosen), or $\beta_1' > \beta_1$ (if the sign $-$ is chosen); and the angles β_1 and β_1' are permuted iff the signs $\pm$ and $\mp$ are permuted. All these formulae immediately follow from indicated above general formula for $\mathbf{tanh}\gamma_{13} = \mathbf{e}_\sigma$.

For J. Bradley formula (1727), A. Einstein introduced relativistic time-correcting factor sech γ_{12} (here it is in secant form (127A)) and used Lorentzian transformation instead of Galilean ones. The small correction makes the formula of aberration identical in two inertial frames of reference associated either with the Earth, or with the Star: $\mathbf{e}_\alpha$ and $\mathbf{e}_\sigma$ are permuted iff signs $\pm$ and $\mp$ are permuted. The maximal *angular radius of aberration* is achieved if $\beta_1 = \pi/2$, and it is $R_a = \delta^m/2 \approx 10^{-4}$ rad. Note, that the *angle of orthospherical rotation* $\vartheta^{(m)}$ will be calculated below. Some athors do not distinguish in aberration the angles $\delta^\pm$ for $\gamma_{12} \pm \gamma_{23}$ and ϑ for $\gamma_{12} + \gamma_{23}, \gamma_{23} + \gamma_{12}$ (?)

According to (135A) and (136A), the vectors $\mathbf{e}_\sigma$ and $\mathbf{e}_\eta$ are linear combinations of $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$. Hence all the four unit vectors are in the same Euclidean plane $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle$. Similar arguments for inverse ordering of motions give similar results, but *the first* directed vector is $\mathbf{e}_\beta$ and *the second* one is $\mathbf{e}_\alpha$. The new vector of orthogonal increment (for the inverse order of motions) is expressed from (136A) by permutation:

$$\mathbf{e}_{\underset{\nu}{\angle}} = \left\{\frac{\cos\alpha_k - \cos\varepsilon \cdot \cos\beta_k}{\sin\varepsilon}\right\} = \frac{\mathbf{e}_\alpha - \cos\varepsilon \cdot \mathbf{e}_\beta}{\sin\varepsilon}, \tag{139A}$$

$$\mathbf{e}_\alpha = \cos\varepsilon \cdot \mathbf{e}_\beta + \sin\varepsilon \cdot \mathbf{e}_{\underset{\nu}{\angle}}, \qquad \mathbf{e}_\beta' \cdot \mathbf{e}_{\underset{\nu}{\angle}} = 0, \quad \mathbf{e}_\alpha' \cdot \mathbf{e}_{\underset{\nu}{\angle}} = \sin\varepsilon, \quad \mathbf{e}_\nu' \cdot \mathbf{e}_{\underset{\nu}{\angle}} = \cos\varepsilon. \tag{140A}$$

The vectors $\mathbf{tanh}\ \overset{\angle}{\gamma}_{13}$, $\mathbf{sinh}\ \overset{\angle}{\gamma}_{13}$, and $\overset{\angle}{\mathbf{v}}_{13}$ are directed in the subbase $\tilde{E}_1^{(3)}$ along $\mathbf{e}_{\underset{\sigma}{\angle}}$, and their modules do not change. The vectors $\mathbf{e}_\sigma$, $\mathbf{e}_{\underset{\sigma}{\angle}}$, $\mathbf{e}_\nu$ and $\mathbf{e}_{\underset{\nu}{\angle}}$ are linear combinations of $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$, hence they lie in the same plane $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle$. The rotations (113A) and (112A) act in the common trigonometric plane of the matrix *rot* Θ_{13}, hence this plane is identical to $\langle\mathcal{E}^2\rangle$ too. The Euclidean plane includes all the six introduced and considered unity vectors of diagonal cosines: $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$, $\mathbf{e}_\sigma$, $\mathbf{e}_{\underset{\sigma}{\angle}}$, $\mathbf{e}_\nu$, $\mathbf{e}_{\underset{\nu}{\angle}}$.
(In general cases, for internal and external multiplications of unity vectors there holds:

$$\mathbf{e}_\mathbf{1}' \cdot \mathbf{e}_2 = \cos\varphi_{12}, \quad \mathbf{e}_\mathbf{1} \cdot \mathbf{e}_2' = \cos\varphi_{12} \cdot \overleftarrow{\mathbf{e}_1 \cdot \mathbf{e}_2'} = \sec\ \varphi_{12} \cdot \overleftarrow{\mathbf{e}_1 \cdot \mathbf{e}_1'} \cdot \overleftarrow{\mathbf{e}_\mathbf{2} \cdot \mathbf{e}_2'}.$$

They may be also useful. The last formulae are the special cases of (196) in Ch. 5.)

The matrix *rot* Θ_{13} can be calculated not only from multiplicative formula (115A). In $\langle \mathcal{P}^{3+1} \rangle$, it may be directly calculated in canonical form (497) due to (499). Indeed, the normal unity axis of *any* orthospherical rotation $\overrightarrow{\mathbf{e}_N}$ is found in terms of vectorial product for any two of all the six independent coplanar vectors as, for example, as:

$$\overrightarrow{\mathbf{r}_N}(\theta) = \mathbf{e}_{\underset{\sigma}{\angle}} \otimes \mathbf{e}_\sigma = \mp \sin\theta \cdot \overrightarrow{\mathbf{e}_N}, \quad \overrightarrow{\mathbf{r}_N}(\varepsilon) = \mathbf{e}_\alpha \otimes \mathbf{e}_\beta = \pm \sin\varepsilon \cdot \overrightarrow{\mathbf{e}_N} \text{ (for } n = 3). \quad (141A)$$

Rotation $\mp\theta$ acts in $\tilde{E}_{1h} = roth \overset{\angle}{\Gamma} \cdot \tilde{E}_1$ – see (111A), (at $n = 2$ in the plane $\langle \mathcal{E}^2 \rangle^{(1h)}$). Here the signs are opposite to each other according to (113A) and (119A)! These values of $\overrightarrow{\mathbf{r}_N}(\theta)$ and $\cos\theta$ give us the matrix *rot* Θ_{13} in canonical form (497) if $n = 3$. According to (499) and (120A), we have the additional variants *for shifting* θ_{13}:

$$\cos\theta_{13} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\underset{\sigma}{\angle}} = tr\ rot\ \Theta/2 - 1 = (tr[rot\ \Theta]_{3\times 3} - 1)/2, |\sin\theta_{13}| = |\overrightarrow{\mathbf{r}_N}(\theta_{13})|. \quad (142A)$$

Speaking strictly they must supplement the pure hyperbolic law of summing motions (velocities), in that number in the different trigonometric forms. So, the orthospherical rotation is the cause of non-commutativity of the law in vector and tensor sides.

Due to **General Signs Rule** (see in (113A) and in sect. 12.2) in hyperbolic geometry and STR, $\boxed{sgn\ \theta_{13} = -sgn\ \varepsilon\ !}$: if $\varepsilon > 0$, then $\theta_{13} < 0$, and if $\varepsilon < 0$, then $\theta_{13} > 0$, i. e., the leg 13 is shifted orthospherically towards the angle $A_{123} = \pi - \varepsilon$ always with decreasing the sum of angles in the hyperbolic triangle (see more further)!

The vectors $\mathbf{e}_{\underset{\sigma}{\angle}}, \mathbf{e}_\sigma, \overrightarrow{\mathbf{e}_N}$ as well as the vectors $\mathbf{e}_\alpha, \mathbf{e}_\beta, \overrightarrow{\mathbf{e}_N}$ form the right triple due to (113A), this corresponds to counting scalar angles as counter-clockwise ones in the *right-handed* bases, and the oriented vector $\overrightarrow{\mathbf{e}_N}$ determines the right screw of rotations. The triple $\mathbf{e}_{\underset{\sigma}{\angle}}, \mathbf{e}_\sigma, \overrightarrow{\mathbf{r}_N}(\theta)$ is universal for analysis of multistep motions.

All the six vectors $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$, $\mathbf{e}_\nu$, $\mathbf{e}_\sigma$, $\overset{\angle}{\mathbf{e}}_\nu$, $\overset{\angle}{\mathbf{e}}_\sigma$ are inside an angle of magnitude π in the plane $\langle \mathcal{E}^2 \rangle \equiv \langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle$. From (136A), (139A), taking into account (122A), we obtain

$$\mathbf{e}'_\nu \cdot \mathbf{e}_{\underset{\nu}{\angle}} = -\cos\varepsilon = +\cos(\pi - \varepsilon) = \cos A_{123},$$

$$\mathbf{e}_\alpha \cdot \mathbf{e}_{\underset{\nu}{\angle}} = \mathbf{e}_\beta \cdot \mathbf{e}_\nu = +\sin\varepsilon = +\sin(\pi - \varepsilon) = \sin A_{123}.$$

The value of $\cos\theta_{13}$ is computed with the use of (120A), and in addition vectorial variant of (135A) and its reverse analog! With respect to the original base $\tilde{E}_1$ we have

$$\cos\theta_{13} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\underset{\sigma}{\angle}} = \frac{A + \cos\varepsilon \cdot B + \cos^2\varepsilon \cdot C + \cos^3\varepsilon \cdot D}{\sinh^2\gamma_{13}} > 0; \quad (143A)$$

$$A = (\cosh\gamma_{12} \cdot \cosh\gamma_{23} - 1)(\cosh\gamma_{12} + \cosh\gamma_{23}) > 0,$$

$$B = \sinh\gamma_{12} \cdot \sinh\gamma_{23} \cdot (\cosh\gamma_{12} \cdot \cosh\gamma_{23} + \cosh\gamma_{12} + \cosh\gamma_{23} - 1) > 0,$$

$$C = \sinh^2\gamma_{12} \cdot \cosh\gamma_{23} \cdot (\cosh\gamma_{23} - 1) + \sinh^2\gamma_{23} \cdot \gamma_{12} \cdot (\cosh\gamma_{12} - 1) > 0,$$

$$D = \sinh\gamma_{12} \cdot \sinh\gamma_{23} \cdot (\cosh\gamma_{12} - 1) \cdot (\cosh\gamma_{23} - 1) > 0.$$

If $\cos\varepsilon = +1$, then $A + B + C + D = \sinh^2\gamma_{13} = \sinh^2(\gamma_{12} + \gamma_{23})$ with $\theta_{13} = 0$.
If $\cos\varepsilon = -1$, then $A - B + C - D = \sinh^2\gamma_{13} = \sinh^2(\gamma_{12} - \gamma_{23})$ with $\theta_{13} = 0$.
These two cases are trivial, they correspond to conventionally collinear two motions.

Theoretically maximal relativistic shift $\theta_{13} = \mp\pi/2$ takes place if conventionally orthogonal velocities are equal to the speed of light c ! Moreover, function (143A) in $\cos\varepsilon$ has three extrema: maximal value $\cos\theta_{13} = 1$ if $\cos\varepsilon = \pm 1$ and minimal value $\cos\theta_{13} = A/\sin^2\gamma_{13}$ if $\cos\varepsilon = 0$. The latter corresponds to conventionally orthogonal two motions with quadratic scalar formulae (128A)-(130A) for their summation in terms of different trigonometric functions. Below we consider in details the last variant. At first, transform scalar sine quadratic formula (129A) into the form:

$$\sinh^2\gamma_{13} = (\cosh\gamma_{12}\cdot\cosh\gamma_{23})^2 - 1 = (\cosh\gamma_{12}\cdot\cosh\gamma_{23} + 1)(\cosh\gamma_{12}\cdot\cosh\gamma_{23} - 1).$$

The *absolute value* of $\cos\theta_{13}$ is minimal iff $|\theta_{13}|$ is maximal, this is equivalent to conventional orthogonality of $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$. For the sum of two hyperbolic motions, provided that $\varepsilon = \pm\pi/2$ ($\sin\varepsilon = \pm 1$), from (143A) and (135A) we obtain:

$$\cos\theta_{13} = \frac{A}{\sinh^2\gamma_{13}} = \frac{\cosh\gamma_{12} + \cosh\gamma_{23}}{\cosh\gamma_{12}\cdot\cosh\gamma_{23} + 1} > 0,$$

$$\sin\theta_{13} = \frac{\mp\tanh\ \gamma_{12}\cdot\tanh\ \gamma_{23}}{1 + \mathrm{sech}\ \gamma_{12}\cdot\mathrm{sech}\ \gamma_{23}} = \frac{\mp\sinh\gamma_{12}\cdot\sinh\gamma_{23}}{\cosh\gamma_{12}\cdot\cosh\gamma_{23} + 1}.$$

The last sine formula was obtained by A. Sommerfeld (1931) [71] for summing two orthogonal velocities as if on a hypothetic *sphere of imaginary radius* with evaluation of the summation result by methods of hyperbolic geometry. This gave visual trigonometric interpretation of coefficient 1/2 in the Thomas precession [70] under condition that $\gamma_{ij} \to 0$ ($v_{ij} \to 0$) in this sine formula. In the scalar and vectorial *tangent* variants of summing two orthogonal segments, we obtain:

$$\tan\theta_{13} = \frac{\mp\sinh\ \gamma_{12}\cdot\sinh\ \gamma_{23}}{\cosh\gamma_{12} + \cosh\gamma_{23}}, \quad (\mathbf{e}'_\alpha\cdot\mathbf{e}_\beta = \mathbf{e}'_\beta\cdot\mathbf{e}_\alpha = \cos\varepsilon = 0, \sin\varepsilon = \pm 1).$$

$$\tanh\gamma_{13}\cdot\mathbf{e}_\sigma = \tanh\gamma_{12}\cdot\mathbf{e}_\alpha + \tanh\gamma_{23}\cdot\mathrm{sech}\ \gamma_{12}\cdot\mathbf{e}_\beta,$$

$$\tanh\gamma_{13}\cdot\mathbf{e}_{\underset{\sigma}{\measuredangle}} = \tanh\gamma_{23}\cdot\mathbf{e}_\beta + \tanh\gamma_{12}\cdot\mathrm{sech}\ \gamma_{23}\cdot\mathbf{e}_\alpha,$$

These three particular formulae for the angle of orthospherical shift θ in cosine, sine and tangent variants must also supplement the pure hyperbolic formulae for summing two conventionally orthogonal motions (velocities) in cosine (128A), sine (129A) and tangent (130A) variants with maximal orthospherical shifting for their complement! In general, this angle is concomitant for non-collinear two- and multistep principal motions in pseudo-Euclidean, quasi-Euclidean and non-Euclidean geometries. It has own real meaning, in that number, for applications in physics and mechanics.

If one of the velocities is $\pm c$, for example, $\tanh\gamma_{23} = \pm 1$, then $\cos\theta_{13} = \mathrm{sech}\gamma_{12}$, $\sin\theta_{13} = \mp\tanh\gamma_{12}$, $\mathbf{e}_\sigma = \pm\tanh\gamma_{12}\cdot\mathbf{e}_\alpha + \mathrm{sech}\ \gamma_{12}\cdot\mathbf{e}_\beta$, $(|\mathbf{e}_\sigma| = 1)$; $\mathbf{e}_{\underset{\sigma}{\measuredangle}} = \pm\mathbf{e}_\beta$.

Such case corresponds to the orthogonal variant of *aberration* with the pseudo-Euclidean right triangle of aberration on the hyperboloid II of radius "ic". First leg is the angle γ_{12} generated due to motion of the Earth relatively to the "immovable" Star. Second leg γ_{23} under the right angle ε (in its Euclidean orthoprojection) is generated due to motion of the light ray from the Star to the Earth. The hypotenuse is sum γ_{13} directed along $\mathbf{e}_\sigma$. The *angular defect of this geodesic right triangle* 123 (see further) is determined with the use of permutation of these two legs by the formula (142A):

$$\cos\theta_{13} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\measuredangle}{\sigma}} = 1 - \frac{(1 - \operatorname{sech}\gamma_{12}) \cdot \sin^2\beta_1}{1 \pm \cos\beta_1 \cdot \tanh\gamma_{12}}.$$

Besides, θ_{13} is here the *angular defect of a geodesic right triangle* 123 – see further. If ordering of the legs is inversed, then their hyperbolic sum is rotated at angle $-\theta_{13}$. Vector $\mathbf{e}_\alpha$ inverses direction each half a year, that is why $\cos\alpha_1 = \pm 1$ and then $\cos\varepsilon = \pm\cos\beta_1$. The angle θ_{13} is realized only in rotation of the plane $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{v},\mathbf{c}\rangle$.

* * *

From squared sine representations of summing two-step and three-step hyperbolic conventionally orthogonal motions, expressed by formulae (129A), (132A), we may get the pure hyperbolic and also the two-step and three-step 1-st metric Euclidean spherically normal forms at the hyperboloid II at n=2 and $n = 3$, with simultaneous determination of the orthospherical shift at such non-collinear differential increments. Construct the forms on two- and three-dimensional unity hyperboloids II embedded in pseudo-Euclidean spaces $\langle\mathcal{P}^{2+1}\rangle$ or $\langle\mathcal{P}^{3+1}\rangle$, trigonometric objects of STR.

Choose the initial pseudo-Cartesian base $\tilde{E}_1$ with origin C_{II} on the hyperboloid II (upper part) as reference point for hyperbolic angle γ.

• The 1-st case corresponds to the two-dimensional hyperboloid II. At the point M we have the values $\gamma = [\gamma_{12}]_M$ in $\tilde{E}_1$, $(d\gamma_{12})_M$ in $\tilde{E}_2$, and further orthogonal to the latter $(d\gamma_{23})_{M'}$. Applying (129A) in the vicinity of the initial point M, we obtain a quadric of the 1-st metrical form in the *normal semi-geodesic hyperbolic coordinates* with two independent orthogonal differentials:

$$(d\gamma)^2_M = (d\gamma_{12})^2_M + \cosh^2[\gamma_{12}]_M\ (d\gamma_{23})^2_{M'}\ (\gamma = \lambda/R),$$

where $[\gamma_{12}]_M$ is the length of the geodesic "OM" as meridian in $\tilde{E}_1$, $d\gamma_{12}$ and $d\gamma_{23}$ are the partial differentials along the geodesic γ_{12} in $\tilde{E}_2$. In STR, the multiplier $\cosh\gamma_{12}$ translates time from $\tilde{E}_2$ into $\tilde{E}_1$ (or from $\tilde{E}_m$ into $\tilde{E}_1$). As result, the orthospherical shift may be calculated how above.

• The 2-nd case corresponds to the three-dimensional hyperboloid II. Suppose that in (132A) $d\gamma_{12}$, $d\gamma_{23}$, $d\gamma_{34}$ are the independent 1-st differentials of γ_{12}, γ_{23}, γ_{34}. We have in these *normal semi-geodesic coordinates* in the vicinity of M the 1-st metrical form:

$$(d\gamma)^2_M = (d\gamma_{12})^2_M + \cosh^2[\gamma_{12}]_M\ (d\gamma_{23})^2_{M'} + \cosh^2[\gamma_{12}]_M \cdot \cosh^2[\gamma_{23}]_{M'}\ (d\gamma_{34})^2_{M''}.$$

This process can be continued in $\langle\mathcal{P}^{n+1}\rangle$ with the similar sums up to n quadrats.

Cosine formula (143A) can be used for other important evaluations. As before, in infinitesimal considerations we take advantage of the useful formula for the cosine of first angular differential (with exactness up to second power of the angular differential). $\boxed{\cosh d\gamma = 1 + (d\gamma)^2/2}$ and $\boxed{\cos d\theta = 1 - (d\theta)^2/2}$ in hyperbolic and spherical forms.

In (135A), with direct and inverse ordering, put $\gamma_{12} = \gamma,\ \gamma_{23} = d\gamma$. The latter is the differential of an arc γ under angle ε to the angle γ. With the use of two cosine formulae (see above), we obtain orthospherical shift as differential, with General Signs Rule from (113A) for hyperbolic geometry (at $n \le 3$) and STR. And further, with the use of formulae (498) and (499), and also this Signs Rule, we translate the scalar product into vectorial one in $\langle \mathcal{P}^{2+1} \rangle \equiv \langle \mathcal{E}^2 \rangle \boxtimes \overrightarrow{y}$, as the rotation of $\langle \mathcal{E}^2 \rangle$ around a third space-like orthogonal axis $\overrightarrow{\mathbf{e}_N}$, complementary till $\mathcal{E}^3$, with value of sign $\boxed{sgn\ \theta_{13} = -sgn\ \varepsilon}$:

$$\mathbf{e}_{\underset{\sigma}{\angle}} \times \mathbf{e}_\sigma = \mp d\theta \cdot \overrightarrow{\mathbf{e}_N} = \pm \frac{\sinh\gamma \cdot \mathbf{e}_\alpha}{\cosh\gamma + 1} \times d\gamma \cdot \mathbf{e}_\beta = \pm \sin\varepsilon \cdot \frac{\sinh\gamma}{\cosh\gamma + 1} \times d\gamma \cdot \overrightarrow{\mathbf{e}_N} =$$

$$= \pm \sin\varepsilon \cdot \frac{\tanh\gamma}{1 + \mathrm{sech}\gamma}\, d\gamma \cdot \overrightarrow{\mathbf{e}_N} = \pm \sin\varepsilon \cdot \tanh\frac{\gamma}{2} d\gamma \cdot \overrightarrow{\mathbf{e}_N} = \tanh\frac{\gamma}{2}\, \overset{\perp}{d\gamma} \cdot \overrightarrow{\mathbf{e}_N}. \quad (144A)$$

Here the angle γ or Γ is expressed in the original base $\tilde{E}_1$. In STR, it is the universal base with relatively immovable Observer N_1 in the space-time $\langle \mathcal{P}^{3+1} \rangle$; the differential $d\gamma \cdot \mathbf{e}_\beta$ is expressed in the base $\tilde{E}_m = roth\, \Gamma \cdot \tilde{E}_1$. In its tangent variant, we see again the correcting coefficient 1/2 (*gotten from experimental data*) in the Thomas precession. See in (172A) the kinematic and dynamic applications of vectorial formulae (144A). (Note, that, for two arcs at point M, the *single* normal $\overrightarrow{\mathbf{e}_N}$ exists only in $\langle \mathcal{P}^{3+1} \rangle$!)

The special case is orthogonal (*now nonconventionally*) summation of motions when their angles in their first differentials are *infinitesimally small.* Let, for example, in (128A) and (144A) the infinitesimal values of both hyperbolic angles. On the unity hyperboloid II in $\langle \mathcal{P}^{2+1} \rangle$, for the right triangle 123 at $\gamma_{12} \to 0, \gamma_{23} \to 0$, we obtain:

$$\gamma_{13} = \sqrt{\gamma_{12}^2 + \gamma_{23}^2}\ ; \quad \theta_{13} = \mp\ \frac{\gamma_{12} \cdot \gamma_{23}}{2} = \mp\ \frac{a_{12} \cdot a_{23}}{2R^2} = S_{123} \cdot K_G \to 0, (\varepsilon_{ij} = \pm\pi/2);$$

$$\delta_{123} = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) - 2\pi = [3\pi - (\pi - \delta_{123})] - 2\pi = S_{123} \cdot K_G \to 0.$$

These are the infinitesimal formulae of the plane Euclidean geometry. This confirms the *infinitesimally Euclidean metric* on Minkowskian hyperboloid II. In the case of $\langle \mathcal{P}^{3+1} \rangle$, from (144A) for a triangle 123, formed by $d\gamma_{12}$ and $d\gamma_{23}$ with their external angle ε, we infer the differential formulae for the vector-element of its area (see [16, p. 526]):

$$d\theta_{13} \cdot \overrightarrow{\mathbf{e}_N} = \mp \sin\varepsilon \cdot \frac{(d\gamma_{12}) \cdot (d\gamma_{23})}{2} \cdot \overrightarrow{\mathbf{e}_N} = \mp \sin\varepsilon \cdot \frac{(da_{12}) \cdot (da_{23})}{2R^2} \cdot \overrightarrow{\mathbf{e}_N} = \mp\ \frac{dS_{123}}{R^2} \cdot \overrightarrow{\mathbf{e}_N}.$$

Due to Signs Rule in the hyperbolic case: if $\varepsilon > 0$, then $\theta_{13} < 0$; if $\varepsilon < 0$, then $\theta_{13} > 0$. Thus we got the differential interdependent $d\theta_{13}$ and the area of the triangle dS_{123}!

However, due to the Lambert's hyperbolic result or, in general, to the Gauss–Bonnet Theorem [16, p. 533], the area of the geodesic triangle 123 (here on a surface of negative constant Gaussian curvature $K_G = -1/R^2 = \text{const}$) and the angular defect of the triangle $d\delta_{123} = \varepsilon_1+\varepsilon_2+\varepsilon_3-2\pi$ are connected as $d\delta_{123} = -dS_{123}/R^2 = K_G\, dS_{123} < 0$. As a result, we get the differential and integral formulae for connection of these angles

$$d\theta_{13} = \pm d\delta_{123} = \mp\,\frac{dS_{123}}{R^2} = \pm K_G\; dS_{123} \;\Rightarrow\; \theta_{13} = \pm\delta_{123} = \mp\,\frac{S_{123}}{R^2} = \pm K_G\cdot S_{123}$$

in geodesic triangles on the Minkowskian hyperboloid II, and in the homeomorphic to it hyperbolic non-Euclidean spaces (see in Ch. 12). The formulae mean: *the angle θ_{13} of orthospherical shifting and Lambert's angular defect δ_{123} in a hyperbolic triangle are equal*! The assertion is true also for other figures as polygons formed from triangles, this is inferred through their decomposition into triangles. (If such triangle is on a hyperspheroid in $\langle\mathcal{Q}^{n+1}\rangle$, the similar formula for orthospherical shifting θ contains the sign $\pm$, see in Ch. 8A.) (Note, that the orthospherical shifting is more general notion, than the angular deviation for geodesic two-dimensional figures, because the former takes place for non-geodesic curvilinear figures too.) Orthospherical tensor angle of rotation Θ_{13}, due to matrix formula (115A), is identical to tensor angular defect of a geodesic triangle on hyperboloid II. Angular deviations (scalar and tensor) take place due to dependence of parallel displacement on a surface with curvature on its way.

Conclusion. *Orthospherical shifting Θ gives the clear mathematical explanation to the Lambert's angular defect of figures in the hyperbolic geometry!*

In Ch. 5A we introduced through relation (79A) in the instantaneous subbase $\tilde{E}_m^{(3)}$ an *inner* 3-acceleration $\mathbf{g}$ directed along the axis $x^{(m)}$, but only for rectilinear physical movements (or collinear absolute motions). For non-collinear motions, in the complete pseudo-Cartesian base $\tilde{E}_m$, the inner acceleration formally is 4-vector, but such one with zero orthoprojection onto time-axis $\overrightarrow{ct}^{(m)}$ is 3-vector. Put in sine-cosine spherically orthogonal decompositions (135A) and (124A) the values $\gamma_{12} = 0, \gamma_{23} = d\gamma$ for the 1-st and 2-nd motions. Take into account (79A) and (137A), we obtain:

$$d\gamma\cdot\mathbf{e}_\beta = \cos\varepsilon\; d\gamma\cdot\mathbf{e}_\alpha + \sin\varepsilon\; d\gamma\cdot\mathbf{e}_\nu = \overline{\overline{d\gamma}}\cdot\mathbf{e}_\alpha + \overset{\perp}{d\gamma}\cdot\mathbf{e}_\nu \to (d\gamma)^2 = \left(\overline{\overline{d\gamma}}\right)_E^2 + \left(\overset{\perp}{d\gamma}\right)_E^2,$$

$$g_\beta\cdot\mathbf{e}_\beta = \cos\varepsilon\cdot g_\beta\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot g_\beta\cdot\mathbf{e}_\nu = \overline{\overline{g_\beta}}\cdot\mathbf{e}_\alpha + \overset{\perp}{g_\beta}\cdot\mathbf{e}_\nu \to g_\beta^2 = \left(\overline{\overline{g_\beta}}\right)_E^2 + \left(\overset{\perp}{g_\beta}\right)_E^2. \quad (145A)$$

This is $3D$ *Absolute Pythagorean theorems* for spherically orthogonal decompositions in the Cartesian base $\tilde{E}_m^{(2)}$ of current differential $d\gamma\cdot\mathbf{e}_\beta$ and inner acceleration $g_\beta\cdot\mathbf{e}_\beta$ with respect to the directional vector $\mathbf{e}_\alpha$ of hyperbolic angle of motion γ or velocity $\mathbf{v}$ at the point M in $\tilde{E}_m^{(2)}$, with the use (137A). Both these $3D$ Absolute Euclidean theorems in Euclidean plane are being generalized into $4D$ Absolute non-Euclidean Pythagorean theorems in the Absolute pseudo-Cartesian base $\tilde{E}_m$ of space-time $\langle\mathcal{P}^{3+1}\rangle$. See in details in Ch. 10A.

Relativistic formulae of the **Doppler effect** for the oscillations frequency of light [53, p. 39], from the trigonometric point of view, have simple hyperbolic interpretation. It is necessary in the classical formulae to change spherical tangent $\tan\varphi_R = v/c$ for hyperbolic one $\tanh\gamma = v/c$ as velocities tangent relation in STR, and to introduce the relativistic secant factor (127A) for the proper time either for the time of moving source of a light or for the time of moving Observer of a light source radiation. In STR only a relative velocity v has importance! With the tangent-tangent analogy, we obtain:

$$\nu^{(2)} \cdot c\tau = \nu^{(1)} \cdot \Delta ct^{(1)} = \nu^{(1)} \cdot ct^{(1)} \cdot (1 - \cos\alpha \cdot \tanh\gamma) \Rightarrow$$

$$\Rightarrow \quad \nu^{(1)} = \nu^{(2)} \cdot \mathrm{sech}\ \gamma \ /(1 - \cos\alpha \cdot \tanh\gamma),$$

where $\nu^{(2)}$ is the oscillations frequency of light in the source, $\nu^{(1)}$ is its frequency felt by the Observer N_1, α is the angle between a light ray and a velocity vector, $\mathrm{sech}\ \gamma$ is the relativistic factor, $t^{(1)}$ and τ are the equivalent time intervals in both these systems.

There are four special variants:

A. *Longitudinal meeting effect*: $\alpha = 0,\ \cos\alpha = +1$, i. e., the source becomes nearer. Then the "blue shift" of light frequency is observed.

B. *Longitudinal opposite effect*: $\alpha = \pi,\ \cos\alpha = -1$, i. e., the source becomes farer. Then the "red shift" of light frequency is observed.

C. *Transversal effect*: $\alpha = \pm\pi/2, \cos\alpha = 0$. Then Observer N_1 fixes the "red shift" too, but it is less than in case B due to Einsteinian time dilation in the moving source.

D. *The Doppler effect is absent* if $\cos\alpha = (1 - \mathrm{sech}\ \gamma)/\tanh(\pm\gamma) \approx \tanh(\pm\gamma)/2$.

And the **Hubble Law** [56, p. 25] can be expressed in the ancestral form through the relative change of the photons frequency as $-\Delta\nu/\nu = \tanh\gamma = v/c = Hl/c = Ht$.

* * *

Further consider tensor and vector trigonometry of the unity hyperboloids.

Hyperboloid II (see Figure 4) has $R = i$. R may be the 4-velocity $\overrightarrow{\mathbf{c}} = c \cdot \mathbf{i}$.

Consider the pseudounity 4×1-radius-vector of a point on a hyperboloid II in $\tilde{E}_1$:

$$\mathbf{i}_k = \begin{bmatrix} \mathbf{sinh}\ \gamma \\ \cosh\gamma \end{bmatrix} = \begin{bmatrix} \sinh\ \gamma \cdot \mathbf{e}_\alpha \\ \cosh\gamma \end{bmatrix} \qquad (\gamma > 0 \text{ if } \Delta ct > 0), \quad \mathbf{i}_1 = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}. \qquad (146A)$$

Its time-like trigonometric invariant is

$$\mathbf{i}'_k \cdot I^{\pm} \cdot \mathbf{i}_k = \mathbf{sinh}'\gamma_{1k} \cdot \mathbf{sinh}\ \gamma_{1k} - \cosh^2\gamma_{1k} =$$

$$= \sinh^2\gamma_{1k} \cdot \mathbf{e}'_\alpha \mathbf{e}_\alpha - \cosh^2\gamma_{1k} = -1 = i^2. \qquad (147A)$$

Here

$\mathbf{sinh}\ \gamma_{1k}$ is the 3×1-vector projection of $\mathbf{i}_k$ into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(1)}$,
$\cosh\gamma_{1k}$ is the scalar projection of $\mathbf{i}_k$ into $\overrightarrow{ct}^{(1)}$ parallel to $\langle\mathcal{E}^3\rangle^{(1)}$. In addition,
$\mathbf{tanh}\ \gamma_{1k}$ is the cross 3×1-vector projection of $\mathbf{i}_k$ into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(k)}$,
$\mathrm{sech}\ \gamma_{1k}$ is the cross scalar projection of $\mathbf{i}_k$ into $\overrightarrow{ct}^{(1)}$ parallel to $\langle\mathcal{E}^3\rangle^{(k)}$.

Consider pure hyperbolic motion $\mathbf{i}_2 \leftrightarrow \mathbf{i}_3$ of a point element on the unity hyperboloid II along two hyperbolae in $\tilde{E}_1$ and $\tilde{E}_2$, with its polar clear description:

$$\{roth\Gamma_{23}\cdot\}_{\tilde{E}_2} \cdot \overset{\mathbf{i}_2}{\begin{bmatrix} \sinh\ \gamma_{12}\cdot \mathbf{e}_\alpha \\ \cosh\gamma_{12} \end{bmatrix}} = \{roth\ \Gamma_{23}\}_{\tilde{E}_2} \cdot roth\ \Gamma_{12} \cdot \overset{\mathbf{i}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = \qquad (148A)$$

$$= \{roth\ \Gamma_{12}\cdot(roth\ \Gamma_{23})_{\tilde{E}_1}\cdot roth^{-1}\ \Gamma_{12}\}_{\tilde{E}_2}\cdot roth\ \Gamma_{12}\cdot \overset{\mathbf{i}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = roth\ \Gamma_{12}\cdot roth\ \Gamma_{23}\cdot \overset{\mathbf{i}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} =$$

$$= roth\ \Gamma_{13}\cdot rot\ \Theta_{13}\cdot \overset{\mathbf{i}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} \equiv\ roth\ \Gamma_{13}\cdot \overset{\mathbf{i}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = \overset{\mathbf{i}_3}{\begin{bmatrix} \sinh\ \gamma_{13}\cdot \mathbf{e}_\sigma \\ \cosh\gamma_{13} \end{bmatrix}}.$$

So, this means solution: *Exactly one geodesic line passes through two points 2 and 3.*

The trajectory of hyperbolic (geodesic) motion $\mathbf{i}_2 \to \mathbf{i}_3$ is in the cut of the hyperboloid by the pseudoplane of rotational matrix $\{roth\ \Gamma_{12}\cdot roth\ \Gamma_{23}\cdot roth^{-1}\Gamma_{12}\cdot\}_{\tilde{E}_2}$. A hyperbolic triangle on the hyperboloid is realized as a cycle of three geodesic motions:

$$roth\ \Gamma_{12}\ \mathbf{u}_1 = \mathbf{u}_2,\quad \{roth\ \Gamma_{23}\}_{\tilde{E}_2}\mathbf{u}_2 = \mathbf{u}_3,\quad \{roth\ \Gamma_{31}\}_{\tilde{E}_3}\mathbf{u}_3 = \mathbf{u}_1.$$

By (148A), for a point element $\mathbf{u}$, rotation Θ_{13} annihilates. The triangle cycle returns a nonpoint object into the start, but it or real body is turned in the base $\tilde{E}_3$ at angle Θ_{13}. Therefore, the point of application of this non-point object moves as $\mathbf{u}_1 \to \mathbf{u}_2 \to \mathbf{u}_3$ along hyperbolic geodesic lines γ_{12} and γ_{23}.

Summing two-step motion, due to polar decomposition *in the original base* $\tilde{E}_1$, is represented as the motion along geodesic line γ_{13} in direction $\mathbf{e}_\sigma$ and then the orthospherical rotation $rot\ \Theta$! See this in details above in (111A) and below in (154A).

* * *

Hyperboloid I (see Figure 4) has $R = \pm 1$. R may be the 4-antivelocity $\overrightarrow{\mathbf{c}^\star} = c\cdot\mathbf{j}$.

Consider the pseudounity 4×1-radius-vector of a point on a hyperboloid I in $\tilde{E}_1$:

$$\mathbf{j}_2 = \begin{bmatrix} \mathbf{cosh}\ \gamma \\ \sinh\gamma \end{bmatrix} = \begin{bmatrix} \cosh\ \gamma\cdot \mathbf{e}_\beta \\ \sinh\gamma \end{bmatrix} \quad (\gamma > 0 \text{ if } \Delta ct > 0),\quad \mathbf{j}_{1(\beta)} = \begin{bmatrix} \mathbf{e}_\beta \\ 0 \end{bmatrix}. \quad (149A)$$

If $\mathbf{e}_\beta = \mathbf{e}_\alpha$, then $\mathbf{i}$ and $\mathbf{j}$ on II and I are conjugate, Figure 4! Its space-like invariant is

$$\mathbf{j}'\cdot I^\pm\cdot\mathbf{j} = \mathbf{cosh}'\gamma\cdot\mathbf{cosh}\ \gamma - \sinh^2\gamma = \cosh^2\gamma\cdot\mathbf{e}'_\beta\mathbf{e}_\beta - \sinh^2\gamma = 1 = 1^2. \quad (150A)$$

Here

$\mathbf{cosh}\ \gamma_{1k}$ is the 3×1-vector projection of $\mathbf{j}_k$ into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(1)}$,

$\sinh\gamma_{1k}$ is the scalar projection of $\mathbf{j}_k$ into $\overrightarrow{ct}^{(1)}$ parallel to $\langle\mathcal{E}^3\rangle^{(1)}$. In addition,

$\mathbf{coth}\ \gamma_{1k}$ is the cross 3×1-vector projection of $\mathbf{j}_k$ into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(k)}$,

$\mathrm{csch}\ \gamma_{1k}$ is the cross scalar projection of $\mathbf{j}_k$ into $\overrightarrow{ct}^{(1)}$ parallel to $\langle\mathcal{E}^3\rangle^{(k)}$.

For hyperboloid I, pure hyperbolic motion $roth\ \Gamma$ is performed at γ with the directional cosine vector $\mathbf{e}'_\beta$, which is orthospherically shifted with respect to the original vector $\mathbf{e}_\beta$. The two geodesic motions of a point element $\mathbf{j}_2 \to \mathbf{j}_3$ on the unity hyperboloid I are realized with topological constraints corresponding to the cotangent plane model or more visually to the tangent cylindrical model outside the Cayley oval (sect. 12.1):

$$\{roth\ \Gamma_{23}\}_{\tilde{E}_2} \cdot \overset{\mathbf{j}_2}{\begin{bmatrix} \cosh\ \gamma_{12} \cdot \mathbf{e}_\beta \\ \sinh \gamma_{12} \end{bmatrix}} = \{roth\ \Gamma_{12} \cdot (roth\ \Gamma_{23})_{\tilde{E}_1} \cdot roth^{-1}\ \Gamma_{12}\}_{\tilde{E}_2} \cdot\ roth\ \Gamma_{12} \cdot \overset{\mathbf{j}_1}{\begin{bmatrix} \mathbf{e}_\beta \\ 0 \end{bmatrix}} = \qquad (151A)$$

$$= roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot \overset{\mathbf{j}_1}{\begin{bmatrix} \mathbf{e}_\beta \\ 0 \end{bmatrix}} = roth\ \Gamma_{13} \cdot rot\ \Theta_{13} \cdot \overset{\mathbf{j}_1}{\begin{bmatrix} \mathbf{e}_\beta \\ 0 \end{bmatrix}} = roth\ \Gamma_{13} \cdot \overset{\mathbf{j}'_1}{\begin{bmatrix} \mathbf{e}'_\beta \\ 0 \end{bmatrix}} = \overset{\mathbf{j}_3}{\begin{bmatrix} \cosh\ \gamma_{13} \cdot \mathbf{e}'_\sigma \\ \sinh \gamma_{13} \end{bmatrix}}.$$

Motion from $\mathbf{j}_2$ to $\mathbf{j}_3$ is possible iff plane cotangent or cylindrical tangent projections of $\mathbf{j}_2$ and $\mathbf{j}_3$ outside the oval absolute may be connected by a straight cotangent ($\mathbf{coth}\ \gamma_{23}$) or tangent ($\mathbf{tanh}\ \gamma_{23}$) segment without the topological obstacle.

Note, that points of Minkowskian hyperboloids II and I have the cotangent–cosecant pseudo-Euclidean invariants II and I concomitant to sine-cosine ones (sect. 6.4, 12.1):

$$\mathbf{i}' \cdot I^\pm \cdot \mathbf{i} = \mathbf{csch}'\gamma \cdot \mathbf{csch}\ \gamma - \coth^2 \gamma = -1 = i^2. \qquad (II)$$

$$\mathbf{j}' \cdot I^\pm \cdot \mathbf{j} = \mathbf{coth}'\gamma \cdot \mathbf{coth}\ \gamma - \mathrm{csch}^2\gamma = +1 = 1^2. \qquad (I)$$

The rotational cotangent-cosecant matrix function in Γ in canonical E-form corresponds to sine-cosine one in the complementary angle Υ as follows (sect. 6.5, 12.1):

$$\overline{roth}\ \Gamma \qquad = \qquad roth\ \Upsilon$$

$$\left| \begin{array}{c|c} \dfrac{\coth\gamma \cdot \overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha{}'} + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha{}'}}{\mathrm{csch}\ \gamma \cdot \mathbf{e}'_\alpha} & \dfrac{\mathrm{csch}\ \gamma \cdot \mathbf{e}_\alpha}{\coth\gamma} \end{array} \right| \begin{array}{c} \cdots \\ \cdots \end{array} \left| \begin{array}{c|c} \dfrac{\cosh\upsilon \cdot \overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha{}'} + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha{}'}}{\sinh\upsilon \cdot \mathbf{e}'_\alpha} & \dfrac{\sinh\upsilon \cdot \mathbf{e}_\alpha}{\cosh\upsilon} \end{array} \right|. \qquad (152A)$$

Recall, that due to the formulae of pseudo-Euclidean and hyperbolic non-Euclidean trigonometry, we have the correspondences for the complementary hyperbolic angles:

$$\sinh(\Gamma,\Upsilon) = \mathrm{csch}\ (\Upsilon,\Gamma) \ \Leftrightarrow\ \sinh(\Gamma,\Upsilon)\cdot\sinh(\Upsilon,\Gamma) = I,$$

$$\cosh(\Gamma,\Upsilon) = \coth\ (\pm\Upsilon,\Gamma) \ \Leftrightarrow\ \tanh(\pm\Gamma,\Upsilon) = \mathrm{sech}(\Upsilon,\Gamma),$$

$$\cosh^2(\Gamma,\Upsilon) - \sinh^2(\Gamma,\Upsilon) = I = \coth^2(\Upsilon,\Gamma) - \mathrm{csch}^2(\Upsilon,\Gamma) - \text{invariants for } \{I^\pm\}.$$

This determines strictly the geometric interdependence of these complementary angles shown at Figure 4 (Ch. 12), i. e., cotangent and cosecant cross projections of the angle Γ or Υ may be interpret as the usual orthoprojections of the angles Υ or Γ! Moreover, the rotational cotangent-cosecant matrix function in the angle Γ or Υ realizes rotation of linear objects (vectors, lineors etc.) at the complementary angle Υ or Γ!

In both the cases, for hyperboloids II and I, one may interpret hyperbolic angles and principal motions as their trigonometric projections by tangent and cotangent models either on the projective hyperplane or on the projective hypercylinder with respect to the trigonometric ball equivalent geometrically to the Cayley oval absolute.

Further, we describe in general form an algorithm for evaluating main characteristics of summary multistep motion in $\langle \mathcal{P}^{n+1} \rangle$ and $\langle \mathcal{P}^{3+1} \rangle \equiv \langle \mathcal{E}^3 \boxtimes \overrightarrow{ct} \rangle$ (see sect. 11.3, 11.4 and Ch. 7A) in the scalar, vectorial, and tensor forms. The algorithm starts with application of formula (485) for right transformation of the original unity base $\tilde{E}_1$. On the final step of the algorithm, the polar representation according to (474)–(476) and (111A)–(120A) is used. On these stages, the homogeneous modal transformations are

$$\tilde{E}_t = roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdots roth\ \Gamma_{(t-1),t} \cdot \tilde{E}_1 = T_{1t} \cdot \tilde{E}_1,$$

$$T_{1t} = roth\ \Gamma_{1t} \cdot rot\ \Theta_{1t} = rot\ \Theta_{1t} \cdot roth\ \overset{\angle}{\Gamma}_{1t}\ .$$

$$T_{1t} \cdot T'_{1t} = roth^2\ \Gamma_{1t} = roth\ 2\Gamma_{1t}, \qquad T'_{1t} \cdot T_{1t} = roth^2\ \overset{\angle}{\Gamma}_{1t} = roth\ 2\ \overset{\angle}{\Gamma}_{1t},$$

$$rot\ \Theta_{1t} = roth^{-1}\Gamma_{1t} \cdot T_{1t} = roth\ (-\Gamma_{1t}) \cdot T_{1t} = roth\ \Gamma_{t1} \cdot T_{1t}.$$

The latter gives $rot\ \Theta_{1t}$ as defect Θ_{1t} of the *Closed cycle of principal rotations!*. We use $\mathbf{e}_\sigma$ and $\mathbf{e}_{\overset{\angle}{\sigma}}$, they are the directional vectors in structures (362), (363) for Γ_{ij} and $\overset{\angle}{\Gamma}_{ij}$; $\cos\theta_{1t} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\angle}{\sigma}} = \dfrac{tr\ rot\ \Theta_{1t} - 2}{n-1}$ is the cosine form of orthospherical scalar shift θ in canonical structure (497). This formula is valid in $\langle \mathcal{P}^{n+1} \rangle$, see (497) and (120A).

The matrices $roth\ \Gamma_{1t}$ and $rot\ \Theta_{1t}$ are evaluated in canonical forms (363) and (497). Lorentzian contraction is evaluated with the use of the angle Γ_{1t} for summarized motion and the hyperbolic deformational matrix with canonical structures (364), (365), in particular, for objects of Ch. 4A. However, $\tanh\Gamma_{1t}$ (the velocity) and $\mathrm{sech}\ \Gamma_{1t}$ (the relativistic factor) may be computed directly from $\sinh\Gamma_{1t}$ and $\cosh\Gamma_{1t}$.

The canonical expression of Lorentzian homogeneous transformation is

$$T = roth\Gamma \cdot rot\ \Theta = rot\ \Theta \cdot roth\ \overset{\angle}{\Gamma} = \qquad (153A), (154A)$$

$$= \left[\begin{array}{c|c} \cosh\gamma \cdot \overleftarrow{\mathbf{e}_\sigma\mathbf{e}'_\sigma} + \overrightarrow{\mathbf{e}_\sigma\mathbf{e}'_\sigma} & \sinh\gamma \cdot \mathbf{e}_\sigma \\ \hline \sinh\gamma \cdot \mathbf{e}'_\sigma & \cosh\gamma \end{array}\right] \cdot \left[\begin{array}{c|c} [rot\ \Theta]_{3\times3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array}\right] =$$

$$= \left[\begin{array}{c|c} [rot\ \Theta]_{3\times3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} \cosh\gamma \cdot \overleftarrow{\mathbf{e}_{\overset{\angle}{\sigma}}\mathbf{e}'_{\overset{\angle}{\sigma}}} + \overrightarrow{\mathbf{e}_{\overset{\angle}{\sigma}}\mathbf{e}'_{\overset{\angle}{\sigma}}} & \sinh\gamma \cdot \mathbf{e}_{\overset{\angle}{\sigma}} \\ \hline \sinh\gamma \cdot \mathbf{e}'_{\overset{\angle}{\sigma}} & \cosh\gamma \end{array}\right] =$$

$$= \left[\begin{array}{c|c} (\cosh\gamma - 1) \cdot \mathbf{e}_\sigma\mathbf{e}'_{\overset{\angle}{\sigma}} + [rot\ \Theta]_{3\times3} & \sinh\gamma \cdot \mathbf{e}_\sigma \\ \hline \sinh\gamma \cdot \mathbf{e}'_{\overset{\angle}{\sigma}} & \cosh\gamma \end{array}\right].$$

Here $\mathbf{e}_\sigma\mathbf{e}'_{\overset{\angle}{\sigma}} = \cos\theta \cdot \overleftarrow{\mathbf{e}_\sigma\mathbf{e}'_{\overset{\angle}{\sigma}}}$. (If some $roth\ \Gamma_{ij}$ are collinear, they are grouped.)
Formula (154A) gives the **General Law of summing principal motions** in $\langle \mathcal{P}^{n+1} \rangle$, expressed in canonical form (364) or for $n = 3$ in (363) in the original base $\tilde{E}_1 = \{I\}$.

Now, with (154A), the readers may one time again be convinced in truety of all the formulae for summing *two-step motions* inferred at first by explicit multiplications.

The matrix $S = roth\,\Gamma$ is emanated, for example, from the last and lowest elements t_{44} and t_{k4} for general matrix T in (154A). They permit one to express the matrix S in the base $\tilde{E}_1$ in canonical forms (363), (364) in $\langle \mathcal{P}^{n+1} \rangle$ and evaluate scalar and vector trigonometric functions in the angle γ with its directional vector $\mathbf{e}_\sigma$ and the angle θ. The matrix $rot\,\Theta$ in $\langle \mathcal{P}^{3+1} \rangle$ is computed in canonical form (497) with the use of (499) for $\sin\theta_{13}$ with the sign of θ, and $\mathbf{e}_N$. For $n = 3$ and k=1, 2, 3 we obtain the following

$$\left.\begin{array}{l} \cosh\gamma = t_{44}, \sinh\gamma = +\sqrt{\cosh^2\gamma - 1},\ \tanh\gamma = v/c; \tanh\gamma_k = t_{k4}/t_{44}; \\ \cos\sigma_k = t_{k4}/\sinh\gamma,\ \cos\overset{\angle}{\sigma}_k = t_{4k}/\sinh\gamma,\ \mathbf{e}_\sigma = \{\cos\sigma_k\}, \mathbf{e}_{\overset{\angle}{\sigma}} = \{\cos\overset{\angle}{\sigma}_k\}. \\ \cos\theta_{13} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\angle}{\sigma}};\ \ \overrightarrow{\mathbf{r}_N}(\theta_{13}) = \mathbf{e}_{\overset{\angle}{\sigma}} \otimes \mathbf{e}_\sigma = \mp\sin\theta_{13} \cdot \overrightarrow{\mathbf{e}_N}\ \text{(last for } n = 3). \end{array}\right\} \quad (155A)$$

Scalar final results do not change under the mirror permutation of particular motions. It leads merely to substitution in (153A), (154A): $T \to T'$ with $\Theta \to -\Theta$, $\mathbf{e}_\sigma \to \mathbf{e}_{\overset{\angle}{\sigma}}$.

Corollary. *In general, multistep noncollinear hyperbolic motions $roth\,\Gamma_{ij}$ in $\langle \mathcal{P}^{n+1} \rangle$ or on the hyperboloids are represented as hyperbolic one and then orthospherical shift.* Such interpretation of law (154A) of summing hyperbolic motions in the base $\tilde{E}_1 = \{I\}$ is confirmed, with polar decomposition in the pseudo-Euclidean space, by the fact, that $rot\,\Theta$ is revealed in the hyperbolically shifting base $\tilde{E}_{1h} = roth\,\Gamma_{1t} \cdot \tilde{E}_1$, where Euclidean geometry is distorted in hyperbolic one! In the physical space-time $\langle \mathcal{E}^3 \boxtimes \overrightarrow{ct} \rangle$ it is confirmed experimentally by the Thomas precession of the electron spin in $\tilde{E}_{1h}^{(m)}$.

* * *

In the sequel, in accordance with our trigonometric approach, we shall use Cartesian tangent subbase $\tilde{E}_1(3)$ in $\tilde{E}_1 = \{I\}$ analogous to projective *homogeneous coordinates* in the Euclidean projective space $\langle\langle \mathcal{E}^3 \rangle\rangle$ (see in Ch. 12). Consider again the tangents (velocities) summation in scalar and vectorial trigonometric forms (138A) and (125A) inside the trigonometric ball analogous to the Cayley oval absolute with radii $R = 1$ for tangents and $R = c$ for velocities. Hyperbolic tangent models of principal motions are preferred, because they are limited by finite parameter 1 or R! This scale factor belongs to the tangent flat model of a hyperboloid II and to the tangent cylindrical model of a hyperboloid I. Indeed, there holds: $\tanh\gamma << \gamma < \sinh\gamma$. The cotangent model is infinite as well as sine one. We shall consider in details the tangent flat model of principal topologically unlimited motions on a hyperboloid II (Figure 4A). It is identical to the projective Klein's model of the real-valued hyperbolic space, see in sect. 12.1. Though the analogous tangent model of a hyperboloid I is realized on the cylindrical model with taking into account topological constraints!

We choose the origin O of this tangent subbase $\tilde{E}_1^{(3)}$ as the start point (1) of first tangent projection [12], the origin O' in the subbase is the following point (2) of second tangent projection [23], where both the projections are summarized, and so on up to the last summand. There is one to one correspondence between all these origins O in this limited tangent subbase $\tilde{E}_1^{(3)}$ and all these points k inside the Cayley oval.

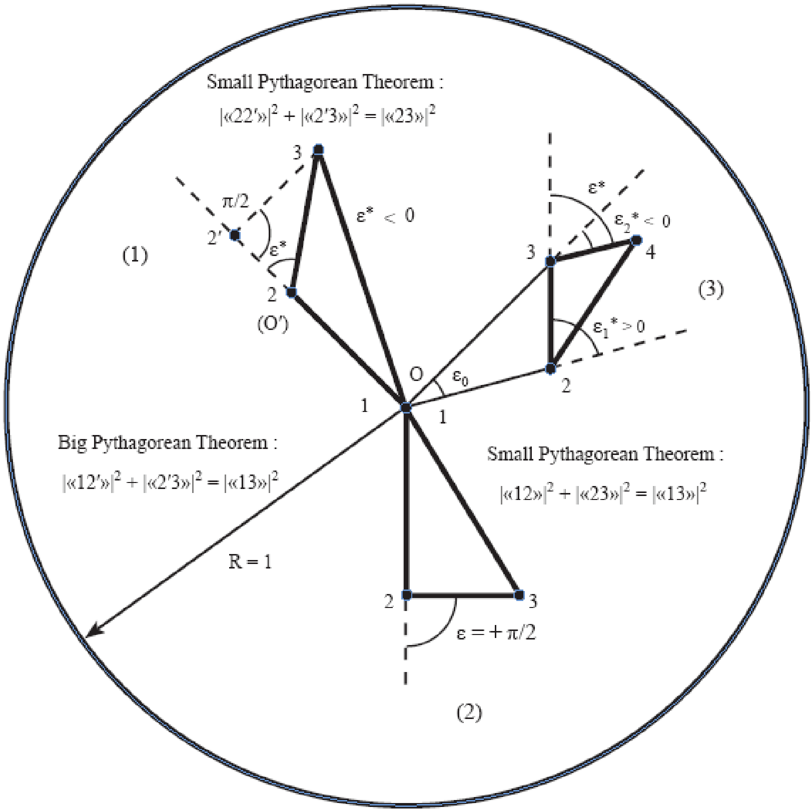

Figure 4A. Summing tangent projections of two hyperbolic motions in the tangent (Klein's) model according to the theorem on representation of their sum in biorthogonal Pythagorean form.

Variant 1. *Centered* triangle in $\tilde{E}_1^{(3)}$:

$$[12] = \mathbf{tanh}\,\gamma_{12}, \quad [23] = \mathbf{tanh}\,\gamma_{23}\cdot k_1^*\cdot k_2\cdot k_3^*, \quad [13] = \mathbf{tanh}\,\gamma_{13},$$

$$[22'] = \mathbf{tanh}\,\overline{\overline{\gamma}}_{23}, \quad [2'3] = \mathbf{tanh}\,\overset{\perp}{\gamma}_{23}, \quad \varepsilon^* = \pi - A_{123}^*, \quad A_{132}^* = \varepsilon^* - \varepsilon_0.$$

Variant 2. Centered right triangle in $\tilde{E}_1^{(3)}$:

$$[12] = \mathbf{tanh}\,\gamma_{12}, \quad [23] = \mathbf{tanh}\,\gamma_{23}\cdot \operatorname{sech}\gamma_{12}, \quad [13] = \mathbf{tanh}\,\gamma_{13}, \quad \varepsilon = A_{123} = \pi/2.$$

Variant 3. Decentered triangle coplanar with center O in $\tilde{E}_1^{(3)}$: $\varepsilon_0 = A_{213}$,

$$[23] = \mathbf{tanh}\,\gamma_{23}, \quad \varepsilon_1^* = \pi - A_{123}^*, \quad [34] = \mathbf{tanh}\,\gamma_{34}, \quad \varepsilon_2^* = \pi - A_{134}^*, \quad [24] = \mathbf{tanh}\,\gamma_{24},$$

$$\varepsilon^* = \pi - A_{234}^* = \varepsilon_1^* + \varepsilon_2^* - \varepsilon_0 = \pi - \{\pi - \varepsilon_2^* - [\pi - \varepsilon_0 - (\pi - \varepsilon_1^*)]\}.$$

The matrix of pure hyperbolic rotation in the base of its own determination $\tilde{E}_1$ can be considered as matrix-function *roth* $\Gamma_{12} = F(\gamma, \mathbf{e}_\alpha)$ due to its canonical form (363). Each such matrix with these two parameters γ and the vector of directional cosine $\mathbf{e}_\alpha$ implements motion of point (1) and determines any other point (k) inside the oval.

All centered tangent projections **tanh** γ_{12} are radiated from the point (1), i. e., center O of the tangent subbase $\tilde{E}_1^{(3)}$ (for example, along $\mathbf{e}_\alpha$). They are not distorted in Euclidean metric of $\langle\langle\mathcal{E}^3\rangle\rangle$, i. e., its Euclidean length in $\tilde{E}_1^{(3)}$ corresponds exactly to $\tanh\gamma_{12}$. Moreover, the central spherical angles ε_0 between $\tanh\gamma_{1i}$ and $\tanh\gamma_{1j}$ in the tangent model are not distorted too. We shall take advantage of these facts!

Following motion γ_{23} starts at point (2). If it is directed along $\mathbf{e}_\alpha$, then in $\langle\langle\mathcal{E}^3\rangle\rangle$ the second motion in its tangent projection $\{\tanh\gamma_{23}\}_{\tilde{E}_1}$ is expressed in the same tangent subbase $\tilde{E}_1^{(3)}$ with these three coefficients of distortions in Euclidean subspace $\langle\langle\mathcal{E}^3\rangle\rangle$:

$$k_1 = \frac{\{\tanh\gamma_{13}\}_{\tilde{E}_1}}{\{\tanh\gamma_{12}\}_{\tilde{E}_1} + \{\tanh\gamma_{23}\}_{\tilde{E}_2}} = 1/(1 + \tanh\gamma_{23}\cdot\tanh\gamma_{12}) < 1.$$

$$k_2\cdot k_3 = \frac{\{\tanh\gamma_{13}\}_{\tilde{E}_1} - \{\tanh\gamma_{12}\}_{\tilde{E}_1}}{\{\tanh\gamma_{23}\}_{\tilde{E}_2}} = \frac{\{\tanh\gamma_{23}\}_{\tilde{E}_1}}{\{\tanh\gamma_{23}\}_{\tilde{E}_2}} = \mathrm{sech}^2\gamma_{12} << 1,$$

where $k_2 = k_3 = \mathrm{sech}\ \gamma_{12}$. The first distortion is caused by *hyperbolic* summation of segments γ_{12} and γ_{23} as one for two *collinear segments.* The sequential distortion is combined from two factors. The first one $k_2 = \mathrm{sech}\ \gamma_{12}$ is Einsteinian dilation of time in the base $\tilde{E}_2$, the second one $k_3 = \mathrm{sech}\ \gamma_{12}$ is contraction of distance as result of *cross projecting at tangent mapping of distance* between two cross-bases (it is *only formally* analogous in result to Lorentzian contraction of extent, when a distance in $\langle\mathcal{E}^3\rangle^{(2)}$ is reduced in $\tilde{E}_1$ due to its cross projecting into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(2)}$, see in Ch. 4A).

In the triangle 123 (Figure 4A(1)), only the term [23] is distorted by k_2, k_3. Due to Pythagorean Theorem (138A) in the big right triangle $12'3$, its parallel projection [22'] is the difference of distorted parallel projection [12'] and undistorted term [12], i. e., [22'] is distorted by k_1^*, k_2, k_3; its normal projection [2'3] is distorted only by k_1^*, k_2:

$$\tanh\overline{\overline{\gamma}}_{23} = \frac{\cos\varepsilon\cdot\tanh\gamma_{23}\cdot\mathrm{sech}^2\gamma_{12}}{1+\cos\varepsilon\cdot\tanh\gamma_{23}\cdot\tanh\gamma_{12}} = \cos\varepsilon\cdot\tanh\gamma_{23}\cdot k_1^*\cdot k_2\cdot k_3. \qquad (156A)$$

$$\tanh\overset{\perp}{\gamma}_{23} = \frac{\sin\varepsilon\cdot\tanh\gamma_{23}\cdot\mathrm{sech}\ \gamma_{12}}{1+\cos\varepsilon\cdot\tanh\gamma_{23}\cdot\tanh\gamma_{12}} = \sin\varepsilon\cdot\tanh\gamma_{23}\cdot k_1^*\cdot k_2. \qquad (157A)$$

(Further distorting coefficients k_i^* depend on the angle ε.) The coefficient k_3 acts only for the parallel projection of **tanh** γ_{23} according to the Herglotz Principle. Due to the *Big Pythagorean Theorem* (125A), (138A) in the right triangle 12'3 in $\tilde{E}_1$, there hold

$$\tanh^2\gamma_{13} = \tanh^2[\gamma_{12} + \overline{\overline{\gamma}}_{23}] + \tanh^2\overset{\perp}{\gamma}_{23},$$

$$\cos\varepsilon_0 = \mathbf{e}'_\sigma\cdot\mathbf{e}_\alpha = \tanh\ [\gamma_{12} + \overline{\overline{\gamma}}_{23}]/\tanh\gamma_{13}, \quad \sin\varepsilon_0 = \mathbf{e}'_\sigma\cdot\mathbf{e}_\beta = \tanh\overset{\perp}{\gamma}_{23}\ /\tanh\gamma_{13}.$$

With squared (156A) and (157A), we obtain in $\tilde{E}_1^{(3)}$ the *Small Pythagorean Theorem* for the *right triangles* $22'3$ and 123 as (130A), due to variants (1) and (2) at Figure 4A:

$$\mathbf{tanh}\gamma_{23} = \mathbf{tanh}\gamma_{13} - \mathbf{tanh}\gamma_{12} \to \{\tanh\ \gamma_{23}\}_{\tilde{E}_1} = \{\tanh\ \gamma_{23}\}_{\tilde{E}_2} \cdot k_1^* \cdot k_2 \cdot k_3^* =$$

$$= \sqrt{\tanh^2 \overline{\overline{\gamma}}_{23} + \tanh^2 \overset{\perp}{\gamma}_{23}} = \tanh\gamma_{23} \cdot k_1^* \cdot \text{sech} \cdot \sqrt{\cos^2\varepsilon \cdot \text{sech}^2\ \gamma_{12} + \sin^2\varepsilon}.$$

(Compare k_2 and k_3^* with coefficients of Lorentzian contraction – collinear (53A) and non-collinear (54A) ones.) The Small Pythagorean Theorem gives the general variant at Figure 4A(1) and the simplest one at Figure 4A(2). For the sine and tangent orthogonal summation, both Small Pythagorean theorems were inferred in (129A), (130A). Note, that we can use geometrically the sine vectorial summation (without k_3) due to Pythagorean Theorem (124A), (135A). But sine projections are non-limited by R. However, in the spherical geometry (Ch. 8A) the sine projections are limited by R !

The decentered angles subject to distortions too. We consider distortion of the angle ε^* between $\mathbf{tanh}\ \gamma_{12}$ and $\mathbf{tanh}\ \gamma_{23}$ (Figure 4A(1)). Cross projecting transfers the origin of distorted vector 23 into O'. The distorted angle ε^* is expressed in terms of the distorted projection $\mathbf{tanh}\ \gamma_{23}$ due to formulae of Euclidean scalar trigonometry:

$$\cos\varepsilon^* = \frac{\tanh\overline{\overline{\gamma}}_{23}}{\{\tanh\ \gamma_{23}\}_{\tilde{E}_2}} = \frac{\cos\varepsilon \cdot \text{sech}\ \gamma_{12}}{\sqrt{\cos^2\varepsilon \cdot \text{sech}^2\ \gamma_{12} + \sin^2\varepsilon}} = \cos\varepsilon \cdot k_3/k_3^* < \cos\varepsilon. \qquad (158A)$$

In STR ε^* is a distorted spherical angle between velocities $\mathbf{v}_{12}$ and $\mathbf{v}_{23}$ in the space $\langle\langle\mathcal{E}^3\rangle\rangle$. If $\varepsilon = \pi/2$, there is no distortion: $\cos\varepsilon^* = \cos\varepsilon = 0$, see this variant at Figure 4A(2).

For coplanar decentered motions in the plane $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle)$ at Figure 4A(3), such angle ε^* is expressed in terms of distorted partial angles ε_1^*, ε_2^* and undistorted central angle ε_0 between $\mathbf{tanh}\ \gamma_{12}$ and $\mathbf{tanh}\ \gamma_{23}$. These *open angles* π are not distorted too, that follows from (158A). By theorems of Euclidean scalar trigonometry, there holds:

$$\varepsilon^* = \varepsilon_1^* + \varepsilon_2^* - \varepsilon_0 = \pi - A_{234}^* = \pi - \{\pi - \varepsilon_2^* - [\pi - \varepsilon_0 - (\pi - \varepsilon_1^*)]\}, \qquad (159A)$$

$$\cos\varepsilon_1^* = \frac{\cos\varepsilon_1 \cdot \text{sech}\ \gamma_{12}}{\sqrt{\cos^2\varepsilon_1 \cdot \text{sech}^2\ \gamma_{12} + \sin^2\varepsilon_1}}, \quad \sin\varepsilon_1^* = \frac{\sin\varepsilon_1}{\sqrt{\cos^2\varepsilon_1 \cdot \text{sech}^2\ \gamma_{12} + \sin^2\varepsilon_1}};$$

$$\cos\varepsilon_2^* = \frac{\cos\varepsilon_2 \cdot \text{sech}\ \gamma_{13}}{\sqrt{\cos^2\varepsilon_2 \cdot \text{sech}^2\ \gamma_{13} + \sin^2\varepsilon_2}}, \quad \sin\varepsilon_2^* = \frac{\sin\varepsilon_2}{\sqrt{\cos^2\varepsilon_2 \cdot \text{sech}^2\ \gamma_{13} + \sin^2\varepsilon_2}};$$

$$\left.\begin{array}{c} \cos\varepsilon^* = \cos[\varepsilon_1^* + \varepsilon_2^* - \varepsilon_0] = \\ = \dfrac{\begin{array}{l}[\cos\varepsilon_0 \cdot (\cos\varepsilon_1 \cdot \cos\varepsilon_2 \cdot \text{sech}\ \gamma_{12} \cdot \text{sech}\ \gamma_{13} - \sin\varepsilon_1 \cdot \sin\varepsilon_2) + \\ + \sin\varepsilon_0 \cdot (\sin\varepsilon_1 \cdot \cos\varepsilon_2 \cdot \text{sech}\ \gamma_{13} + \cos\varepsilon_1 \cdot \sin\varepsilon_2 \cdot \text{sech}\ \gamma_{12})]\end{array}}{\sqrt{(\cos^2\varepsilon_1 \cdot \text{sech}^2\ \gamma_{12} + \sin^2\varepsilon_1) \cdot (\cos^2\varepsilon_2 \cdot \text{sech}^2\ \gamma_{13} + \sin^2\varepsilon_2)}}. \end{array}\right\} \qquad (160A)$$

Summation of $\mathbf{tanh}\ \gamma_{23}$ and $\mathbf{tanh}\ \gamma_{34}$ is realized as $[12] + [23]^* = [13]$ under ε_1^* and $[13] + [34]^* = [14]$ under ε_2^*, see at Figure 4A(3). Further we have again variant 4A(1).

Generally, with *non-coplanar summands*, for example, $\tanh\gamma_{34} \notin \langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle$, for the summation in $\langle\mathcal{E}^3\rangle$, (159A) do not hold. We choose $\tanh\gamma_{13}\cdot\mathbf{e}_{\sigma(13)} = \tanh\gamma_{13}\cdot\mathbf{e}_\sigma$ due to (138A) as the first segment and $\tanh\gamma_{34} \cdot \mathbf{e}_{\beta(34)}$ as the *third* segment. Further, we use (156A)-(160A) for this two-step motion in $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_{\sigma(13)}, \mathbf{e}_{\beta(34)}\rangle$, etc.!

* * *

Kinematics of a material body progressive movement is determined by kinematics of the material point M, which is the barycenter of homogeneous body. For the point M, distinction between non-relativistic and relativistic kinematics can be seen in projective representations of the point movement in the universal base $\tilde{E}_1 = \{I\}$ as original one. (For the current coordinate of the proper distance along the movement, we use in $\tilde{E}_1$ the greek notation $\chi = x^{(1)}$, introduced in (73A), by analogy with the proper time!) In Lagrange space-time $\langle \mathcal{L}^{3+1} \rangle \equiv \langle \mathcal{E}^3 \oplus \overrightarrow{t} \rangle$, the increment and differentials of movement, with decomposition (137A), along a world line of point M do change as follows:

$$\Delta \mathbf{x}^{(1)} = d\mathbf{x}^{(1)} + d^2\mathbf{x}^{(1)}/2! + \cdots = dx^{(1)} \cdot \mathbf{e}_\alpha + d^2x^{(1)} \cdot \mathbf{e}_\beta/2! + \cdots, \ d\mathbf{x}^{(1)} = d\chi \cdot \mathbf{e}_\alpha,$$

$$d^2\mathbf{x}^{(1)} = d^2\chi \cdot \mathbf{e}_\beta = d^2\chi \cdot (\cos\varepsilon \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \mathbf{e}_\nu) = \overline{\overline{d^2\chi}} \cdot \mathbf{e}_\alpha + \overset{\perp}{d^2\chi} \cdot \mathbf{e}_\nu \equiv$$

$$\equiv d(d\chi \cdot \mathbf{e}_\alpha) = [\partial d\chi]_\alpha \cdot \mathbf{e}_\alpha + d\chi[\partial \mathbf{e}_\alpha]_{dx} = [\partial d\chi]_\alpha \cdot \mathbf{e}_\alpha + d\chi \left\{ ||\partial \mathbf{e}_\alpha|| \cdot \frac{\partial \mathbf{e}_\alpha}{||\partial \mathbf{e}_\alpha||} \right\}_{dx} =$$

$= [\partial d\chi]_\alpha \cdot \mathbf{e}_\alpha + d\chi \cdot [\partial\alpha]_{dx} \cdot \mathbf{e}_\nu$. We used for $\mathbf{e}_\beta$ decomposition (137A). That is why

$$[\partial d\chi]_\alpha = \cos\varepsilon \cdot d^2\chi = \overline{\overline{d^2\chi}}, \qquad d\chi \cdot [\partial\alpha]_{dx} = \sin\varepsilon \cdot d^2\chi = \overset{\perp}{d^2\chi};$$

$$\mathbf{v}(t) = \frac{d\mathbf{x}^{(1)}}{dt} = v_0 \cdot \mathbf{e}_\alpha(t_0) + \int_{t_0}^{t} \mathbf{g}(t)dt;$$

$$\mathbf{g}(t) = \frac{d^2\mathbf{x}^{(1)}}{dt^2} = g(t) \cdot \mathbf{e}_\beta(t) = \frac{\overline{\overline{d^2\chi}}}{dt^2} \cdot \mathbf{e}_\alpha(t) + \frac{\overset{\perp}{d^2\chi}}{dt^2} \cdot \mathbf{e}_\nu(t) = \overline{\overline{g}}(t) \cdot \mathbf{e}_\alpha(t) + \overset{\perp}{g}(t) \cdot \mathbf{e}_\nu(t),$$

$$\overline{\overline{g}}(t) = \cos\varepsilon(t) \cdot g(t) = \left[\frac{\partial d\chi}{dt^2}\right]_\alpha, \ \overset{\perp}{g}(t) = \sin\varepsilon(t) \cdot g(t) = \frac{d\chi}{dt} \cdot \left[\frac{\partial\alpha}{dt}\right]_{dx} = v(t) \cdot w(t), \text{etc.}$$

Orthospherical rotation $w_\alpha(t)$ does not change the progressive nature of the movement. The Cosine Law of Energy Conservation holds as $[\cos\varepsilon F](t)d\chi(t) = d[mv^2/2](t)$. The Sine Law of Momentum Conservation holds as $[\sin\varepsilon F \cdot \mathbf{e}_\nu](t)dt = d[mv \cdot \mathbf{e}_\alpha](t)$.

* * *

In Minkowski space-time $\langle \mathcal{P}^{3+1} \rangle \equiv \langle \mathcal{E}^3 \boxtimes \overrightarrow{ct} \rangle$, with (80A), (137A), (145A), there hold:

In $\tilde{E}_1$: $\quad \Delta \mathbf{x}^{(1)} \neq d\mathbf{x}^{(1)} + d^2\mathbf{x}^{(m)}/2! + \cdots, \quad d\mathbf{x}^{(1)} = dx \cdot \mathbf{e}_\alpha = d\chi \cdot \mathbf{e}_\alpha$;

$$\left.\begin{aligned} &\text{In } \tilde{E}_m: \quad d^2\mathbf{x}^{(m)} = d^2x^{(m)} \cdot \mathbf{e}_\beta = d\gamma^{(m)} \cdot d(c\tau) \cdot \mathbf{e}_\beta = \\ &= d^2x^{(m)} \cdot (\cos\varepsilon \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \mathbf{e}_\nu) = \overline{\overline{d^2x^{(m)}}} \cdot \mathbf{e}_\alpha + \overset{\perp}{d^2x^{(m)}} \cdot \mathbf{e}_\nu \Leftrightarrow \\ &\Leftrightarrow d\gamma = d\gamma \cdot \mathbf{e}_\beta = \cos\varepsilon \ d\gamma \ \mathbf{e}_\alpha + \sin\varepsilon \ d\gamma \cdot \mathbf{e}_\nu) = \overline{\overline{d\gamma}} \cdot \mathbf{e}_\alpha + \overset{\perp}{d\gamma} \cdot \mathbf{e}_\nu; \end{aligned}\right\} \quad (161A)$$

In $\langle \mathcal{P}^{3+1} \rangle$, the differentials $d\mathbf{x}^{(1)}$ and $d^2\mathbf{x}^{(m)}$ are not summed immediately unlike the case of $\langle \mathcal{L}^{3+1} \rangle$, as they are situated in different subspaces $\langle \mathcal{E}^3 \rangle$ and thus should be summed hyperbolically with the use of motion angle γ and its differentials $d\gamma^{(m)} = d\gamma$.

For integral non-collinear motions of a point object, the angle $\gamma = \gamma^{(1)}$ (a scalar) and these 3×1-vectors of directional cosines vary continuously in these bases. Parameters of motion are set in $\tilde{E}_1$, but the original angular motion differential $d\gamma$ is represented in the instantaneous base $\tilde{E}_m$. From (161A), we have $\frac{d^2x^{(m)}}{d\tau^2} = \frac{dv^{(m)}}{d\tau} = c \cdot \frac{d\gamma}{d\tau} = g(\tau)$ as the inner 3-acceleration in $\tilde{E}_m$ along the current axis $x^{(m)}$ without its hyperbolic time projection. However, in $\tilde{E}_m$ we can have preliminary its spherically orthogonal projections in the Cartesian subbase $\tilde{E}^3_m$, the directional cosine vectors $\mathbf{e}_\beta$, $\mathbf{e}_\alpha$ and $\mathbf{e}_\nu$ are conventionally expressed in $\tilde{E}_m$ and $\tilde{E}_1$ without their changing. There hold:

$c\frac{d\gamma}{d\tau} = \frac{d^2x^{(m)}}{d\tau^2} = \frac{dv^{(m)}}{d\tau} = g(\tau) = F/m_0$ is the *inner 3-acceleration* with $\mathbf{e}_\beta$ in $\tilde{E}^3_m$,

$c\frac{\overline{\overline{d\gamma}}}{d\tau} = \frac{\overline{\overline{dv^{(m)}}}}{d\tau} = \overline{\overline{g^*_c}}(\tau) = \cos\varepsilon \cdot g(\tau) = \overline{\overline{F}}/m_0$ is the *parallel proper 3-acceleration* with $\mathbf{e}_\alpha$ in $\tilde{E}^3_m$,

$c\frac{\overset{\perp}{d\gamma}}{d\tau} = \frac{\overset{\perp}{dv^{(m)}}}{d\tau} = \overset{\perp}{g}(\tau) = \sin\varepsilon \cdot g(\tau) = \overset{\perp}{F}/m_0$ is the *normal proper 3-acceleration* with $\mathbf{e}_\nu$ in $\tilde{E}^3_m$ and $\tilde{E}^3_1$, according to the Herglotz Principle – see below.

They satisfy $3D$ Pythagorean Theorems (145A) in the instantaneous base $\tilde{E}_m$. From the physical point of view, Pythagorean theorems of such type relate to 3-accelerations. But sine and tangent Big and Small Pythagorean Theorems relate to velocities.

By (119A), we have $\cos\varepsilon = \mathbf{e}'_\beta\mathbf{e}_\alpha = \mathbf{e}'_\alpha\mathbf{e}_\beta$, $0 \le \varepsilon \le \pi$. The values ε in $[0; \pi/2)$ correspond to accelerations, but ε in $(\pi/2; \pi]$ correspond to decelerations. Further, evaluate on the hyperboloid II variations of main trigonometric functions with velocities, accelerations and projections into $\overrightarrow{\mathcal{Y}}$ and $\langle \mathcal{E}^3 \rangle$ – strictly and in details see in Ch. 10A.

$$\cosh\gamma = \frac{d(ct)}{d(c\tau)} = \frac{dt}{d\tau}; \quad \rightarrow \quad \mathbf{sinh}\ \gamma = \cosh\gamma \cdot \mathbf{tanh}\ \gamma, \quad \mathbf{tanh}\ \gamma = \mathbf{sinh}\ \gamma / \cosh\gamma,$$

$$\mathbf{sinh}\ \gamma = \frac{d\mathbf{x}^{(1)}}{d(c\tau)} = \frac{d\chi}{d(c\tau)} \cdot \mathbf{e}_\alpha = \sinh\gamma \cdot \mathbf{e}_\alpha, \quad \mathbf{tanh}\ \gamma = \frac{d\mathbf{x}}{d(ct)} = \frac{d\chi}{d(ct)} \cdot \mathbf{e}_\alpha = \tanh\gamma \cdot \mathbf{e}_\alpha.$$

From (122A), (135A), (138A) at $\gamma_{12} = \gamma$, $\gamma_{23} = d\gamma$ we get in $\tilde{E}_1$ all their differentials.

$$\left.\begin{aligned} &d\cosh\gamma = \sinh\gamma\ d\gamma = \cos\varepsilon \cdot \sinh\gamma'\ d\gamma' = \sinh\gamma'\ \overline{\overline{d\gamma'}} = d\frac{d(ct)}{d(c\tau)} = d\frac{dt}{d\tau}, \\ &\cosh\gamma = \cosh\gamma_0 + \textstyle\int_{\gamma_0}^{\gamma} \sinh\gamma\ d\gamma = \frac{d(ct)}{d(c\tau)} = \frac{dt}{d\tau}. \end{aligned}\right\} \quad (162A)$$

For $\mathbf{sinh}\ \gamma = \mathbf{v}^*/c$ from *orthogonal* (135A) and with (137A) at differentiation in $\tilde{E}^3_1$:

$$\left.\begin{aligned} &d\mathbf{sinh}\ \gamma = d(\sinh\gamma \cdot \mathbf{e}_\alpha) = \cosh\gamma\ d\gamma \cdot \mathbf{e}_\alpha + \sinh\gamma\ d\alpha \cdot \mathbf{e}_\nu = \\ &= \cosh\gamma'\ d\gamma' \cdot \mathbf{e}_\sigma = \cosh\gamma' \cdot [\cos\varepsilon\ d\gamma' \cdot \mathbf{e}_\alpha + \sin\varepsilon\ d\gamma' \cdot \mathbf{e}_\nu], \\ \\ &|d\mathbf{sinh}\ \gamma|^2 = \cosh^2\gamma\ d\gamma^2 + \sinh^2\gamma\ d\alpha^2 = (\cosh\gamma'\ d\gamma')^2 = \\ &= \cosh^2\gamma' \cdot [(\cos\varepsilon\ d\gamma')^2 + (\sin\varepsilon\ d\gamma')^2] = \cosh^2\gamma'\ [(\overline{\overline{d\gamma'}})^2 + (\overset{\perp}{d\gamma'})^2], \\ \\ &\mathbf{sinh}\ \gamma = \sinh\gamma \cdot \mathbf{e}_\alpha(\gamma) = \\ &= \sinh\gamma_0 \cdot \mathbf{e}_{\alpha(0)} + \textstyle\int_{\gamma_0}^{\gamma} [\cosh\gamma\ d\gamma \cdot \mathbf{e}_\alpha + \sinh\gamma\ d\alpha \cdot \mathbf{e}_\nu]. \end{aligned}\right\} \quad (163A)$$

For $\mathbf{tanh}\ \gamma = \mathbf{v}/c$ from *orthogonal* (138A) and with (137A) at differentiation in $\tilde{E}_1^3$:

$$\left.\begin{array}{l} d\mathbf{tanh}\ \gamma = d(\tanh\gamma\cdot\mathbf{e}_\alpha) = \mathrm{sech}^2\gamma\ d\gamma\cdot\mathbf{e}_\alpha + \tanh\gamma\ d\alpha\cdot\mathbf{e}_\nu = \\ = \mathrm{sech}^2\gamma'\ d\gamma'\cdot\mathbf{e}_\sigma = \mathrm{sech}^2\gamma'\cdot[\cos\varepsilon\ d\gamma'\cdot\mathbf{e}_\alpha + \sin\varepsilon\ d\gamma'\cdot\mathbf{e}_\nu], \\ \\ |d\mathbf{tanh}\ \gamma|^2 = \mathrm{sech}^4\ d\gamma^2 + \tanh^2\gamma\ d\alpha^2 = (\mathrm{sech}^2\gamma'\ d\gamma')^2 = \\ = \mathrm{sech}^4\gamma'\cdot[(\cos\varepsilon\ d\gamma')^2 + (\sin\varepsilon\ d\gamma')^2] = \mathrm{sech}^4\gamma'\cdot[(\overline{\overline{d\gamma'}})^2 + (\overset{\perp}{d}\gamma')^2]; \\ \\ \mathbf{tanh}\ \gamma = \tanh\gamma\cdot\mathbf{e}_\alpha(\gamma) = \\ = \tanh\gamma_0\cdot\mathbf{e}_{\alpha(0)} + \int_{\gamma_0}^{\gamma}[\mathrm{sech}^2\gamma\ d\gamma\cdot\mathbf{e}_\alpha + \tanh\gamma\ d\alpha\cdot\mathbf{e}_\nu]. \end{array}\right\} \quad (164A)$$

We see $|d\mathbf{tanh}\ \gamma| \ll |d\gamma'|$, which is caused by the limitation of the tangent motion model to $R = 1$ of the trigonometric circle (Figure 4A). From (162A), (163A), one may easy infer the metrical invariant on the hyperboloid II in these two forms, indicated above and in the end of Ch. 6A, but here in clear vector interpretations, with (145A).

The angle γ (with vector of the directional cosines) is main angular argument of the motion models. In them their *quadratic forms* are applicable on the hyperboloid II. By this way it is easy decompose $d\mathbf{cosh}\ \gamma = d(\cosh\gamma\cdot\mathbf{e}_\alpha) = \sinh\gamma'\ d\gamma'\cdot\mathbf{e}_\sigma$ from (149A) for trigonometric mapping of motions on a hyperboloid I, constrained by its topology. As a result, one may get analogous decompositions on the one sheet hyperboloid I.

The *vector of proper velocity* of a particle M or the barycenter of a body M can be interpreted from (163A) as the *sine hyperbolic projection* of 4-velocity into $\langle\langle\mathcal{E}^3\rangle\rangle$

$$\mathbf{v}^*(\tau) - \mathbf{v}^*(\tau_0) = c\cdot(\mathbf{sinh}\ \gamma - \mathbf{sinh}\ \gamma_0) = v^*(\tau)\cdot\mathbf{e}_\alpha(\tau) - v^*(\tau_0)\cdot\mathbf{e}_\alpha(\tau_0) = \quad (165A)$$

$$= c\int_{\tau_0}^{\tau}\cos\varepsilon(\tau)\cdot\cosh\gamma'(\tau)\cdot\frac{d\gamma'}{d\tau}\ d\tau\cdot\mathbf{e}_\alpha(\tau) + c\int_{\tau_0}^{\tau}\sin\varepsilon(\tau)\cdot\cosh\gamma'(\tau)\cdot\frac{d\gamma'}{d\tau}\ d\tau\cdot\mathbf{e}_\nu(\tau) =$$

$$= \int_{\tau_0}^{\tau}\cosh\gamma(\tau)\cdot\left[c\cdot\frac{d\gamma}{d\tau}\right]\ d\tau\cdot\mathbf{e}_\alpha(\tau) + \int_{\tau_0}^{\tau}\left[c\cdot\sinh\gamma(\tau)\cdot\frac{d\alpha}{d\tau}\right]\ d\tau\cdot\mathbf{e}_\nu(\tau) =$$

$$= \int_{\tau_0}^{\tau}\cosh\gamma(\tau)\cdot\overline{\overline{g}}(\tau)\ d\tau\cdot\mathbf{e}_\alpha(\tau) + \int_{\tau_0}^{\tau}\overset{\perp}{g}(\tau)\ d\tau\cdot\mathbf{e}_\nu(\tau) =$$

$$= \int_{\tau_0}^{\tau}\frac{\overline{\overline{dv^*}}}{d\tau}\ d\tau\cdot\mathbf{e}_\alpha(\tau) + \int_{\tau_0}^{\tau}v^*(\tau)\cdot w_\alpha^*(\tau)\ d\tau\cdot\mathbf{e}_\nu(\tau),$$

where: $d\alpha$ - is the 1-st differential of the orthospherical rotation of $\mathbf{e}_\alpha(\tau)$;
$\cosh\gamma\cdot\overline{\overline{g}}(\tau) = \dfrac{\overline{\overline{dv^*}}}{d\tau} = \overline{\overline{g}}\ ^*(\tau)$ is *tangential proper acceleration*, see (82A);
$c\dfrac{\overset{\perp}{d}\gamma}{d\tau} = \dfrac{\overset{\perp}{d}v^{(m)}}{d\tau} = \overset{\perp}{g}{}^*(\tau) = \overset{\perp}{g}(\tau) = v^*(\tau)\cdot w_\alpha^*(\tau)$ is *normal proper acceleration* (Ch. (10A);
$w_\alpha^*(\tau)$ is the *proper angular velocity of a world line in a point M in the local base* $\tilde{E}_m$.
Both satisfy to the Herglotz Principle as normal ones; g is an inner 4-acceleration in $\tilde{E}_m$.
These proper accelerations satisfy the $3D$ *Relative Pythagorean theorem* (Ch. 10A).

The *vector of coordinate velocity* ($||\mathbf{v}(t)|| < c$) of a point M is strictly inferred here with (164A), but taking into account the fact that under normal hyperbolic projecting, including tangent cross-projecting, the time for the normal motion direction streams as proper time (without the secant factor) as in (165A) too. In $\tilde{E}_1$, there holds:

$$\mathbf{v}(t) - \mathbf{v}(t_0) = c \cdot (\mathbf{tanh}\ \gamma - \mathbf{tanh}\ \gamma_0) = v(t) \cdot \mathbf{e}_\alpha(t) - v(t_0) \cdot \mathbf{e}_\alpha(t_0) = \qquad (166A)$$

$$= c \int_{t_0}^{t} \cos\varepsilon \cdot \mathrm{sech}^2\gamma'(t) \cdot \frac{d\gamma'}{dt}\ dt \cdot \mathbf{e}_\alpha(t) + c \int_{t_0}^{t} \sin\varepsilon \cdot \mathrm{sech}^2\gamma'(t) \cdot \frac{d\gamma'}{dt}\ dt \cdot \mathbf{e}_\nu(t) =$$

$$= \int_{\tau_0}^{\tau} \mathrm{sech}^2\gamma(\tau) \cdot \left[c \cdot \frac{d\gamma}{d\tau}\right]\ d\tau \cdot \mathbf{e}_\alpha + \int_{\tau_0}^{\tau} \mathrm{sech}\gamma(\tau) \cdot \left[c \cdot \sinh\gamma(\tau) \cdot \frac{d\alpha}{d\tau}\right] d\tau \cdot \mathbf{e}_\nu(\tau) =$$

$$= c \int_{t_0}^{t} \mathrm{sech}^3\gamma(t) \cdot \overline{\overline{g}}[\tau(t)]\ dt \cdot \mathbf{e}_\alpha(t) + c \int_{t_0}^{t} \mathrm{sech}\gamma[\tau(t)] \cdot \overset{\perp}{g}\ [\tau(t)]\ d[\tau(t)] \cdot \mathbf{e}_\nu[\tau(t)] =$$

$$= \int_{t_0}^{t} \frac{\overline{\overline{dv}}}{dt} dt \cdot \mathbf{e}_\alpha(t) + \int_{t_0}^{t} v(t) \cdot w_\alpha(t)\ dt \cdot \mathbf{e}_\nu(t),$$

where $t_0 = \tau_0,\ t = t(\tau)$ are due to (85A), and $v(t) \cdot w_\alpha(t)\ dt = v(t) \cdot w^*_\alpha[\tau(t)]\ d[\tau(t)] = \mathrm{sech}\gamma \cdot v^*(\tau) \cdot w^*_\alpha(\tau)\ d\tau$. The tangential and normal *coordinate accelerations* are

$$\mathrm{sech}^3\ \gamma \cdot \overline{\overline{g}}[\tau(t)] = \frac{\overline{\overline{dv}}}{dt} = \overline{\overline{g}}^{(1)}(t),\ \mathrm{sech}\ \gamma \cdot \overset{\perp}{g}\ [\tau(t)] = \frac{\overset{\perp}{dv}}{dt} = \overset{\perp}{g}^{(1)}\ (t) = v(t) \cdot w_\alpha(t), \qquad (167A, 168A)$$

where v is a velocity, w^*_α is an angular velocity of a world line, orthogonal to $\mathbf{v}(t)$. So, from (166A) we obtain the well-known formulae of STR for these two coordinate accelerations in the original base $\tilde{E}_1$ very simply and in clear trigonometric forms:

$$\overline{\overline{F}} = \cos\varepsilon \cdot m_0\ g = m_0 \cdot \cosh^3\gamma \cdot \overline{\overline{g}}^{(1)}(t), \quad \overset{\perp}{F} = \sin\varepsilon \cdot m_0\ g = m_0 \cdot \cosh\gamma \cdot \overset{\perp}{g}^{(1)}.$$

The current *proper distance* is evaluated by analogous two ways with the separation in two time parameters $t_0 = \tau_0$, and $t = t(\tau)$ under condition (84A), (85A) of simultaneity. In the base $\tilde{E}_1$, from (165A) and (166A) we obtain two identical integrals:

$$\mathbf{x}_\tau(\tau) - \mathbf{x}_0 \equiv \mathbf{x}_t(t) - \mathbf{x}_0 = \int_{\tau_0}^{\tau} v^*(\tau) \cdot \mathbf{e}_\alpha(\tau)\ d\tau \equiv \int_{t_0}^{t} v(t) \cdot \mathbf{e}_\alpha(t)\ dt \equiv$$

$$\equiv \int_{\tau_0}^{\tau} \left[v^*_0 \cdot \mathbf{e}_\alpha(\tau_0) + \int_{\tau_0}^{\tau} \cosh\gamma(\tau) \cdot \overline{\overline{g}}(\tau)\ d\tau \cdot \mathbf{e}_\alpha(\tau) + \int_{\tau_0}^{\tau} \overset{\perp}{g}\ (\tau)\ d\tau \cdot \mathbf{e}_\nu(\tau)\right] d\tau =$$

$$= \int_{\tau_0}^{\tau} \left[v^*_0 \cdot \mathbf{e}_\alpha(\tau_0) + \int_{\tau_0}^{\tau} \overline{\overline{g}}\ ^*(\tau)\ d\tau \cdot \mathbf{e}_\alpha(\tau) + \int_{\tau_0}^{\tau} \overset{\perp}{g}\ (\tau)\ d\tau \cdot \mathbf{e}_\nu(\tau)\right] d\tau \equiv \qquad (169A)$$

$$\equiv \int_{t_0}^{t} \left[v_0 \cdot \mathbf{e}_\alpha(t_0) + \int_{t_0}^{t} \mathrm{sech}^3\gamma(t) \cdot \overline{\overline{g}}(t)\ dt \cdot \mathbf{e}_\alpha(t) + \int_{t_0}^{t} \mathrm{sech}\gamma(t) \cdot \overset{\perp}{g}\ (t)\ dt \cdot \mathbf{e}_\nu(t)\right] dt =$$

$$= \int_{t_0}^{t} \left[v_0 \cdot \mathbf{e}_\alpha(t_0) + \int_{t_0}^{t} \overline{\overline{g}}^{(1)}(t)\ dt \cdot \mathbf{e}_\alpha(t) + \int_{t_0}^{t} \overset{\perp}{g}^{(1)}\ (t)\ dt \cdot \mathbf{e}_\nu(t)\right] dt. \qquad (170A)$$

Variation of the scalar cosine, in its turn, is, by (162A), proportional to the current work of the *inner force* (81A) during progressive movement of a material point M:

$$\cosh\gamma - \cosh\gamma_0 = \left.\frac{d(ct)}{d(c\tau)}\right|_{\tau_0}^{\tau} = \int_{\gamma_0'}^{\gamma'} \cos\varepsilon(\tau)\cdot\sinh\gamma'\, d\gamma' = \frac{1}{c^2}\cdot\int_{\tau_0}^{\tau}\cos\varepsilon(\tau)\cdot v^*(\tau)\cdot g(\tau)d\tau =$$

$$= \frac{1}{c^2}\cdot\int_{t_0}^{t}\cos\varepsilon[\tau(t)]\cdot v[\tau(t)]\cdot g[\tau(t)]\; dt = \frac{1}{c^2}\cdot\int_{\chi_0}^{\chi}\cos\varepsilon(\chi)\cdot g(\chi)\; d\chi =$$

$$= \frac{1}{m_0 c^2}\cdot\int_{\chi_0}^{\chi}\cos\varepsilon(\chi)\cdot F(\chi)\; d\chi = \frac{1}{E_0}\cdot\int_{\chi_0}^{\chi}\overline{\overline{F}}(\chi)\; d\chi = \frac{A}{E_0}, \qquad (171A)$$

where we use the notations: $E_0 = m_0c^2,\ A = \int_{\chi_0}^{\chi}\cos\varepsilon(\chi)\cdot F(\chi)\; d\chi = \int_{\chi_0}^{\chi}\overline{\overline{F(\chi)}}\; d\chi.$

If $\gamma_0 = 0$, $(v_0 = 0)$, then $\cosh\gamma = 1 + A/E_0$, i. e., $\boxed{E = \cosh\gamma\cdot E_0 = E_0 + A = mc^2}$. This illustrates the fact, that during progressive movement the total Einsteinian energy of a material body in $\tilde{E}_1$ is the *cosine projection* of energy–momentum tensor (101A) onto the axis $\overrightarrow{ct}^{(1)}$. The tensor is conservative under $\{\mathbf{F} = \mathbf{0} \leftrightarrow \mathcal{T}_E = \text{CONST}\}$.

Historically Joseph Thomson (discovered the electron) was first who attempted to connect the energy and the mass in 1881, when he used the electromagnetic mass [72]. In 1900 Henry Poincaré inferred first the relation $E = mc^2$ for the light as a kind of electromagnetic radiation [83]. In 1905 Albert Einstein inferred also the relation, but in the form $m = E/c^2$ on the base of the Planck quantum theory of radiation by massive body [50]. Gilbert Lewis in 1908 [68] confirmed the same correspondence, but between increments of the inertial mass and the kinetic energy of a material body. But only after Lise Meitner has revealed the fact the fission of uranium and explained by this relation the defect of mass at this process, physics and not only they drew attention to this previously abstract formula with well-known further consequences.

In Ch. 5A, we marked that as a true progenitor of concepts momentum and energy, in relativistic sense, should be considered the *own* 4-*momentum* $\mathbf{P}_0 = m_0\mathbf{c}$. Here it is 4-th column of the hyperbolic tensor of momentum–energy (101A) in space-time $\langle\mathcal{P}^{3+1}\rangle$. The physical tensor is proportional with "c" to dimensionless trigonometric tensor of motion (100A). The own 4-momentum gives two pseudo-Euclidean orthoprojections:

$P_0\cdot\mathbf{i} = P\cdot\mathbf{i}_1 + p\cdot\mathbf{j} \Rightarrow (iP_0)^2 = (iP)^2 + p^2$ – for tensor $I^{\pm}$ i. e., in right form (37A).

Then we adopt that $m = P/c,\ E = P\cdot c,\ \mathbf{p} = m\mathbf{v}$. Right column of the *momentum tensor* $\mathcal{T}_P = roth\,\Gamma\cdot P_0$ of progressive movement, as $\mathbf{P}_0 = P_0\cdot\mathbf{i} = m_0\mathbf{c}$, is the *geometric invariant* along a world line to Lorentzian transformations, where $\mathbf{c} = c\cdot\mathbf{i}$ is 4-velocity of Poincaré. This momentum is tangential to a world line. *Its variable cosine projection* onto direction of the time arrow $\overrightarrow{ct}^{(1)}$ is the *total impulse* $\mathbf{P} = P\cdot\mathbf{i}_1 = P_0\cdot\cosh\gamma\cdot\mathbf{i}_1$ expressed in the base $\tilde{E}_1$. *Its variable vectorial sine projection* into the subspace $\langle\mathcal{E}^3\rangle^{(1)}$ is the *real momentum* $\mathbf{p} = p\cdot\mathbf{j} = P_0\cdot\mathbf{sinh}\,\gamma = P_0\cdot\sinh\gamma\cdot\mathbf{e}_\alpha = m_0\mathbf{v}^* = m\mathbf{v}$ in the base $\tilde{E}_1$ in the Minkowskian space-time $\langle\mathcal{P}^{3+1}\rangle$, or the space-time of Poincaré $\langle\mathcal{Q}^{3+1}\rangle_c$.

Geometrically $\mathbf{P}_0 = m_0\mathbf{c}$ is a hypotenuse of the *pseudo-Euclidean right triangle of momenta* in the *pseudoplane of motion* $\langle \mathbf{e}_\alpha, \mathbf{i}_1\rangle$. Its sides are similar to ones of the interior right triangle at Figure 1A(1) because $\mathcal{T}_P = P_0 \cdot roth\ \Gamma$. Due to this trigonometric approach with the *Absolute Pythagorean Theorem*, we may see origination of Einsteinian relativistic formula for the non-invariant total energy $E^2 = E_0^2 + (pc)^2$ [50].

Vectorial trigonometric functions in the hyperbolic angle of motion and with common $\mathbf{e}_\beta$ vary compatibly:

$$\cosh\gamma = \frac{\sinh\gamma\cdot\mathbf{e}_\beta}{\tanh\gamma\cdot\mathbf{e}_\beta} = \frac{\sin\gamma_0 + \int_{\gamma_0}^{\gamma} d\sinh\gamma}{\tanh\gamma_0 + \int_{\gamma_0}^{\gamma} d\tanh\gamma} = \cosh\gamma_0 + \int_{\gamma_0}^{\gamma} d\cosh\gamma.$$

Recall, the current vector $\mathbf{e}_\beta$ may be orthogonally splitted with the use of (135A). The current real momentum $\mathbf{p}(\tau)$ as sine projection of $\mathbf{P}_0$ into $\langle\mathcal{E}^3\rangle^{(1)}$ is expressed as:

$$\mathbf{p}(\tau) = \mathbf{p}(\tau_0) + m_0\int_{\tau_0}^{\tau} dv^{(m)}(\tau)\cdot\mathbf{e}_\beta(\tau) = p(\tau_0)\cdot\mathbf{e}_\alpha(\tau_0) + m_0\int_{\tau_0}^{\tau} g(\tau)\cdot\mathbf{e}_\beta(\tau)d\tau \equiv$$

$$\equiv \mathbf{p}(t) = p[\tau_0(t_0)]\cdot\mathbf{e}_\alpha[\tau_0(t_0)] + \int_{t_0}^{t}\{\ \overline{\overline{F^*}}[\tau(t)]\cdot\mathbf{e}_\alpha[\tau(t)] + \overset{\perp}{F}\ [\tau(t)]\cdot\mathbf{e}_\nu[\tau(t)\ \}dt.$$

The tensor of momentum–energy is conservative under $\{\mathbf{F} = \mathbf{0} \leftrightarrow \mathcal{T}_P = \mathrm{CONST}\}$.

In STR and external non-Euclidean geometry, progressive non-collinear movements of a nonpoint object in some Euclidean plane of $\langle\mathcal{P}^{3+1}\rangle \equiv \langle\mathcal{E}^3\rangle \boxtimes \overrightarrow{y}$ is accompanied by *orthospherical shift* as the rotation of the plane $\langle\mathcal{E}^2\rangle$, with Cartesian subbase $\tilde{E}_m^{(2)}$ and the object, around a third space-like orthogonal axis $\overrightarrow{\mathbf{e}_N}$, complementary till $\mathcal{E}^3$. In time, this shift causes the *orthospherical precession* of the object together with $\langle\mathcal{E}^2\rangle^{(m)} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle$ in the space $\langle\mathcal{E}^3\rangle^{(m)}$ at the point M and its subbase $\tilde{E}_m^{(2)}$, see (499), (99A), (144A), (161A). They are expressed in terms of $\mathbf{tanh}\ (\gamma/2) = \tanh(\gamma/2)\cdot\mathbf{e}_\alpha$ and $\mathbf{d}\gamma = d\gamma\cdot\mathbf{e}_\beta$ in $\tilde{E}_1$ and $\tilde{E}_m$:

$$\left.\begin{array}{l}
\mathbf{d}\theta = \mathbf{tanh}\ (\gamma/2)\times\mathbf{d}\gamma = \dfrac{\mathbf{tanh}\ (\gamma)}{1+\mathrm{sech}\ \gamma}\times\mathbf{d}\gamma = \dfrac{1-\mathrm{sech}\ (\gamma)}{\tanh\gamma}\cdot\mathbf{e}_\alpha\times\mathbf{d}\gamma = \\
= \tanh(\gamma/2)\ d\gamma\cdot\overrightarrow{\mathbf{r}_N} = -\tanh(\gamma/2)\ \sin\varepsilon(\gamma)\ d\gamma\cdot\overrightarrow{\mathbf{e}_N}, \\
(\text{here there hold: } \sin\varepsilon(\gamma)\ d\gamma = \overset{\perp}{d\gamma},\quad \dfrac{d\gamma}{d\tau} = \eta_K(\tau) = g^{(m)}/c), \\
\dfrac{\mathbf{d}\theta}{d\tau} = w_\theta^*\cdot\overrightarrow{\mathbf{e}_N} = -\sin\varepsilon\cdot\tanh(\gamma/2)\cdot\eta_K\cdot\overrightarrow{\mathbf{e}_N} = -\dfrac{\sin\varepsilon\cdot v^{(1)}\cdot g^{(m)}}{c^2\cdot(1+\mathrm{sech}\ \gamma)}\cdot\overrightarrow{\mathbf{e}_N}.
\end{array}\right\} \quad (172A)$$

Particular case at $\varepsilon = \pi/2$ is the Thomas precession of electron spin [70, p. 514] with trigonometric view of A. Sommerfeld [71]. Precession, due to (172A), is approximated by area of the triangle with sides v/c, g/c and the angle ε between them, that is why precession changes during movement. Vectorial formula (172A) is represented exactly in the physical relativistic form, without "c" and "$\sin\varepsilon$", through angular velocities:

$$\frac{\mathbf{d}\theta}{d\tau} = \mathbf{e}_\alpha\times\left[\frac{\mathbf{g}}{v} - \frac{\mathbf{g}}{v^*}\right] = \mathbf{e}_\alpha\times\mathbf{e}_\beta\left[\frac{g}{v} - \frac{g}{v^*}\right] = -\sin\varepsilon\cdot\left[\frac{g}{v} - \frac{g}{v^*}\right]\cdot\overrightarrow{\mathbf{e}_N} = -\left[\frac{\overset{\perp}{g}}{v} - \frac{\overset{\perp}{g}}{v^*}\right]\cdot\overrightarrow{\mathbf{e}_N} = -\sin\varepsilon\cdot(w_\alpha - w_\alpha^*)\cdot\overrightarrow{\mathbf{e}_N}.$$

Orthospherical rotation and precession (172A) are explained by formulae (111A) of summing non-collinear motions in $\langle\mathcal{P}^{3+1}\rangle$ with appearance of the additional motion *rot* $d\Theta$, which restores inertiality of this *total moving* description. From the point of view of Observer N_1 in the original base $\tilde{E}_1$, the body receives as if a torque (torsion moment) at point M relatively to the unity axis $\overrightarrow{\mathbf{e}_N}$. It generates principal momentum $\mathbf{M}$ of the body with moment of inertia $\mathcal{J}$ (as in Newtonian mechanics). If $v(t) << c$, then, take into account $1 + \operatorname{sech}\gamma \approx 2$, relatively to the normal axis $\overrightarrow{\mathbf{e}_N}$:

$$\frac{\mathbf{d}\theta}{d\tau} \approx \frac{\mathbf{v}[t(\tau)] \times \mathbf{g}^{(m)}(\tau)}{2c^2};\;\; \overrightarrow{\mathbf{L}} = \mathcal{J}_0 \cdot \frac{\mathbf{d}\theta}{d\tau};\;\; \overrightarrow{\mathbf{M}} = \frac{d\mathbf{L}}{d\tau} = J_0 \cdot \frac{\mathbf{d}^2\theta}{d\tau^2} \approx \frac{J_0}{2P_0} \cdot \left[\mathbf{v}[t(\tau)] \times \frac{d\mathbf{F}^{(m)}}{dc\tau}\right],$$

where $\mathbf{g}^{(1)} \times \mathbf{g}^{(m)} = \mathbf{0}$ for parallel (to $\mathbf{e}_\beta$) vectors. Let $\mathbf{F}^{(m)} = F^{(m)} \cdot \mathbf{e}_\beta$ be the inner force (as in (81A)) producing the inner acceleration $\mathbf{g}^{(m)}$ of the body velocity $\mathbf{v}$ (with own mass m_0 and own momentum $P_0 = m_0 c$), $\mathbf{L} = \mathcal{J} \cdot \eta_\theta$ be the angular momentum, $\overrightarrow{\mathbf{M}} = \mathcal{M} \cdot \mathbf{e}_N$ be principal momentum expressing kinetic energy of the body orthospherical rotation (precession) around axis $\overrightarrow{\mathbf{e}_N}$ as if created by some torsion moment. The sign "$-$" in the vectorial formulae with rotation around $\overrightarrow{\mathbf{e}_N}$ illustrates the following **Rule** $\boxed{sgn\ \theta_{13} = -sgn\ \varepsilon}$ in pseudo-Euclidean and hyperbolic geometries, and STR.

On the other side, from the point of view of Observer N_1 in the inertial frame of reference $\tilde{E}_1$, this orthospherical rotation of the uninertial subbase $\tilde{E}_m^{(3)}$ with M in $\langle\mathcal{E}^3\rangle^{(m)}$, may be interpreted as if the manifestation of Coriolis acceleration g_C from force $F_C = mg_C$ in $\tilde{E}_m$. Then, from (172A), we obtain the absolute Coriolis acceleration of the body or particle M by the exact formula and with approximation to the time t:

$$\mathbf{g}_C = \left[c \cdot \frac{\mathbf{d}\theta}{d\tau} + c \cdot \frac{\mathbf{d}\theta}{dt}\right] = -\sin\varepsilon \cdot v^{(1)} \cdot \eta_K \cdot \overrightarrow{\mathbf{e}_N} \approx 2c \cdot \frac{d\theta}{dt} \cdot \overrightarrow{\mathbf{e}_N} = 2[c \cdot w_\theta^*] \cdot \overrightarrow{\mathbf{e}_N}.$$

The orthospherical rotation may have a relative (ephemeral) character. This is true if $\mathbf{e}_\beta = \text{const}$, and it is not obligatory that $\mathbf{e}_\beta = \mathbf{e}_\alpha$. This is accelerated (decelerated) physical movement in the plane $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_{\alpha(0)}, \mathbf{e}_\beta\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle \equiv \text{Const}$ with $\mathbf{v}_0$ under the angle ε_0 to $\mathbf{e}_\beta = \text{const}$. In the origin of the base $\tilde{E}_1$, the world line slope corresponds to $\tanh\gamma_0 = v_0/c$ with $\mathbf{e}_{\alpha(0)}$. Execute the pure hyperbolic modal transformation of the base as *roth* $\Gamma \cdot \tilde{E}_1 = \tilde{E}_{1h}$ with $\gamma = \gamma_0$ and $\mathbf{e}_\alpha = \mathbf{e}_{\alpha(0)}$. Then, in the new base, we annihilate the rotation $d\theta$, because in it $\mathbf{tanh}\ \gamma$ ($\mathbf{v}$) and $\mathbf{d}\gamma$ ($\mathbf{g}$) are collinear vectors ($\sin\varepsilon = 0$). Such transformation is equivalent to translation in the frame of reference with $\mathbf{v} = \mathbf{v}_0$. As a result, we obtain the same, but collinear motion in the new base $\tilde{E}_{1h}$ without its ephemeral orthospherical rotation and torsion.

Kinematics and dynamics of absolute movement were considered in Chs. 5A–7A in the *relative* trigonometric interpretations. Particle (barycenter) M moving was represented with cosine scalar and sine 3-dimensional vector projections in $\langle\mathcal{P}^{3+1}\rangle$. The absolute tensor trigonometric interpretation of the movement of M in terms of its invariant geometric and physical parameters will be given in Ch. 10A.

Chapter 8A

Trigonometric models of two-step and multistep non-collinear motions in quasi-Euclidean space and in spherical geometry

Definition of the *quasi-Euclidean* oriented space $\langle \mathcal{Q}^{n+1} \rangle$ (sect. 5.7) is similar in a certain extent to that of the pseudo-Euclidean Minkowskian space $\langle \mathcal{P}^{n+1} \rangle$ (sect. 11.2). The reflector-tensor $I^{\pm}$ is very important in $\langle \mathcal{Q}^{n+1} \rangle$ too. It determines orientation and admitted transformations in this space. However, the metric of a quasi-Euclidean space is Euclidean. In geometry of $\langle \mathcal{Q}^{n+1} \rangle$, the essential part is quasi-Euclidean trigonometry with tensor spherical functions. They are defined in their canonical forms with respect to the unity base $\{I\}$ as principal rotations $rot\ \Phi$ with frame axis (313), (314), and secondary ones $rot\ \Theta$ in form as (259) or as (184A) for $n \geq 2$ or as (497) for $n = 3$.

The oriented *hyperspheroid* of radius R in $\langle \mathcal{Q}^{n+1} \rangle$ can be considered as a spherical geometric object and trigonometric one at $R = 1$ similar to hyperboloids in $\langle \mathcal{P}^{n+1} \rangle$. Its internal and external geometries are determined by radius R and directed axis $\overrightarrow{y}$ of reference for principal rotations. The center of this hyperspheroid is the origin of all orthospherically connected universal quasi-Cartesian bases $\tilde{E}_{1u}$. Respectively to the hyperspheroid, the rotation $rot\ \Theta$ expresses, in the *external way*, the orthospherical shift under summing non-collinear principal rotations $rot\ \Phi ij$, and, in the *internal way*, the angular excess for geometric figures composed of geodesic lines (large circles) on its surface as results of these rotations. The space $\langle \mathcal{Q}^{n+1} \rangle$ can be represented in a quasi-Cartesian base $\tilde{E}$ as the spherically orthogonal direct sum similar to (462)

$$\langle \mathcal{Q}^{n+1} \rangle \equiv \langle \mathcal{E}^{n} \rangle \boxplus \overrightarrow{y} \equiv CONST, \tag{173A}$$

where $\langle \mathcal{E}^{n} \rangle$ is an Euclidean hyperplane, $\overrightarrow{y}$ is the oriented frame axis for angles φ.

From the point of view of quasi-Euclidean trigonometry, the subspace $\langle \mathcal{E}^{n} \rangle^{(k)}$ is a sine hyperplane, and $\overrightarrow{y}^{(k)}$ is a cosine axis. Imaginarization of the axis $\overrightarrow{y}$ transforms $\langle \mathcal{Q}^{n+1} \rangle$ into a complex-valued quasi-Euclidean space of index $q = 1$ (see in sect. 6.1) isomorphic to the real-valued pseudo-Euclidean space with the same reflector and metric (!) tensor $I^{\pm}$. The following operations are admitted in the space $\langle \mathcal{Q}^{n+1} \rangle$:
1) rotations of the two types: as principal spherical $rot\ \Phi$ and orthospherical $rot\ \Theta$;
2) parallel translations preserving the space structure (173A) with reflector tensor $I^{\pm}$.
The *principal tensor of motion* in a spherical angle Φ in $\langle \mathcal{Q}^{n+1} \rangle$ due to (257), (267A) are

$$\langle rot\ \Phi \rangle : rot\ \Phi = \cos\Phi + i \cdot \sin\Phi, \quad rot\ \Phi \cdot I^{\pm} \cdot rot\ \Phi = I^{\pm}. \tag{174A}$$

They execute principal spherical rotations with the frame axis $\overrightarrow{y}$. The *secondary orthospherical rotations* (as in the space $\langle P^{n+1} \rangle$ with the same reflector tensor) are

$$\langle rot\ \Theta \rangle : \ rot'\ \Theta \cdot I^{\pm} \cdot rot\ \Theta = I^{\pm} = rot\ \Theta \cdot I^{\pm} \cdot rot'\ \Theta. \tag{175A}$$

They execute spherical rotations of a geometric object in the subspace $\langle \mathcal{E}^{n} \rangle$ from (173A). Both are a tool for solving a lot of problems on the hyperspheroid too.

That is why, for analysis of homogeneous composite motion T, we shall use the polar decomposition (the right-oriented universal base should be chosen as original one):

$$\tilde{E} = T \cdot \tilde{E}_1 = rot\ \Phi \cdot rot\ \Theta \cdot \tilde{E}_1 = rot\ \Theta \cdot rot\ \overset{\angle}{\Phi} \cdot \tilde{E}_1. \tag{176A}$$

$$T = rot\ \Phi \cdot rot\ \Theta = rot\ \Theta \cdot rot\ \overset{\angle}{\Phi}, \qquad det\ T = +1. \tag{177A}$$

The hyperspheroid of *fixed radius* R embedded into $\langle \mathcal{Q}^{n+1} \rangle$ is a suitable object where *internal* spherical geometry is in one-to-one correspondence with the space tensor trigonometry having the same orientation. Abstract spherical-hyperbolic analogy (322) in $\tilde{E}_{(01)}$, see (443), and (323) in $\tilde{E}_{(02)}$, see (444), takes place; and the concrete analogy, mainly as sine-tangent one (331), can be used in universal bases, see sect. 6.1, 6.2. Principal spherical rotations are expressed in $\tilde{E}_1$ according to abstract analogy (323):

$$\Gamma \leftrightarrow i\Gamma \leftrightarrow \Phi, \quad roth\ \Gamma \leftrightarrow rot\ i\Gamma \leftrightarrow rot\ \Phi, \quad (\tilde{E}_{(1h)} \leftrightarrow \tilde{E}_{(02)} \leftrightarrow \tilde{E}_{(1s)}), \tag{178A}$$

On the base, we expose the materials of this Chapter in parallel with ones of Ch. 7A ! The *spherical tensor of motion rot* Φ with the frame axis $\overrightarrow{y}$ in $\langle \mathcal{Q}^{n+1} \rangle$ has, due to (314), the following canonical structure in $\tilde{E}_1$ corresponding to the reflector tensor:

$$\{rot\Phi\}_{3\times 3} = \cos\Phi + i \cdot \sin\Phi \qquad rot\ \Theta \qquad I^{\pm}$$

$$\begin{array}{|c|c|}\hline \cos\varphi_i \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' & \mp\sin\varphi_i \cdot \mathbf{e}_\alpha \\ \hline \pm\sin\varphi_i \cdot \mathbf{e}'_\alpha & \cos\varphi_i \\ \hline \end{array} \cdots\cdots \begin{array}{|c|c|}\hline \{rot\ \Theta\}_{3\times 3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \\ \hline \end{array} \cdots\cdots \begin{array}{|c|c|}\hline I_{3\times 3} & \mathbf{0} \\ \hline \mathbf{0}' & -1 \\ \hline \end{array}\ . \tag{179A}$$

The orthospherical rotation in the angle Θ as a rule is secondary for principal angle. Due to the analogy in the universal base, formulae of hyperbolic geometry in Ch. 7A with (119A) are transformed into analogs in the spherical geometry. By correspondence between angles of principal motions in the hyperbolic and spherical Lambert measures

$$a_{(H)} = \gamma \cdot R, \qquad a_{(S)} = \varphi \cdot R, \tag{180A}$$

the formulae of these two geometries in the small are transformed into each other in the internal and external interpretations. Further, we infer formulae of the elementary spherical tensor trigonometry ($q = 1$) with the use of spherical–hyperbolic analogy (with corresponding to it commentaries). For two-step noncollinear motions, by (176A, 177A), we obtain the modal transformations as spherical analogs of (111A):

$$\tilde{E}_3 = rot\ \Phi_{12} \cdot rot\ \Phi_{23} \cdot \tilde{E}_1 = \{rot\ \Phi_{12} \cdot rot\ \Phi_{23} \cdot rot'\ \Phi_{12}\}_{\tilde{E}_2} \cdot rot\ \Phi_{12} \cdot \tilde{E}_1 =$$

$$= rot\ \Phi_{13} \cdot rot\ \Theta_{13} \cdot \tilde{E}_1 = \{rot\ \Phi_{13} \cdot rot\ \Theta_{13} \cdot rot'\ \Phi_{13}\}_{\tilde{E}_{1s}} \cdot rot\ \Phi_{13} \cdot \tilde{E}_1 = \tag{181A}$$

$$= rot\ \Theta_{13} \cdot rot\ \overset{\angle}{\Phi}_{13} \cdot \tilde{E}_1 = \{rot\ \Theta_{13} \cdot rot\ \overset{\angle}{\Phi}_{13} \cdot rot'\ \Theta_{13}\}_{\tilde{E}_{1u}} \cdot rot\ \Theta_{13} \cdot \tilde{E}_1 = T_{13} \cdot \tilde{E}_1.$$

These formulae are given for the direct order of the two principal motions.

Corollary. *Two-step noncollinear spherical motions rot* Φ_{ij} *in* $\langle \mathcal{Q}^{n+1} \rangle$ *or on the hyperspheroid may be represented as a pair of spherical and orthospherical rotations.*

Some characteristics of such motions in direct and inverse orders are expressed as

$$rot\ \overset{\angle}{\Phi}_{13} = rot'\ \Theta_{13} \cdot rot\ \Phi_{13} \cdot rot\Theta_{13} = rot\ (-\Theta_{13}) \cdot rot\ \Phi_{13} \cdot rot\ (+\Theta_{13}), \quad (182A)$$

with $\mathbf{e}_{\overset{\angle}{\sigma}} = \{rot\ (+\Theta_{13})\}_{3\times 3} \cdot \mathbf{e}_\sigma$ (under rule $\varepsilon > 0 \to \theta_{13} > 0$) $\Rightarrow \cos\theta_{13} = \mathbf{e}'_{\overset{\angle}{\sigma}} \cdot \mathbf{e}_\sigma$.
Rotation $\pm\theta$ is expressed in $\tilde{E}_{1s} = rot\ \overset{\angle}{\Phi} \cdot \tilde{E}_1$. (If $n = 2$, it acts in the plane $\langle \mathcal{E}^2 \rangle^{(1s)}$). If $n = 3$, we have $\overrightarrow{\mathbf{r}_N}(\theta) = \mathbf{e}_{\overset{\angle}{\sigma}} \otimes \mathbf{e}_\sigma = \pm \sin\theta \cdot \overrightarrow{\mathbf{e}_N}, \quad \overrightarrow{\mathbf{r}_N}(\varepsilon) = \mathbf{e}_\alpha \otimes \mathbf{e}_\beta = \pm \sin\varepsilon \cdot \overrightarrow{\mathbf{e}_N}$.

There is the essential difference between the angles Γ and Φ: in $\tilde{E}_1$, Γ is symmetric, Φ is antisymmetric. In their diagonal forms, Γ is real-valued, Φ is imaginary-valued. As consequence, all the trigonometric formulae are identical, when these angles are represented in symmetric forms: Γ in the base $\tilde{E}_1$, $-i\Phi$ in the base $\tilde{E}_{(01)}$, see (271). The next formula holds due to this peculiarity in the real-valued original base $\tilde{E}_1$:

$$rot\ \Phi_{13} = \sqrt{rot\ \Phi_{12} \cdot rot\ (2\Phi_{23}) \cdot rot\ \Phi_{12}} = \sqrt{rot\ (2\Phi_{13})} =$$
$$= \sqrt{[rot\ \Phi_{12} \cdot rot\ \Phi_{23}] \cdot [rot\ \Phi_{23} \cdot rot\ \Phi_{12}]} = \sqrt{T\ T^*}, \quad (183A)$$

The formula is analogous to (114A), but square roots are *trigonometric* (what is it see in sect. 5.6). Here we have the peculiarity, which relates to spherical case for permutation of particular motions with change of their order into contrary one. From the original $\tilde{E}_1 = \{I\}$, as in (181A), this leads now to the base $\tilde{E}_3^* = rot\ \Phi_{23} \cdot rot\ \Phi_{12} = T_{13}^* \cdot \tilde{E}_1$, where matrix T^* is quasi-analogous to $T\ '$ in (116A), but here $T_{13}^* \neq T'_{13}$!

From the direct formulae (181A), we obtain the orthospherical analog of (115A):

$$rot\ (+\Theta_{13}) = rot\ \Phi_{12} \cdot rot\ \Phi_{23} \cdot rot\ \overset{\angle}{\Phi}_{31} = rot\ \Phi_{31} \cdot rot\ \Phi_{12} \cdot rot\ \Phi_{23}. \quad (184A)$$

It represents this secondary orthospherical rotation as the result of the closed cycle of motions $rot\ \Phi_{ij}$ in the spherical triangle 123. It is executed also as in (115A) from points 1 and 3 in the bases of particular rotations actions along of the triangle sides!

In order that a result of (183A) was $rot\ \Phi_{13}$, we adopted for this two-step motion inverse to (181A) the expression analogous to (116A) (without transition in $\tilde{E}_{(01)}$!):

$$\tilde{E}_3^* = rot\ \Phi_{23} \cdot rot\ \Phi_{12} \cdot \tilde{E}_1 = T_{13}^* \cdot \tilde{E}_1 = rot\ (-\Theta_{13}) \cdot rot\ \Phi_{13} \cdot \tilde{E}_1 =$$

$$= rot\ \overset{\angle}{\Phi}_{13} \cdot rot\ (-\Theta_{13}) \cdot \tilde{E}_1 = \{rot\ \overset{\angle}{\Phi}_{13} \cdot rot\ (-\Theta_{13}) \cdot rot'\ \overset{\angle}{\Phi}_{13}\}_{\tilde{E}_1} \cdot rot\ \overset{\angle}{\Phi}_{13} \cdot \tilde{E}_1. \quad (185A)$$

This expression is completely compatible with (182A), gotten from (181A)! For inverse order of motions, we obtain the analogs of (117A), (118A) with inverse cycle (184A):

$$rot\ \overset{\angle}{\Phi}_{13} = \sqrt{rot\ \Phi_{23} \cdot rot\ (2\Phi_{12}) \cdot rot\ \Phi_{23}} = \sqrt{rot\ (2\ \overset{\angle}{\Phi}_{13})} = \sqrt{T^*\ T}, \quad (186A)$$

$$rot\ (-\Theta_{13}) = rot'\ \Theta_{13} = rot^{-1}\Theta_{13} = rot\ \Theta_{31} = rot\ \Phi_{32} \cdot rot\ \Phi_{21} \cdot rot\ \Phi_{13}. \quad (187A)$$

We have initially the *invariant* as radius $R = 1$ for all these trigonometric rotations.

The angles Φ_{13} and $\overset{\angle}{\Phi}_{13}$ differ only by their vectors of directional cosines. Due to (182A), its scalar summarized angle φ_{13} (in that number for multistep motion) does not depend on ordering of summands (direct or inverse). The case when the directional cosines of motions are either equal or additively opposite to each other, corresponds to collinear motions. Choice of direct or inverse order of summands in two-step spherical motion (T or T^*) is redused to these partial angles substitution analogous to (121A):

$$\varphi_{12} \leftrightarrow \varphi_{23}, \qquad \alpha_k \leftrightarrow \beta_k, \; k = 1, 2. \tag{188A}$$

Formulae of two-step motion in $\langle Q^{n+1}\rangle$ are obtained with multiplying the matrices in (183A) or (186A), or alternative applying *abstract spherical-hyperbolic analogy* (178A). In particular, the *scalar* cosine of summarized angle φ_{13} is expressed as follows [21]:

$$\cos\varphi_{13} = \cos\varphi_{12}\cdot\cos\varphi_{23} - \cos\varepsilon\cdot\sin\varphi_{12}\cdot\sin\varphi_{23} =$$

$$= \cos\varphi_{12}\cdot\cos\varphi_{23} + \cos A_{123}\cdot\sin\varphi_{12}\cdot\sin\varphi_{23}, \quad A_{123} = \pi - \varepsilon. \tag{189A}$$

This formula shows that the *scalar* angle of summarized motion does not depend on ordering of partial motions φ_{12}, φ_{23}. This is the classical formula of spherical geometry for the cosine of the scalar angle φ_{13}. Motion on the surface of a hyperspheroid with increasing y-coordinate preserves angles φ_{ij} positive. That is why, for positive angles of motions and distances in the *spherical Lambert measure*, the "parallelogram rule" takes place (as in Euclidean geometry and non-Euclidean hyperbolic geometry):

$$|\varphi_{12} - \varphi_{23}| \le \varphi_{13} \le \varphi_{12} + \varphi_{23}.$$

It is analogous to (123A) and follows from (189A). Due to inequalities (190A) and $\varphi_{ij} > 0$, distance in spherical geometry is a norm. The whole quasi-Euclidean space has Euclidean metric, that is why the length of a geodesic spherical arc $d\varphi$ and an orthospherical arc $d\theta$ are Euclidean. In its *sine model*, the hyperspheroid is mapped onto the whole two-side closed sine projective hyperplane (with topology of a sphere). In internal geometry of the hyperspheroid in $\langle Q^{n+1}\rangle$, the following *scalar* formulae for the sine and tangent of the arcs sum hold in direct and contrary orders of motions.

Two *quadratic* formulae for the scalar sine follow from (189A) as analogs of (124A):

$$\sin^2\varphi_{13} =$$

$$\left.\begin{aligned} &= (\sin\varphi_{12}\cdot\cos\varphi_{23} + \cos\varepsilon\cdot\sin\varphi_{23}\cdot\cos\varphi_{12})^2 + (\sin\varepsilon\cdot\sin\varphi_{23})^2 = \\ &= (\sin\varphi_{23}\cdot\cos\varphi_{12} + \cos\varepsilon\cdot\sin\varphi_{12}\cdot\cos\varphi_{23})^2 + (\sin\varepsilon\cdot\sin\varphi_{12})^2. \end{aligned}\right\} \tag{190A}$$

Tangent direct formula follows from (190A) and (189A) as analog of (125A):

$$\tan^2\varphi_{23} = \tag{191A}$$

$$= \left[\frac{\tan\varphi_{12} + \cos\varepsilon\cdot\tan\varphi_{23}}{1 - \cos\varepsilon\cdot\tan\varphi_{23}\cdot\tan\varphi_{12}}\right]^2 + \left[\frac{\sin\varepsilon\cdot\tan\varphi_{23}\cdot\sec\ \varphi_{12}}{1 - \cos\varepsilon\cdot\tan\varphi_{23}\cdot\tan\varphi_{12}}\right]^2.$$

They express the spherical *Big and Small Pythagorean Theorems* in $\langle \mathcal{Q}^{n+1}\rangle$, which act in quasi-Euclidean and spherical geometries also for sine and tangent segments as projections into $\langle\langle \mathcal{E}^n\rangle\rangle$. Each theorem acts in two variants: for direct and inverse orders of these segments.

Due to the *sine-tangent analogy* with tangent's differentials (164A), in $\tilde{E}_1$ we obtain:

$$\left.\begin{array}{l}
d\mathbf{tanh}\,\gamma = d(\tanh\gamma\cdot\mathbf{e}_\alpha) = \mathrm{sech}^2\,d\gamma\cdot\mathbf{e}_\alpha + \tanh\gamma\;d\alpha\cdot\mathbf{e}_\nu = \\
= \mathrm{sech}^2\gamma'\;d\gamma'\cdot\mathbf{e}_\sigma = \mathrm{sech}^2\gamma'\cdot[\cos\varepsilon\;d\gamma'\cdot\mathbf{e}_\alpha + \sin\varepsilon\;d\gamma'\cdot\mathbf{e}_\nu],\\[2ex]
|d\mathbf{tanh}\,\gamma|^2 = \mathrm{sech}^4\,d\gamma^2 + \tanh^2\gamma\;d\alpha^2 = (\mathrm{sech}^2\gamma'\;d\gamma')^2 = \\
= \mathrm{sech}^4\gamma'\cdot[(\cos\varepsilon\;d\gamma')^2 + (\sin\varepsilon\;d\gamma')^2] = \mathrm{sech}^4\gamma'\cdot[(\overline{\overline{d\gamma'}})^2 + (\overset{\perp}{d\gamma'})^2] \equiv \\[2ex]
\equiv |d\mathbf{sin}\,\varphi(\gamma)|^2 = \cos^2\varphi\;d\varphi^2 + \sin^2\varphi\;d\alpha^2 = \cos^2\varphi'\;(d\varphi')^2 = \\
= \cos^2\varphi'\cdot[(\cos\varepsilon\;d\varphi')^2 + (\sin\varepsilon\;d\varphi')^2] = \cos^2\varphi'\cdot[\overline{\overline{d\varphi'}}^2 + \overset{\perp}{d\varphi'}^2] < 1;\\[2ex]
|d\mathbf{cos}\,\xi(\upsilon)|^2 = \sin^2\xi\;d\xi^2 + \cos^2\xi\;d\alpha^2 = \sin^2\xi'\;(d\xi')^2 = \\
= \sin^2\xi'\cdot[(\cos\varepsilon\;d\xi')^2 + (\sin\varepsilon\;d\xi')^2] = \sin^2\xi'\cdot[\overline{\overline{d\xi'}}^2 + \overset{\perp}{d\xi'}^2] < 1;\\[2ex]
\sin\varphi\cdot\mathbf{e}_\alpha(\gamma) = \sin\varphi_0\cdot\mathbf{e}_{\alpha(0)} + \int_{\varphi_0}^{\varphi}[\cos\varphi\;d\varphi\cdot\mathbf{e}_\alpha + \sin\varphi\;d\alpha\cdot\mathbf{e}_\nu].
\end{array}\right\} \qquad (192A)$$

Sine-tangent analogy acts for the 1-st differentials above, because angle's counting is executing off zero! Besides, $\gamma(\varphi) \leftrightarrow \varphi(\gamma)$ are *covariant parallel angles* in (26A), Ch. 1A. They are accompanied in (192A) by *contravariant parallel and complementary* angles $\xi(\upsilon) \leftrightarrow \upsilon(\xi)$. All relations between them were inferred in the end of Ch. 6. Similar simplest bond of complementary angles is peculiarity of the spherical geometry.

This decomposition of $d\mathbf{sin}\,\varphi$ is executed for trigonometric mapping of motions on a hyperspheroid into the ring, limited in the Euclidean projective plane by radius $R = 1$, by analogy with tangent projective model of principal motions in the hyperbolic non-Euclidean geometry (Ch. 12).

From (189A), for summing conventionally orthogonal particular spherical segments or motions, the scalar cosine multiplicative formula hold, with its generalization:

$$\cos\varphi_{13} = \cos\varphi_{12}\cdot\cos\varphi_{23}, \qquad (\varepsilon = \pm\pi/2),$$

$$\cos\varphi = \prod_{k=1}^{t}\cos\varphi_{(k)}, \quad \varepsilon_{ij} = \pm\pi/2, \quad 1 \le i,j \le t \le n, \quad i \ne j. \qquad (193A)$$

The final *scalar* angle φ and the distance $a = R\cdot\varphi$ do not depend on ordering of conventionally orthogonal particular angles. If all t orthogonal segments are infinitesimal, then the *Infinitesimal Pythagorean Theorem* holds for such (now non-conventionally) orthogonal infinitesimal spherical segments with the measure of Lambert φ.

For the sine of conventionally orthogonal motions sum, we obtain:

$$\sin^2\varphi_{13} = \sin^2\varphi_{12} + (\sin\varphi_{23}\cdot\cos\varphi_{12})^2 = \sin^2\varphi_{23} + (\sin\varphi_{12}\cdot\cos\varphi_{23})^2,\ (\varphi = l/R).$$

Suppose that $d\varphi_{12}$ is differential of an angular geodesic segment φ_{12} at a point M and $d\varphi_{23}$ is conventionally orthogonal to it in the point M differential. Then, for their geometric sum, we obtain the spherical 1-st metric Euclidean form on the hyperspheroid:

$$(d\varphi)^2_M = (d\varphi_{12})^2_M + \cos^2[\varphi_{12}]_M\ (d\varphi_{23})^2_{M'}\ (\varphi = l/R),$$

where $[\varphi_{12}]_M$ is the length of the segment "OM" as meridian in $\tilde{E}_1$, $d\varphi_{23}$ acts in $\tilde{E}_2$. The formulae for orthospherical shifts in sine-cosine variants are given below.

The vector sine is analogous to (135A) with direct (181A) and inverse (185A) orders of summing (and with further explicitly determined secondary orthospherical rotation):

$$\left.\begin{array}{l}\mathbf{sin}\ \varphi_{13} = \sin\varphi_{13}\cdot\mathbf{e}_\sigma = \\ = (\sin\varphi_{12}\cdot\cos\varphi_{23} + \cos\varepsilon\cdot\sin\varphi_{23}\cdot\cos\varphi_{12})\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot\sin\varphi_{23}\cdot\mathbf{e}_\nu, \\ \mathbf{sin}\ \varphi_{13} = \sin\varphi_{13}\cdot\mathbf{e}_{\underset{\sigma}{\angle}} = \\ = (\sin\varphi_{23}\cdot\cos\varphi_{12} + \cos\varepsilon\cdot\sin\varphi_{12}\cdot\cos\varphi_{23})\cdot\mathbf{e}_\beta + \sin\varepsilon\cdot\sin\varphi_{12}\cdot\mathbf{e}_{\underset{\nu}{\angle}}.\end{array}\right\}\quad(194A)$$

$$\mathbf{sin}\ \varphi_{13} = [\sin\varphi_{12}\cdot\cos\varphi_{23} - \cos\varepsilon\cdot\sin\varphi_{23}\cdot(1-\cos\varphi_{12})]\cdot\mathbf{e}_\alpha + \sin\varphi_{23}\cdot\mathbf{e}_\beta\ \text{(direct)}.$$

Projective sine measure $R\sinh\lambda/R$ may be used in the plane model of a hyperspheroid, which also follows the spherical Big and Small Pythagorean Theorems (see above).

Formula for the vector tangent is analogous to (138A) and given only for completeness:

$$\mathbf{tan}\ \varphi_{13} = \tan\varphi_{13}\cdot\mathbf{e}_\sigma = \left(\frac{\tan\varphi_{12} + \cos\varepsilon\cdot\tan\varphi_{23}}{1-\cos\varepsilon\cdot\tan\varphi_{23}\cdot\tan\varphi_{12}}\right)\cdot\mathbf{e}_\alpha + \left(\frac{\sin\varepsilon\cdot\tan\varphi_{23}\cdot sec\ \varphi_{12}}{1-\cos\varepsilon\cdot\tan\varphi_{23}\cdot\tan\varphi_{12}}\right)\cdot\mathbf{e}_\nu.\quad(195A)$$

We can use here the same formulae (136A) and (139A) for the vectors of directional cosines:

$$\mathbf{e}_\nu = \frac{\mathbf{e}_\beta - \cos\varepsilon\cdot\mathbf{e}_\alpha}{\sin\varepsilon} = \frac{\overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{e}_\beta}{||\overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{e}_\beta||},\quad \mathbf{e}_{\underset{\nu}{\angle}} = \frac{\mathbf{e}_\alpha - \cos\varepsilon\cdot\mathbf{e}_\beta}{\sin\varepsilon} = \frac{\overrightarrow{\mathbf{e}_\beta\mathbf{e}'_\beta}\cdot\mathbf{e}_\alpha}{||\overrightarrow{\mathbf{e}_\beta\mathbf{e}'_\beta}\cdot\mathbf{e}_\alpha||}.$$

We obtain: $\cos\theta_{13} = \mathbf{e}'_{\underset{\sigma}{\angle}}\cdot\mathbf{e}_\sigma$; $\mathbf{e}_\beta = \cos\varepsilon\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot\mathbf{e}_\nu \leftrightarrow \mathbf{e}_\alpha = \cos\varepsilon\cdot\mathbf{e}_\beta + \sin\varepsilon\cdot\mathbf{e}_{\underset{\nu}{\angle}}$,

$$\mathbf{e}'_\nu\cdot\mathbf{e}_{\underset{\nu}{\angle}} = -\cos\varepsilon = +\cos A_{123},\quad \mathbf{e}_\alpha\cdot\mathbf{e}_{\underset{\nu}{\angle}} = \mathbf{e}_\beta\cdot\mathbf{e}_\nu = +\sin\varepsilon = +\sin A_{123}.$$

Vectors $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$, $\mathbf{e}_\nu$, $\mathbf{e}_\sigma$, $\mathbf{e}_{\underset{\nu}{\angle}}$ x $\mathbf{e}_{\underset{\sigma}{\angle}}$ are inside an angle π in the plane $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle$.

Due to **General Signs Rule**, see in (182A) and in sect. 12.2, for spherical geometry we have: $\boxed{sgn\ \theta_{13} = +sgn\ \varepsilon\ !}$. If $\varepsilon > 0$, then $\theta_{13} > 0$, and if $\varepsilon < 0$, then $\theta_{13} < 0$, i. e., the leg 13 is shifted orthospherically in direction off the angle $A_{123} = \pi - \varepsilon$ always with increasing the sum of angles in the spherical triangle 123. Plane of this orthospherical rotation is $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle$. If $n = 3$, then vectors $\mathbf{e}_\alpha, \mathbf{e}_\beta, \overrightarrow{\mathbf{e}_N}$ and $\mathbf{e}_{\underset{\sigma}{\angle}}, \mathbf{e}_\sigma, \overrightarrow{\mathbf{e}_N}$ form the right ($\varepsilon > 0$) or left $\varepsilon < 0$ triples. They correspond to counter-clockwise scalar angles in *right-handed* bases. (Oriented vector $\overrightarrow{\mathbf{r}_N}(\theta) = \mathbf{e}_{\underset{\sigma}{\angle}}\otimes\mathbf{e}_\sigma = \pm\sin\theta\cdot\overrightarrow{\mathbf{e}_N}$ determines right screw of rotations if $n = 3$.)

* * *

Formula (143A) for $\cos\theta_{13}$ is transformed by similar way, as it was on hyperboloids (Ch. 7A). For two-step principal spherical motion, formula gives the angular *excess* of geodesic spherical triangle 123 on the hyperspheroid. For two conventionally orthogonal motions, we obtain the maximal by module this orthospherical rotation θ_{13}:

$$\cos\theta_{13} = \frac{\cos\varphi_{12} + \cos\varphi_{23}}{\cos\varphi_{12}\cdot\cos\varphi_{23} + 1} > 0, \quad \sin\theta_{13} = \frac{\pm\sin\varphi_{12}\cdot\sin\varphi_{23}}{\cos\varphi_{12}\cdot\cos\varphi_{23} + 1} \to d\theta = \frac{\pm\sin\varphi\ d\varphi}{1+\cos\varphi} = \pm\tan\ (\varphi/2)\ d\varphi.$$

As before, in infinitesimal considerations we shall apply the useful formulae for the cosine of the first angular differential (with exactness up to 2-nd power of differentials) $\boxed{\cosh d\varphi = 1 - (d\varphi)^2/2}$ and $\boxed{\cos d\theta = 1 - (d\theta)^2/2}$.

In both sine formulae (194A), put these values of angles: $\varphi_{12} = \varphi$, $\varphi_{23} = d\varphi$. The latter is the differential of an arc φ under angle ε to the segment φ. Further, similar to inferring hyperbolic formulae (144A) in Ch. 7A, with the use of the cosine formulae indicated above, and direct and inverse ordering variants of (194A) with angles φ and $d\varphi$, and also vectorial expression (499), if $n = 3$, we obtain the differential of the secondary orthospherical rotation angle as a result of the following vectorial product:

$$\mathbf{d}\theta = \frac{\mathbf{sin}\ \varphi}{1+\cos\varphi}\otimes d\varphi = \tan\frac{\varphi}{2}\cdot\overrightarrow{\mathbf{r}_N} = \pm\tan\frac{\varphi}{2}\cdot\sin\varepsilon\ d\varphi\cdot\overrightarrow{\mathbf{e}_N} = \pm\tan\frac{\varphi}{2}\ \overset{\perp}{d\varphi}\ \cdot\overrightarrow{\mathbf{e}_N}. \quad (196A)$$

It has positive values due to same directions of θ and ε, and vice versa. The angles φ and $d\varphi$ are expressed in the bases $\tilde{E}_1$ and $\tilde{E}_m$ of $\langle\mathcal{Q}^{n+1}\rangle$. Recall, that for two arcs, the *single* normal $\overrightarrow{\mathbf{e}_N}$ exists only in $\langle\mathcal{Q}^{3+1}\rangle$!) This *differential variant* of orthospherical rotation θ is useful in spherical geometry. So, for a triangle 123 in $\langle\mathcal{Q}^{3+1}\rangle$, formed by $d\varphi_{12}$ and $d\varphi_{23}$, with their external angle ε, we infer these formulae (see [16, p. 526]):

$$d\theta_{13}\cdot\overrightarrow{\mathbf{e}_N} = \pm\sin\varepsilon\cdot\frac{(d\varphi_{12})\cdot(d\varphi_{23})}{2}\cdot\overrightarrow{\mathbf{e}_N} = \pm\sin\varepsilon\cdot\frac{(da_{12})\cdot(da_{23})}{2R^2}\cdot\overrightarrow{\mathbf{e}_N} = \pm\ \frac{dS_{123}}{R^2}\cdot\overrightarrow{\mathbf{e}_N}.$$

Thus we got the differential interdependence $d\theta_{13}$ and the area of the triangle dS_{123}!

However, due to the Harriot's spherical result or, in general, to the Gauss–Bonnet Theorem [16, p. 533], the area of the geodesic triangle 123 (here on a surface of positive constant Gaussian curvature $K_G = +1/R^2 = \text{const}$) and the angular excess of the triangle $d\delta_{123} = \varepsilon_1+\varepsilon_2+\varepsilon_3-2\pi$ are connected as $d\delta_{123} = +dS_{123}/R^2 = K_G\ dS_{123} > 0$. As result, we get the differential and integral formulae for connection of these angles

$$d\theta_{13} = \pm d\delta_{123} = \pm\ \frac{dS_{123}}{R^2} = \pm K_G\ dS_{123}\ \Rightarrow\ \theta_{13} = \pm\delta_{123} = \pm\ \frac{S_{123}}{R^2} = \pm K_G\cdot S_{123}$$

in geodesic triangles on the hyperspheroid and, hence, in the other curvilinear spherical non-Euclidean spaces too. These formulae mean: the angle θ_{13} of orthospherical shifting and Harriot's angular excess $+\delta_{123}$ in a spherical triangle 123 are equal, as well as it was for Lambert's angular defect $-\delta_{123}$ in a hyperbolic triangle (Ch. 7A).

Inference of both these expressions consists in contour and surface integrating and applying their infinitesimal identity. This is the internal point of view on the hyperspheroid geometry. It (as well as any sphere) cannot be bent without loss of its metrical properties, and, hence, it is a surface of constant positive radius. (The same is valid for the hyperboloid II as a sphere of imaginary constant radius iR, see in Ch. 12.)

Orthospherical tensor rotation Θ_{13}, in accordance with tensor formulae (184A), (187A), is identical to *tensor angular excess* of a geodesic triangle on the hyperspheroid. Angular deviations (scalar and tensor) take place due to dependence of parallel displacement on a surface with curvature on its way. The scalar or tensor angular excesses are expressed through the orthospherical shift θ or Θ as the result of a closed cycle of geodesic motions along the triangle sides! Take into account analogous results in Ch. 7A, we formulate the following.

Corollary. *Orthospherical rotation Θ gives the clear mathematical explanation to the Harriot, Lambert and, in general, Gauss–Bonnet angular deviations in geometric figures in non-Euclidean geometries, including their spherical and hyperbolic types!*

The special case is summation of two step or multistep motions when particular angles are *infinitesimally small.* Suppose that, for example, in formulae (193A), (196A) with $n = 2$ both the principal spherical angles are infinitesimal. In particular, for right triangle 123 with $\cos\varepsilon = 0$, we obtain as $\varphi_{12} \to 0, \varphi_{23} \to 0$:

$$\varphi_{13} = \sqrt{\varphi_{12}^2 + \varphi_{23}^2}$$

$$\theta_{13} = \pm\,\frac{\varphi_{12}\cdot\varphi_{23}}{2} = \pm\,\frac{a_{12}\cdot a_{23}}{2R^2} = S_{123}\cdot K_G, \quad (\varepsilon = \pm\pi/2);$$

$$\delta_{123} = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) - 2\pi = [3\pi - (\pi - \delta_{123})] - 2\pi = S_{123}\cdot K_G > 0.$$

For k-step motion according to (193) the following generalization holds:

$$\lim_{\varphi_{(j)}\to 0} l = R\cdot\sqrt{\sum_{j=1}^{k}\varphi_{(j)}^2},$$

$$v = \varphi_{(1)}\cdots\varphi_{(k)}\cdot R^k, \qquad k \le n, \quad \varepsilon = \pm\pi/2.$$

They are the simplest infinitesimal formulae of the Euclidean geometry. This confirms the *infinitesimal character of Euclidean metric* on the hyperspheroid of radius R.

Corollary. *Geometry of the hyperspheroid is infinitesimally Euclidean.*

Commutativity of partial angles of motion (arcs) takes place in the scalar variant of conventionally orthogonal summation formulae. In particular, the first differential of the total angle arc is represented on the tangent n-dimensional Euclidean subspace to the n-dimensional hyperspheroid (as on the n-dimensional hyperboloid II in Ch. 7A):

$$(d\varphi)^2 = \sum_{k=1}^{n}[d\varphi_{(k)}]^2, \qquad (da)^2 = \sum_{k=1}^{n}[da_{(k)}]^2, \qquad \varepsilon_{(ij)} = \pm\pi/2, \qquad (197A)$$

According to the Big Pythagorean theorem (see it in sine versions: scalar (190A) and vectorial (194A)), for spherical geometry of the hyperspheroid, it is possible to use Cartesian subbase $\tilde{E}_1^{(n)}$ of the original base $\tilde{E}_1 = \{I\}$, as sine projective *homogeneous coordinates* in the Euclidean projective hyperspace $\langle\langle\mathcal{E}^n\rangle\rangle$, but only inside the trigonometric ball, for example, with radius $R = 1$ (similar to tangent one for hyperbolic geometry of the hyperboloid II in Ch. 12). The sine model of principal motions are preferred here, because they are bounded by finite parameter either 1 as trigonometric one or R as geometric one for considerations of geometric problems.

In $\langle\mathcal{Q}^{2+1}\rangle$, for analysis and interpretation of two-step motion on the hyperspheroid by differential method it is useful to apply decomposition of the total differential $d\varphi$ into two partial orthoprojections, parallel (along $\mathbf{e}_\alpha$) and orthogonal (along $\mathbf{e}_\nu$) ones with respect to the current vector of principal motions $\mathbf{e}_\alpha$ in the current base $\tilde{E}_m$. Put in sine spherically orthogonal decompositions (194A) and (190A) the angles values: $\varphi_{12} = 0, \varphi_{23} = d\varphi$ as 1-st and 2-nd principal spherical motions. By analogy with (145A), we have the vectorial decomposition, if $\mathbf{e}_\alpha \neq \mathbf{const}(da = Rd\varphi)$:

$$\left.\begin{aligned} d\varphi\cdot\mathbf{e}_\beta &= \cos\varepsilon\ d\varphi\cdot\mathbf{e}_\alpha + \sin\varepsilon\ d\varphi\cdot\mathbf{e}_\nu = \overline{\overline{d\varphi}}\cdot\mathbf{e}_\alpha + \overset{\perp}{d\varphi}\cdot\mathbf{e}_\nu \to (d\varphi)^2 = \left(\overline{\overline{d\varphi}}\right)^2 + \left(\overset{\perp}{d\varphi}\right)^2, \\ dl_\beta\cdot\mathbf{e}_\beta &= \cos\varepsilon\cdot dl_\beta\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot dl_\beta\cdot\mathbf{e}_\nu = \overline{\overline{dl_\beta}}\cdot\mathbf{e}_\alpha + \overset{\perp}{dl_\beta}\cdot\mathbf{e}_\nu \to dl_\beta^2 = \left(\overline{\overline{dl_\beta}}\right)^2 + \left(\overset{\perp}{dl_\beta}\right)^2. \end{aligned}\right\} \quad (198A)$$

It is $2D$ *Absolute Euclidean Pythagorean theorems* for orthogonal decompositions in the Cartesian base $\tilde{E}_m^{(2)}$ of $d\varphi\cdot\mathbf{e}_\beta$ and $dl\cdot\mathbf{e}_\beta$ with the use of formula (137A). Both these $2D$ theorems may be generalized in $3D$ *Absolute non-Euclidean Pythagorean theorems* in the quasi-Cartesian base $\tilde{E}_m$, with the use of analogous approach as in Ch. 10A.

* * *

Further we consider external vector trigonometry of the unity hyperspheroid.

Hyperspheroid (see Figure 4) has $R = 1$.

A point $n \times 1$-element on the hyperspheroid has the following unit radius-vector:

$$\mathbf{e}_k = \begin{bmatrix} \mathbf{sin}\ \varphi \\ \cos\varphi \end{bmatrix} = \begin{bmatrix} \sin\varphi\cdot\mathbf{e}_\alpha \\ \cos\varphi \end{bmatrix} \qquad (\varphi > 0 \text{ if } \Delta y > 0). \qquad (199A)$$

The metric invariant is

$$\mathbf{e}_k'\cdot\mathbf{e}_k = \mathbf{sin}'\ \varphi_{1k}\cdot\mathbf{sin}\ \varphi_{1k} + \cosh^2\varphi_{1k} = \sin^2\varphi_{1k}\cdot\mathbf{e}_\alpha'\mathbf{e}_\alpha + \cos^2\varphi_{1k} = 1 = 1^2. \quad (200A)$$

Here

$\mathbf{sin}\ \varphi_{1k}$ is the $n \times 1$-vector orthoprojection of $\mathbf{e}_k$ into $\langle\mathcal{E}^n\rangle^{(1)}$ parallel to $\overrightarrow{y}^{(1)}$,
$\cos\varphi_{1k}$ is the scalar orthoprojection of $\mathbf{e}_k$ into $\overrightarrow{y}^{(1)}$ parallel to $\langle\mathcal{E}^n\rangle^{(1)}$.

Also the following trigonometric functions will be used:
$\mathbf{tan}\ \varphi_{1k}$ is the cross 3×1-orthoprojection of $\mathbf{e}_k$ into $\langle\mathcal{E}^n\rangle^{(1)}$ parallel to $\overrightarrow{y}^{(k)}$,
$\sec\ \varphi_{1k}$ is the cross scalar orthoprojection of $\mathbf{e}_k$ into $\overrightarrow{y}^{(1)}$ parallel to $\langle\mathcal{E}^n\rangle^{(k)}$.

Consider geodesic motion $\mathbf{e}_2 \leftrightarrow \mathbf{e}_3$ of a point element on the unity hyperspheroid along two large circles in $\tilde{E}_1$ and $\tilde{E}_2$ with the following visual polar description:

$$= \{rot\ \Phi_{23}\}_{\tilde{E}_2} \cdot \overset{\mathbf{e}_2}{\begin{bmatrix} \sin\ \varphi_{12} \cdot \mathbf{e}_\alpha \\ \cos\varphi_{12} \end{bmatrix}} = \{rot\ \Phi_{23}\}_{\tilde{E}_2} \cdot rot\ \Phi_{12} \cdot \overset{\mathbf{e}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = \qquad (201A)$$

$$= \{rot\ \Phi_{12} \cdot (rot\ \Phi_{23})_{\tilde{E}_1} \cdot rot'\ \Phi_{12}\}_{\tilde{E}_2} \cdot rot\ \Phi_{12} \cdot \overset{\mathbf{e}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = rot\ \Phi_{12} \cdot rot\ \Phi_{23} \cdot \overset{\mathbf{e}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} =$$

$$= rot\ \Phi_{13} \cdot rot\ \Theta_{13} \cdot \overset{\mathbf{e}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = rot\ \Phi_{13} \cdot \overset{\mathbf{e}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = \overset{\mathbf{e}_3}{\begin{bmatrix} \sin\varphi_{13} \cdot \mathbf{e}_\sigma \\ \cos\varphi_{13} \end{bmatrix}}.$$

The trajectory of spherical geodesic motion $\mathbf{e}_2 \to \mathbf{e}_3$ is in the cut of the hyperspheroid by the quasiplane of rotation with matrix $\{rot\ \Phi_{12} \cdot rot\ \Phi_{23} \cdot rot'\ \Phi_{12}\}_{\tilde{E}_2}$. This means that the trajectory is constructed with continuous transformation $\mathbf{e} \to (\mathbf{e} + d\mathbf{e})$ accomplished with varying the scalar angle in the matrix $\{rot\ \Phi_{23}\}$ from 0 up to φ_{23} and with constant $\mathbf{e}_\beta$. Intersection of the plane of rotation with the projective hyperplane is a straight line segment in $\langle\langle\mathcal{E}^n\rangle\rangle$, it corresponds to this geodesic trajectory. A spherical triangle on a hyperspheroid can be easily implemented as a cycle of three geodesic motions. If the start apex is a central element $\mathbf{u}_1$, then

$$rot\ \Phi_{12} \cdot \mathbf{u}_1 = \mathbf{u}_2,\ \ \{rot\ \Phi_{12} \cdot rot\ \Phi_{23} \cdot rot'\ \Phi_{12}\}_{\tilde{E}_2} \cdot \mathbf{u}_2 = \mathbf{u}_3,\ \ \{rot\ \Phi_{31}\}_{\tilde{E}_3} \cdot \mathbf{u}_3 = \mathbf{u}_1.$$

The triple $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\rangle$ may be converted into a triangle with these transformation of coordinates. Thus *for any two points* $\mathbf{u}_2$ *and* $\mathbf{u}_3$ *on a hyperspheroid of radius R, there exists a unique geodesic line passing through them.* However, there is *a special case*: when certain two points of the hyperspheroid are polar (the Nord pole C_{II} at Figure 4 and South one). Such points produce digons. (They are polygons with two sides and two vertices.) This illustrates the following well-known theorem of spherical geometry: any two points of a semisphere or two nonpolar points of a spheroid may be connected by a unique arc of a large circle (a geodesic line), this arc is the shortest in the Euclidean length measure. Therefore, this gives the *matrix* way for solving the problem in the initial base $\tilde{E}_1$. In base $\tilde{E}_2 = rot\ \Phi_{12} \cdot \tilde{E}_1$, motion $\mathbf{u}_2 \to \mathbf{u}_3$ is going along the shortest arc with length $a_{23} = R \cdot \varphi_{23}$. By (201A), for a point element $\mathbf{u}$, orthospherical rotation Θ_{13}, in fact, annihilates. A triangle cycle of motions returns a nonpoint object into the start point, but this geometric object (or a real body) is turned in the base $\tilde{E}_3$ at angle Θ_{13}. Hence, the application point of this non-point object is transformed here as $\mathbf{e}_1 \to \mathbf{e}_2 \to \mathbf{e}_3$ along spherical geodesic lines φ_{12} and φ_{23}.

Summing two-step motion, due to polar decomposition *in the original base* $\tilde{E}_1$, is represented as the motion along geodesic line φ_{13} in direction $\mathbf{e}_\sigma$ and further orthospherical rotation *rot* Θ! See this in details above in (181A) and below in (202A).

* * *

Now, we describe in general form an algorithm for evaluating main characteristics of summed multistep motion in $\langle \mathcal{Q}^{n+1} \rangle$ and $\langle \mathcal{Q}^{2+1} \rangle \equiv \langle \mathcal{E}^2 \boxplus \overrightarrow{y} \rangle$ in the scalar, vectorial, and tensor forms. The algorithm starts with application of formula (485) for right transformation of the original unity base $\tilde{E}_1$. On the final step of the algorithm, the polar representation according to (474)–(476) and (111A)–(120A) is used. On these stages, with T and T^* from (183A), the homogeneous modal transformations are

$$\tilde{E}_t = rot\ \Phi_{12} \cdot rot\ \Phi_{23} \cdots rot\ \Phi_{(t-1),t} \cdot \tilde{E}_1 = T_{1t} \cdot \tilde{E}_1,$$

$$T_{1t} = rot\ \Phi_{1t} \cdot rot\ \Theta_{1t} = rot\ \Theta_{1t} \cdot rot\ \overset{\angle}{\Phi}_{1t}\ .$$

$$rot\ \overset{\angle}{\Phi}_{1t} = rot'\Theta_{1t} \cdot rot\ \Phi_{1t} \cdot rot\ \Theta_{1t} = rot\ (-\Theta_{1t}) \cdot rot\ \Phi_{1t} \cdot rot\ \Theta_{1t}$$

$$T_{1t} \cdot T^*_{1t} = rot^2\ \Phi_{1t} = rot\ 2\Phi_{1t}, \qquad T^*_{1t} \cdot T_{1t} = rot^2\ \overset{\angle}{\Phi}_{1t} = rot\ 2\ \overset{\angle}{\Phi}_{1t},$$

$$rot\ \Theta_{1t} = rot^{-1}\Phi_{1t} \cdot T_{1t} = rot\ (-\Phi_{1t}) \cdot T_{1t}.$$

(The latter is the *closed cycle of principal rotations with result rot* Θ_{1t}.) The matrices *rot* Φ_{1t} and *rot* Θ_{1t} are evaluated in canonical forms (313), (314) and (259), (497).

Quasipolar representation (177A), (178A) is used for inferring the general law of summing multistep motions or most general homogeneous rotations in this external spherical trigonometry of $\langle Q^{n+1} \rangle$ analogous to (153A–155A) in $\langle P^{n+1} \rangle$. We obtain:

Canonical formulae of Quasi-Euclidean homogeneous transformation, in that number, for summarized multistep principal motion:

$$T_{1t} = rot\ \Phi \cdot rot\ \Theta = rot\ \Theta \cdot rot\ \overset{\angle}{\Phi} = rot\ \Phi_{12} \cdots rot\ \Phi_{(t-1),t} = \qquad (202A)$$

$$= \left[\begin{array}{c|c} \cos\varphi \cdot \overleftarrow{\mathbf{e}_\sigma \mathbf{e}'_\sigma} + \overrightarrow{\mathbf{e}_\sigma \mathbf{e}'_\sigma} & -\sin\varphi \cdot \mathbf{e}_\sigma \\ \hline +\sin\varphi \cdot \mathbf{e}'_\sigma & \cos\varphi \end{array} \right] \cdot \left[\begin{array}{c|c} [rot\ \Theta]_{n\times n} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array} \right] =$$

$$= \left[\begin{array}{c|c} [rot\ \Theta]_{n\times n} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array} \right] \cdot \left[\begin{array}{c|c} \cos\varphi \cdot \overleftarrow{\mathbf{e}_{\overset{\angle}{\sigma}} \mathbf{e}'_{\overset{\angle}{\sigma}}} + \overrightarrow{\mathbf{e}_{\overset{\angle}{\sigma}} \mathbf{e}'_{\overset{\angle}{\sigma}}} & -\sin\varphi \cdot \mathbf{e}_{\overset{\angle}{\sigma}} \\ \hline +\sin\varphi \cdot \mathbf{e}'_{\overset{\angle}{\sigma}} & \cos\varphi \end{array} \right] =$$

$$= \left[\begin{array}{c|c} (1-\cos\varphi) \cdot \mathbf{e}_\sigma \mathbf{e}'_{\overset{\angle}{\sigma}} + [rot\ \Theta]_{n\times n} & -\sin\varphi \cdot \mathbf{e}_\sigma \\ \hline +\sin\varphi \cdot \mathbf{e}'_{\overset{\angle}{\sigma}} & \cos\varphi \end{array} \right].$$

Here $\mathbf{e}_\sigma \mathbf{e}'_{\overset{\angle}{\sigma}} = \cos\theta \cdot \overleftarrow{\mathbf{e}_\sigma \mathbf{e}'_{\overset{\angle}{\sigma}}}$ – see Ch. 5). If some *rot* Φ_{ij} are collinear, they are grouped. Formulae (202A) give the **General Law of summing principal motions** in $\langle \mathcal{Q}^{n+1} \rangle$ (in particular, on a sphere in the Euclidean space), expressed *rot* Φ in canonical forms (313) or (314) with respect to unity base $\{I\}$. The simplest case corresponds to $n = 2$, when the canonical structure of the matrix *rot* $\Theta_{3\times 3}$ is expressed with cell (259).

The matrix $R = rot\ \Phi$ is generated, for example, by the last element t_{nn} and all the right elements t_{kn} of matrix T in (202A). They permit one to express the matrix R in the base $\tilde{E}_1$ in canonical forms (313), (314) with the frame axis in $\langle \mathcal{Q}^{n+1} \rangle$ and evaluate scalar and vector trigonometric functions of the angle φ with its directional vector $\mathbf{e}_\sigma$. The matrix $rot\ \Theta$ may be computed in form (259) if $n \geq 2$, or (497) if $n = 3$ with the axis $\mathbf{e}_N$ and the sign of θ, or by matrix formula (184A). If $n = 2, k = 1, 2$, there hold:

$$\left.\begin{array}{l} \cos\varphi = t_{33}, \quad \sin\varphi = +\sqrt{1-\cos^2\varphi} = \| -\sin\varphi \cdot \mathbf{e}_\sigma \|; \ \sin\varphi_k = -t_{k3}; \\ \cos\sigma_k = -t_{k3}/\sin\varphi, \ \cos\overset{\angle}{\sigma}_k = t_{3k}/\sin\varphi, \ \mathbf{e}_\sigma = \{\cos\sigma_k\}, \mathbf{e}_{\overset{\angle}{\sigma}} = \{\cos\overset{\angle}{\sigma}_k\}. \\ \cos\theta = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\angle}{\sigma}} = \mathbf{e}'_{\overset{\angle}{\sigma}} \cdot \mathbf{e}_\sigma, \ \sin\theta = \sqrt{1-\cos^2\theta} > 0 \text{ at } \varepsilon > 0 \text{ and v. v.} \end{array}\right\} \quad (203A)$$

Besides, if $n = 3$, then we use formulae (499): $\overrightarrow{\mathbf{r}_N}(\theta_{13}) = \mathbf{e}_{\overset{\angle}{\sigma}} \otimes \mathbf{e}_\sigma = \pm\sin\theta_{13} \cdot \overrightarrow{\mathbf{e}_N}$.

The scalar final results do not change under the mirror permutation of particular motions. It leads only to the substitution in (202A): $T \to T^*$ with $\Theta \to -\Theta$, $\mathbf{e}_\sigma \to \mathbf{e}_{\overset{\angle}{\sigma}}$.

The specific matrix T^* in (185A) with contrary ordering of partial motions ($T^* \neq T'$, as $\Phi \neq \Phi'$, but $\Phi = -\Phi'$) has the general structure, gotten from T with $\mathbf{e}_\sigma \leftrightarrow \mathbf{e}_{\overset{\angle}{\sigma}}$:

$$T^* = rot\ \Phi_{23} \cdot rot\ \Phi_{12} = rot\ \overset{\angle}{\Phi} \cdot rot\ (-\Theta) = rot\ (-\Theta) \cdot rot\ \Phi = \{rot\ (-\Theta) \cdot T \cdot rot(-\Theta)\}$$

$$= \left[\begin{array}{c|c} (1-\cos\varphi)\cdot \mathbf{e}_{\overset{\angle}{\sigma}}\mathbf{e}'_\sigma + [rot\ (-\Theta)]_{2\times 2} & -\sin\varphi \cdot \mathbf{e}_{\overset{\angle}{\sigma}} \\ \hline +\sin\varphi \cdot \mathbf{e}'_\sigma & \cos\varphi \end{array}\right]. \quad (204A)$$

T and T^* are connected by simple transposing in original complex binary base (271), where they both are Hermitian symmetric (see at beginning of the Chapter).

Corollary. *Multistep non-collinear spherical motion T_{1t} in the space $\langle \mathcal{Q}^{n+1} \rangle$ or on the hyperspheroid is represented as a spherical rotation and further orthospherical one* Such interpretation of law (202A) for summing spherical motions is confirmed in the quasi-Euclidean space by the fact, that $rot\ \Theta$ is revealed in the base $\tilde{E}_{1s} = rot\ \Phi_{1t} \cdot \tilde{E}_1$ by polar decomposition.

First real steps in creating hyperbolic geometry were made by J. H. Lambert [33] and F. A. Taurinus [36]. Lambert assumed its analogy (as $-i\varphi \to \gamma$) with geometry as if on the sphere of imaginary radius iR. Taurinus constructed its first *cosine model* with formula (189A) on a such hypothetic sphere. Later F. Klein [42] and H. Minkowski [49] proved that this hypothetic geometric object is the upper hyperboloid II in $\langle \mathcal{P}^{n+1} \rangle$. Nicolai Lobachevsky [37] and János Bolyai [39] created independently the hyperbolic geometry in sufficiently complete forms by Euclid's axiomatic method. Unfortunately, the Lobachevsky–Bolyai metric plane and space on the whole are unvisual for men.

In the Chapter, laws of hyperbolic geometry motions established in Chs. 5A and 7A were transformed by spherical-hyperbolic analogy (323) $i\Gamma \leftrightarrow \Phi$ into spherical ones! The polar representations were inferred in analogous forms in quasi-Euclidean tensor trigonometry with the use of analogy (322) $-i\Phi \leftrightarrow \Gamma$. Between two types of geometries and tensor trigonometries, we use the analogy $\Phi \leftrightarrow -i\Phi \leftrightarrow \Gamma \leftrightarrow i\Gamma \leftrightarrow \Phi$ entirely!

"Everything must be made as simple as possible. But not simpler." – Albert Einstein.

Chapter 9A

Real and observable space-time in the general relativity[1]

The *Special Theory of Relativity* (STR) formulates the laws of relativistic movement of matter in inertial and uninertial systems under abstract condition that gravitation is supposed to be absent – see, for example, in [53]. The absolute motion takes place in a macroworld and a microworld and does not depend on a nature of active forces. In Chs. 1A–7A, we used tensor trigonometry for describing laws of the motion in clear trigonometric forms. In 1905 H. Poincaré made a revolutionary step: he suggested the idea of united complex-valued space-time with pseudo-Euclidean metric based on Lorentzian transformations of space-time coordinates *as the space group* [47; 53, p. 107] (this idea was not noticed by contemporaries). Poincaré introduced the *imaginary* time coordinate and its scale coefficient c. (The real speed of light can sometimes differ from c, but it never exceeds c.) Later, in 1907–1908 H. Minkowski suggested detailed *real-valued* model of such pseudo-Euclidean space-time [49; 53, p. 41]. He introduced into the relativistic theory the notions of time-like and space-like intervals, isotropic cone, and others. His ideas were quickly accepted, the scientific ground was prepared.

Earlier, problems of inertia and gravitation origin were set in Newtonian mechanics and his theory of gravitation. In order to explain a nature of inertia phenomenon, I. Newton postulated so called *absolute space and absolute time*. So, as a consequence, he gave the absolute character to the notions of inertia and acceleration.

Newton's point of view was criticized in 1883 by E. Mach [87]. However Mach's theory merely gave the concrete sense to such "absolute space and time" by bonding them with the star system of the Universe. Mach's theory, contrary to the Newton's approach, had the qualitative and philosophic character. According to Mach, inertia and acceleration are determined relatively to a certain inertial frame of reference $\tilde{E}_0$. This system is, in its turn, connected with the barycenter of the Universe that is immovable with respect to $\tilde{E}_0$ (*the Mach Principle*). Then inertia and acceleration are regarded absolute as in Newton's approach. The Mach frame of reference $\tilde{E}_0$ determines the infinite set of Galilean inertial pseudo-Cartesian bases $\langle \tilde{E}_j \rangle$. Cartesian subbases $\tilde{E}_j^{(3)}$ move in $\langle \mathcal{E}^3 \rangle^{(0)}$ rectilinearly and uniformly, gravitation is as if absent.

While elaborating the *General Theory of Relativity* [51] in 1913-1916, A. Einstein paid attention to empiriocritical Mach's regards on gnoseology of celestial mechanics uniting dynamics and gravitation, especially on the Law of Gravitational and Inertial Mass Identity, used implicitly. As a usual Principle of Equivalence it holds in classical and relativistic forms. Gravitational acceleration does not depend on substance nature, this was established by I. Newton and confirmed with high precision by L. Eötvös.

[1] The chapter is given in this shortening english variant only as discussional intermezzo between two last chapters.

A. Einstein had proposed the General Principle of Relativity, inside Galilean one, according to which the Laws of Nature have generally covariant forms in any free frames of reference. As a consequence, for its realization into GTR, he introduced in addition the General Principle of Equivalence of inertia an gravitation. This led to curving relativistic basis space-time of GTR. However, the bend relates not only to time-arrow, but and to Euclidean subspace's geometry with its material objects!?

Another explanation of the both masses identity law is closer to E. Mach's approach. For a material body, the Newtonian force of attraction is caused by *active* gravitational action of other objects, while the force of inertia is caused by *passive* gravitational influence of the whole Universe. In the base associated with the barycenter of the body, both the forces are proportional to its mass as a real-valued "gravitational charge". In this interpretation, the 2-nd Newtonian Law of mechanics complements naturally his Gravity Law. In order to obtain its geometric expression in $\langle \mathcal{P}^{3+1} \rangle$, we pass from acceleration to its proportional analog in (99A) – a pseudocurvature of a world line:

$$-F_{(i)} = F = m_0 \cdot g = m_0 \cdot c^2/R_K = E_0/R_K \to R_K = 1/K = E_0/F, \qquad (205A)$$

F is the inner (i. e., applied in the current base $\tilde{E}_m$) active force causing bending trajectory of the *absolute motion* of a material point M in $\langle \mathcal{P}^{3+1} \rangle$;
$F_{(i)}$ is the passive force of inertia counteracting to F in $\tilde{E}_m$;
m_0 and E_0 are the own mass and the own Einsteinian energy of a material point;
R_K is the radius of instantaneous absolute pseudo-Euclidean curvature of the world line at the point M in the *osculating pseudoplane* $\langle \mathcal{P}^{1+1} \rangle_K$;
c is the constant module of 4-*pseudovelocity* of a material point absolutely moving along its world line in $\langle \mathcal{P}^{3+1} \rangle$ (this characteristic was first introduced by H. Poincaré [47], and it has imaginary directional unity vector $\mathbf{i}$).

From "energetic formula" (205A), we have $E_0 = F \cdot R_K$ as the torque of the active force F causing local pseudo-Euclidean rotation of the world line ($F = 0 \leftrightarrow R_K = \infty$). For each body moving rectilinearly with acceleration, "gravitational interpretation" of inertia as in formula (205A) means that $F_{(i)}$ is the centripetal force always directed towards the instantaneous center of a hyperbola (*pseudocircle*) tangential to a world line in $\langle \mathcal{P}^{3+1} \rangle$. Recall here, that as long ago as in the 15-th century Nicholas of Cusa (Nicolaus Cusanus) noted: "The Universe is a sphere, and its Center is everywhere!"

Energetic gravitational form (205A) of the 2-nd Newtonian Law is in accordance (and it is necessary) with his 1-st and 3-rd ones:

$$F = 0 \;\leftrightarrow\; g = 0 \;\leftrightarrow\; K = 0, \; -F_{(i)} = +F.$$

If a material point is subjected to simultaneous actions of a few of active forces with different directions and, may be, different nature, then these forces and generated by them accelerations are summarized as vectors in Euclidean subspace $\langle \mathcal{E}^3 \rangle^{(m)}$ of $\langle \mathcal{P}^{3+1} \rangle$:

$$-\mathbf{F}_{(i)} = \mathbf{F} = \sum_{j=1}^{t} \mathbf{F}_j = \sum_{j=1}^{t} m_0 \cdot \mathbf{g}_j = m_0 \cdot \mathbf{g} \quad \to \quad \mathbf{g} = \sum_{j=1}^{t} \mathbf{g}_j. \qquad (206A.)$$

The partial vectors of absolute hyperbolic pseudocurvature at the same point are summarized by similar *geometric way*, and with the common direction pure additively:

$$\mathbf{k} = \sum_{j=1}^{t} \mathbf{k}_j \quad (k = \sum_{j=1}^{t} \pm k_j,\ \mathbf{e}_\alpha = \text{const}). \tag{207A}$$

According to (207A), at a point M of a world line for a material body, the pseudocurvatures are summarized covariantly to active proper forces and inner accelerations.

Consequently, if these inner forces at the point are coaxial, then these pseudocurvatures are trigonometrically compatible (their eigen pseudoplanes are identical), pseudocurvatures as well as angles (Rule 2 in sect. 5.7 and in sect. 6.2) are algebraically additive. However, if these causing them inner forces are non-coaxial, but applied to the same point M, then the pseudocurvatures and inner accelerations are summarized by geometric Euclidean way as vectors in $\langle \mathcal{E}^3 \rangle^{(m)}$. It is the principal distinction between nonrelativistic Euclidean summing *inner* 4-accelerations in $\tilde{E}_m$ and relativistic non-Euclidean summing different (collinear and non-collinear) physical 3-velocities in $\tilde{E}_1$.

Spherical curvature has similar properties. So, for the radius of curvature of light's way, there holds the additive optical Newtonian formula, with the same consequence:

$$1/R_1 + 1/R_F = 1/R_2, \quad \rightarrow \quad k_1 + k_F = k_2\ (\mathbf{e}_\alpha = \text{const}), \tag{208A}$$

where R_F is the focal distance of a lens or a mirror, it is either negative, or positive. The formula may be applied repeatedly at summation points of a certain light ray as the optical axis (each time). If the light ray is reflected by a mirror, then in addition the direction and the sign of curvature are changed by inverse ones (for a flat mirror, the absolute value of curvature stays the same). We will generalize (207A) in Ch. 10A.

In STR, from the point of view of a *Galilean-inertial* Observer N_j situated in the Euclidean subspace $\langle \mathcal{E}^3 \rangle^{(j)}$, any *accelerated* frame of reference $\tilde{E}_m$, as an *instantaneous base*, preserves formally its inertiality in $\langle \mathcal{P}^{3+1} \rangle$: i. e., $\tilde{E}_m \in \langle \tilde{E}_j \rangle$. This fact was used in Chs. 5A–7A. However, for an accelerated Observer N_m, situated in the current Euclidean subspace $\langle \mathcal{E}^3 \rangle^{(m)}$, its frame of reference noted further as $\tilde{\mathbf{E}}_m$ is Galilean-uninertial with respect to $\langle \tilde{E}_j \rangle$! Mathematically, it is a Gaussian curvilinear coordinate system in the enveloping and absolute space-time $\langle \mathcal{P}^{3+1} \rangle$. Thus we have the *relativistic dualism* and two ways (simplest and complex) for describing accelerated movement in $\langle \mathcal{P}^{3+1} \rangle$. Such a dualism was considered, for example, in [61, p. 121-128].

In $\tilde{\mathbf{E}}_m = \{\tilde{\mathbf{x}}, c\tau\}$ coordinates are curvilinear. Mapping $\tilde{E}_j \leftrightarrow \tilde{\mathbf{E}}_m$ is isomorphism. In particular, for hyperbolic motion of Observer N_m as the center of $\tilde{\mathbf{E}}_m$ (Chs. 5A, 6A), this curvilinear coordinate grid contains in the base $\tilde{E}_1$ the time axis due to (88A) and the space axis due to (106A) of $\tilde{\mathbf{E}}_m$, connected as $\tilde{x} = x^{(m)} = R \cdot \ln\cosh\,(c\tau/R)$.

The connection between the coordinates in the bases $\tilde{E}_m$ and $\tilde{\mathbf{E}}_m$, is expressed also by a smooth function, that is why differentials $d(c\tau)$ and $d\tilde{x}_k$ in $\tilde{\mathbf{E}}_m = \{\tilde{\mathbf{x}}, c\tau\}$ are homogeneous linear functions depending on $dx_k^{(m)}$ and $d(c\tau^{(m)})$ in $\tilde{E}_m = \{\mathbf{x}^{(m)}, c\tau^{(m)}\}$, this is equivalent to the one-valued connection of differentials as $d\tilde{\mathbf{u}} = V_{(i)}^{-1} d\mathbf{u}^{(m)}$.

The arc of a world line at a point M, as invariant scalar element in $\langle\mathcal{P}^{3+1}\rangle$, may be evaluated by these two ways, either in $\tilde{E}_m$, or in $\tilde{\mathbf{E}}_m$:

$$[d(c\tau)]^2 = [d\mathbf{u}^{(m)}]' \cdot I^{\pm} \cdot d\mathbf{u}^{(m)} = d\tilde{\mathbf{u}}' \cdot \{V'_{(i)} \cdot I^{\pm} \cdot V_{(i)}\} \cdot d\tilde{\mathbf{u}} = d\tilde{\mathbf{u}}' \cdot G^{\pm}_{(i)} \cdot d\tilde{\mathbf{u}}.$$

The matrix of local linear transformation $V_{(i)}$ is uniquely determined by this general congruent representation of the *metric tensor of inertia* (see also in sect. 11.1):

$$G^{\pm}_{(i)} = R' \cdot D^{\pm} \cdot R = (\sqrt{D^{\oplus}} \cdot R)' \cdot I^{\pm} \cdot (\sqrt{D^{\oplus}} \cdot R) = V'_{(i)} \cdot I^{\pm} \cdot V_{(i)}.$$

Thus *the initial metric of the basis space of events is preserved under passage into accelerated bases.* In the flat Minkowskian space-time $\langle\mathcal{P}^{3+1}\rangle$, applying Gaussian curvilinear coordinates of the base $\tilde{\mathbf{E}}_m$ for inner analysis of accelerated motions formally leads to the use of Ricci tensor calculus without change of topology of the space-time.

So, in uninertial and inertial bases, differentials of their coordinates for the same arc are affine-connected, this connection is determined by variable tensor $G^{\pm}_{(i)}$ in the Minkowskian space-time (the so-called *metric tensor of inertia*). The tensor acts as a function in coordinates of an arbitrary point M. *It is important that tensor of Riemannian–Christoffelian curvature for $G^{\pm}_{(i)}$ is zero here, as this basis space-time is flat.* In an accelerated frame of reference, bending the coordinate grid takes place just *relatively to Observer N_m*. He is always in the center of his own instantaneous base $\tilde{\mathbf{E}}_m$. But *Galilean-inertial* Observer N_j notices no bend of coordinates $\mathbf{x}$ and $\overrightarrow{c\tau}$ with respect to the instantaneous frame of reference $\tilde{E}_m$ wherever N_m is in $\tilde{E}_m$. In particular, a rod moving with acceleration together with Observer N_m is seen by N_j as rectilinear, since for the Observer at any points of $\tilde{E}_m$ the metric tensor is $I^{\pm}$. However, *uninertial* Observer N_m in $\tilde{\mathbf{E}}_m$ can see the exactly same rod in $\tilde{E}_j$ bent. This relativistic effect has the coordinate nature. There are no additional mechanical stretches in this rod merely *seemed* bent, as the same *active inner forces* may be expressed in any inertial frame of reference and they are identical as absolute characteristics in $\langle\mathcal{P}^{3+1}\rangle$. The metric tensor $G^{\pm}_{(i)}$ is used for representing the quadratic form of a metric interval in the basis space as the scalar product of differentials. Such tensor is determined also in terms of a linear element $\tilde{\mathbf{u}}$ differentials with respect to its mutual coordinates:

$$[dl]^2 = d\mathbf{u}'_{con} \cdot d\mathbf{u}_{cov} \equiv d\mathbf{u}'_{cov} \cdot d\mathbf{u}_{con} = d\mathbf{u}'_{con} \frac{d\mathbf{u}_{cov}}{d\mathbf{u}_{con}} d\mathbf{u}_{con} = d\mathbf{u}'_{con} G_{(i)} d\mathbf{u}_{con} \equiv$$

$$\equiv d\mathbf{u}'_{cov} \frac{d\mathbf{u}_{con}}{d\mathbf{u}_{cov}} d\mathbf{u}_{cov} = d\mathbf{u}'_{cov} \hat{G}_{(i)} d\mathbf{u}_{cov}, \quad \hat{G}_{(i)} = G^{-1}_{(i)}.$$

In accelerated frame of reference $\tilde{\mathbf{E}}_m$, we have distorted Minkowskian geometry with basis $\langle\mathcal{P}^{3+1}\rangle$, the variable metric tensor of inertia $G^{\pm}_{(i)}$, and the zero tensor of the Riemannian-Christoffelian curvature (sect. 11.1). Christoffelian symbols in $\tilde{\mathbf{E}}_m$ play a role of *tensor* analogue of the absolute *vector* acceleration. *As the important inference for describing in both types of coordinates of the same motion in $\langle\mathcal{P}^{3+1}\rangle$ by this natural way, there hold two metric*, though in STR gravitation is not taken into account!

If gravitation is present, then N_j in $\tilde{E}_j$ fixes the distortion of $\tilde{E}_m$ coordinates too, with the metric tensor $G^{\pm}$ (in the 2-nd order), as N_j and N_m are divided by a field of gravitation. The cardinal reason for this distortion of $\tilde{E}_m$ coordinates is that in real cosmic space there is only the Einsteinian tool of estimating geometric and temporal parameters: it is a light ray between the object and its Observer (usually on the Earth in a weak gravitational field). A light ray, due to changes in the potential of the field in a light's path, is subjected to corresponding bending, even due to Newtonian theory of gravitation. The idea of accepting rays of light as straight lines (or geodesics in GTR) in cosmic space was taken by A. Einstein from the experiment of great Carl Gauss with his students (as the head of the astronomical observatory in Göttingen) with measuring the sum of the angles of a triangle formed by three mountain peaks. And, as is often the case in science, it was necessary to solve the dilemma: either what is observed and measured using light rays should be taken for reality (a positivist approach, for which Ernst Mach was the most known philosopher), or the same should not always be considered as real assessment of the present, but as its mapping with possible distortions from the applied measurement remedy (an objectivist approach). A. Einstein accepted the first point of view, and as a result of which, a curvature of the real space-time $\langle \mathcal{R}^{3+1} \rangle$ with its time arrow and *real geometric objects* (?) arose. Then for N_j the tensor of Riemannian–Christoffelian curvature is non-zero. The dualism in description of the same motion by N_j and by N_m is essentially widen, and two scalar products are one-valued functions one of another. In the space-time $\langle \mathcal{R}^{3+1} \rangle$ there is no this deviation of light rays, because in it these rays are straight lines. In the space-time $\langle \mathcal{P}^{3+1} \rangle$ this deviation of light rays is fixed with respect to its pseudo-Euclidean straight lines in $\tilde{E}_j$, and then to reveal this deviation, it is not necessary to refuse basis $\langle \mathcal{P}^{3+1} \rangle$.

An objectivist dualism with difference in descriptions of relativistic movements in a field of gravitation was realized in the modifications of GTR as *Bimetric Theories of Gravitation* (BMT), where both metric tensors $I^{\pm}$ of the Minkowskian space-time and $G^{\pm}$ of the pseudo-Riemannian space-time act with the same signature. All BMT kinds do not refuse the Minkowskian space-time as GTR and use it in a different degree.

The first BMT was constructed in 1940–1975 by Nathan Rosen [75] – Albert Einstein assistant and colleague! In his variant of BMT, metric tensor $I^{\pm}$ describes in $\langle \mathcal{P}^{3+1} \rangle$ as in STR the inertial part connected with absolute matter motion. Further, the tensor of energy–momentum for a field of gravitation is evaluated, it characterizes this field by $G^{\pm}$, which determines $\langle \mathcal{R}^{3+1} \rangle$ with the pseudo-Riemannian geometry for Observers. Geometric parameters and time of an object moving in basis $\langle \mathcal{P}^{3+1} \rangle$ are own. Under translation into $\langle \mathcal{R}^{3+1} \rangle$ by Observer on the Earth, the time slows down; but *geometric parameters are as if distorted, as really kinetic distortion of objects is impossible*. We have a *paradox* in BMT like apparent optical curving a light picture seen through a lens, where $G^{\pm}$ *is a gravitational lens for* $\langle \mathcal{R}^{3+1} \rangle$. The term is used in Astronomy [78], when large cosmic objects are observed on the Earth through a strong field of gravitation.

Riemannian geometry has a differential character, defined initially through the symmetric metric tensor of its space, as the matrix function of a point element. So, it may be the Riemannian space $\langle\mathcal{R}^m\rangle$. The Riemannian space has always some internal local geometry. But the Riemannian geometry as a whole differs significantly from homogeneous geometries, such as quasi- and pseudo-Euclidean geometries, in which the concepts of *group of motions*, *freedom of motion of figures*, and *topological properties* are of particular importance. For the Riemannian space as a whole with its indefinite topology, the notion of "embeddability" with respect to the Euclidean superspace does not make any sense. This causes the uncertainty for it of the minimum dimension of the enveloping superspace n_{min}. But if we restrict ourselves to the study of any topologically affine-equivalent domain of the Riemannian m-dimensional space, then the value of n_{min} is determined entirely by its local differential-geometric properties.

The symmetric tensor of $\langle\mathcal{R}^m\rangle$ contains a maximum of $k = m{\cdot}(m{+}1)/2$ independent functional scalar elements g_{ij}. Hence, the domain $\mathcal{D}$ of the Riemannian m-space is embeddable in flat $\langle\mathcal{E}^k\rangle$ without changing internal geometry. This was inferred strictly by E. J. Cartan [79]. Consider an analytical definition of $\mathcal{D}$ in the superspace $\langle\mathcal{E}^n\rangle$, where $n \geq k$, with its Cartesian base through $n \times 1$-radius-vector $\mathbf{u}$ with m degrees of freedom of translations. Let each degree of freedom $\mathbf{u}$ corresponds to the Gaussian curvilinear coordinate v_t of the Riemannian m-space. Then there is a functional map $\mathbf{v}(\mathbf{u}) \leftrightarrow \mathbf{u}(\mathbf{v})$. At an each point $\mathbf{v}$ of $\mathcal{D}$ in $\langle\mathcal{R}^m\rangle$ there exists $n \times m$ Jacobi matrix $d\mathbf{u}/d\mathbf{v}$ $(n > m)$ as continuous 1-st derivative of $\mathbf{u}$ in $\mathbf{v}$. The internal geometry of $\mathcal{D}$ is defined through the homomultiplication as the $m \times m$ metric tensor of $\langle\mathcal{R}^m\rangle \subset \langle\mathcal{E}^n\rangle$:

$$d\mathbf{v}' \cdot G^{+} \cdot d\mathbf{v} = d\mathbf{u}' \cdot d\mathbf{u} \;\Leftrightarrow\; G^{+} = \left\{\frac{d\mathbf{u}}{d\mathbf{v}}\right\}' \cdot \left\{\frac{d\mathbf{u}}{d\mathbf{v}}\right\},\; det\ G^{+} \neq 0\ (\mathbf{v}, \mathbf{u} \in \mathcal{D}).$$

For $\langle\mathcal{R}^{3+1}\rangle$ due to A. Friedman [80] there is $10D$ space of embedding $\langle\mathcal{P}^{9+1}\rangle$, and then

$$d\mathbf{v}' \cdot G^{\pm} \cdot d\mathbf{v} = d\mathbf{u}' \cdot I^{\pm} \cdot d\mathbf{u} \;\Leftrightarrow\; G^{\pm} = \left\{\frac{d\mathbf{u}}{d\mathbf{v}}\right\}' \cdot I^{\pm} \cdot \left\{\frac{d\mathbf{u}}{d\mathbf{v}}\right\},\; det\ G^{\pm} \neq 0\ (\mathbf{v}, \mathbf{u} \in \mathcal{D}).$$

For the complete functional independence of all k elements of the symmetric metric tensors, it is necessary that the inequality $n \geq k$ holds. Obviously, *in the case of an equal sign, this independence is realized only with the affine topology of the given Riemannian space.* Otherwise, they are connected by some parameters. So, Cartesian coordinates of a sphere are connected by its radius R. For $n > k$, the Gaussian Theorem Egregium allows to lower the order of embedding of a bounded domain of the Riemannian m-space to at least $n_{min} = k$ using bending. By this way, an isomorphic translation of the motions described in k-dimensional pseudo-Euclidean space, but within m-dimensional pseudo-Riemannian space embedded in it, is carried out. For the observational pseudo-Riemannian space-time $\langle\mathcal{R}^{3+1}\rangle$, it is $n_{min} = 10$. $\langle\mathcal{P}^{9+1}\rangle$ can be a flat space-time for mapping motions in a gravitational field by Observer in a weak field. (See more in [17, p. 290-293] and in [85].) Without the field we have $n_{min} = 4$.

However, only for simplest kinds of gravitational fields, it is not difficult to identify actions of gravitation and inertia [84, pp. 233, 264]. A uniformly accelerated rectilinear physical movement under the action of a homogeneous gravitational field is mathematically equivalent to the hyperbolic motion in $\langle\mathcal{P}^{3+1}\rangle$ under the action of a constant tangential inner force. The circular physical movement under the action of a spherically symmetric gravitational field is mathematically equivalent to the pseudoscrewed motion in $\langle\mathcal{P}^{3+1}\rangle$ under the action of a constant normal force. The Einsteinian idea of the strong Principle of Equivalence arose as a basis of GTR, apparently, from here.

Consider the *energetic analogy* between both Einsteinian time dilations for two accelerated movements: in STR under some inner force (Ch. 5A) and in GTR or BMT under decreasing potential with applying differential cosine form (171A) from Ch. 7A.

$$\left.\begin{aligned} &d\cosh\gamma = d\frac{d(ct)}{d(c\tau)} = g_{(i)}d\chi/c^2 = F_{(i)}d\chi/(m_0\cdot c^2) = dE_{(i)}/E_0,\\ &\frac{d(ct)}{d(c\tau)} = \cosh\gamma = 1 + A_{(i)}/(m_0\cdot c^2) = 1 + \Delta E_{(i)}/E_0 > 1. \end{aligned}\right\} \tag{209A}$$

Proper time τ mathematically corresponds to proper time $\overset{\bullet}{\tau}$ in a stationary field of gravity with the constant equivalent intensity $g_{(f)} \equiv g_{(i)}$. Time $\overset{\bullet}{\tau}$ is expressed in $\tilde{E}_m$, as the inertial mass and the gravitational mass m_0 in $\langle\mathcal{P}^{3+1}\rangle$ are equal, and $P < 0$:

$$\left.\begin{aligned} &d\frac{d(ct)}{d(c\,\overset{\bullet}{\tau})} = g_{(f)}d\chi/c^2 = F_{(i)}d\chi/(m_0\cdot c^2) = dE_{(f)}/E_0 = d(-P)/c^2,\\ &\frac{d(ct)}{d(c\,\overset{\bullet}{\tau})} = 1 + A_{(f)}/(m_0\cdot c^2) = 1 + \Delta E_{(f)}/E_0 = 1 + (-P)/c^2 > 1. \end{aligned}\right\} \tag{210A}$$

Comparing (209A) and (210A), one can see that inertia and gravitation in the simplest case are in fact mathematically and energetically identical as both mass are identical. However, Albert Einstein decides that the *Principle of Equivalence* in GTR has the most general sense independent on complexity of a relativistic movement and a gravity field. (Authors of RTG, as the kind of BMT, do no agree with this opinion [60].) From (210A) the well-known estimation for the *Einsteinian gravitational local proper time dilation* follows:

$$\frac{d(c\,\overset{\bullet}{\tau}_1)}{d(c\,\overset{\bullet}{\tau}_2)} = [1 + (-P_2)/c^2]/[1 + (-P_1)/c^2] \approx 1 + [(-P_2) - (-P_1)]/c^2. \tag{211A}$$

If $P_1 = 0 = \max$, then $\overset{\bullet}{\tau}_1 = t^{(1)}$, it is nonrelativistic coordinate time with respect to N_1 on the Earth. But proper time $\overset{\bullet}{\tau}$ evaluated at a point M is slower, because its potential is negative. Formula (211A) with the Newtonian potential at the point M gives rather precise estimation in the near-Solar space:

$$d(c\,\overset{\bullet}{\tau}) = d(ct)/[1 + f\cdot M_0/(r\cdot c^2)] \approx d(ct)\cdot[1 - f\cdot M_0/(r\cdot c^2)]. \tag{212A}$$

Hence, the "gravitational twins paradox" is possible too (for STR, see in Chs. 3A, 5A).

So, in GTR, but with more logical *negative signature for the time-arrow* of W. Pauli [53, p. 206-213], i. e., as initially was introduced by Poincaré in 1905, this Einsteinian local gravitational dilation of proper time (here near the Sun) is laid in the angular element g_{44} of the metric tensor $G^{\pm}$ in $\langle\mathcal{R}^{3+1}\rangle$:

$$\frac{d(c\,\overset{\bullet}{\tau})}{d(ct)} = \sqrt{-g_{44}} \approx 1 - (-P_S)/c^2 < 1 \;\Leftrightarrow\; g_{44} = -[1 - (-P_S)/c^2]^2 \;\rightarrow\; \overset{\bullet}{c} = c\cdot\sqrt{-g_{44}} < c,$$

where $P_S = -(fM_0)/r$, $\overset{\bullet}{c} < c$ is the local speed of a light by Einstein. (In STR we have $g_{44} = -1$, Ch. 1A.)

In BMT, the Riemannian distortion of space-time coordinates in the Sun's field may be interpreted by N_1 on the Earth (in its weak field) as the observable one as a result of mapping by the gravitational lens from the variable potential of the field in a neighborhood of an each point M in $\tilde{E}_m$, when translation from $\langle \mathcal{P}^{3+1} \rangle$ into $\langle \mathcal{R}^{3+1} \rangle$ is made, see also in sect. 11.1. In the gravitation field with decreasing potential, the world-lines $\overrightarrow{c\tau}$ in such $\langle \mathcal{R}^{3+1} \rangle$ are mapped as dilated. Under decreasing potential, the oscillations frequency ν in a light ray gone of outside (!) counterwise increases with increasing each photon kinetic energy $h\nu$. Since the oscillations quantity is invariant in the equivalent time intervals as $Q = \dot{\tau} \cdot \dot{\nu} = t \cdot \nu$, then its wave length decreases:

$$c\,\dot{\tau}\,/(ct) = \nu/\,\dot{\nu} = \dot{\lambda}\,/\lambda \approx 1 - (-P)/c^2 < 1 \;\rightarrow\; (c = \nu \cdot \lambda = \dot{\nu} \cdot \dot{\lambda} = \text{const}). \quad (213A)$$

The photon as a particle of light (they are Newton's corpuscles) was introduced again by Einstein to interpret the photoeffect' laws. Evaluate with an *energetic–refractional approach* the deflection of a light ray near the Sun (so, see an analogy with the optical light refraction in [84, p. 308]). Suppose that a photon of mass m moves with respect to an astronomical mass M at velocity $\mathbf{v}$ under angle ε to the radius-vector $\mathbf{r}$ from the barycenter of M. The current centripetal *inner* force of gravitation F applicated to the particle m is invariant in Galilean bases from $\langle \tilde{E}_j \rangle$. If $M >> m$, then the particle at each moment get the differential of circular movement in $\langle \mathcal{E}^3 \rangle$ around the Sun. By the Newtonian Gravitation Law and STR Laws of dynamics in $\tilde{E}_m$, there holds

$$\mathbf{F} = F \cdot \mathbf{e}_\beta = [(f \cdot M \cdot m)/r^2] \cdot \mathbf{e}_\beta = [(m \cdot c^2)/R] \cdot \mathbf{e}_\beta = \overset{\perp}{F} \cdot \mathbf{e}_\nu + \overline{\overline{F}} \cdot \mathbf{e}_\alpha. \quad (214A)$$

Here the inner force and inner acceleration are decomposed into normal and tangential projections. In the quasi-Cartesian base $\tilde{E}_1$, we have the Euclidean projections of $\mathbf{F}$:

$$\overset{\perp}{F} = \sin\varepsilon \cdot [(f \cdot M \cdot m)/r^2] = (m_0 \cdot c^2) \cdot \overset{\perp}{K} = (m_0 \cdot c^2)/\,\overset{\perp}{R} = (m \cdot v^2)/\rho, \text{(see Ch. 10A)},$$

$$\overline{\overline{F}} = \cos\varepsilon \cdot [(f \cdot M \cdot m)/r^2] = (m_0 \cdot c^2) \cdot \overline{\overline{K}} = (m_0 \cdot c^2)/\,\overline{\overline{R}} = d(mv)/dt, \text{ (see Ch. 5A)}.$$

These orthoprojections of F and K are summarized with the Pythagorean Theorem. The tangential projection causes acceleration of the particle m_0 along the velocity $\mathbf{v}$. For a photon, this projection merely increases or decreases its oscillation frequency and energy E_L during motion. Hence, this projection does not influence on the spherical Newtonian deviation of a light ray. Contrary, the normal projection, as a classical centripetal force, causes such a bend of the light ray trajectory with its local radius ρ.

Therefore, for photons, projections of the gravitational force are the following:

$$\overset{\perp}{F} = \sin\varepsilon \cdot [(f \cdot M \cdot m_L)/r^2] \approx (m_L \cdot c^2)/\rho = E_L/\rho \quad (\text{ as } \rho \approx \overset{\perp}{R}\,,\ v = c\ ),$$

$$\overline{\overline{F}} = \cos\varepsilon \cdot [(f \cdot M \cdot m_L)/r^2] = (m_L \cdot c^2)/\,\overline{\overline{R}} \approx \frac{d(m_L c)}{dt} = c \cdot \frac{dm_L}{dt} = \frac{dE_L}{d(ct)};$$

$\dot{m}_L \approx m_L = h\nu/c^2$ is the mass of a moving photon. We have: $1/\rho = \sin\varepsilon (fM)/(rc)^2$.

In theory [51, 84, p. 351-355], $b = \text{const}$ is the distance between center of M and the intersection point of ray asymptotes: $b \approx r{\cdot}\sin\varepsilon \approx \min(\mathrm{r})$, then $1/\rho = \sin^3\varepsilon (fM)/(bc)^2$. Thus a light ray bend is calculated in the following differential and integral forms:

$$d\delta_I = dl/\rho \approx d(-r{\cdot}\cos\varepsilon)/\rho = b\,d(-\cot\varepsilon)/\rho = [fM/(bc^2)]{\cdot}\sin\varepsilon\,d\varepsilon = [-P(\varepsilon)/c^2]\,d\varepsilon,$$

$$\delta_I \approx [fM/(b\cdot c^2)]\cdot\int_0^\pi \sin\varepsilon\,d\varepsilon = 2fM/(b\cdot c^2) = 2\cdot(-P_{min})/c^2.$$

Just this first estimation was obtained by Albert Einstein in 1911 [53, p. 202]. However Johann von Soldner was the historically first, who evaluated it in 1801 [74; 78, p. 7]. In fact, this estimation follows from Newton's gravitational and corpuscular theories. Moreover, Isaac Newton forecasted discovery of this effect for light corpuscles in 1704.

In 1915, Einstein evaluated the general-relativistic correction for a light ray bend in a spherically symmetric gravitation field, with the use of Absolut Tensor Calculus. New value was proved to be twice larger. In order, to estimate separately 2-nd term of a light ray bend, we use the *mathematical analogy* of light propagation in the optic medium with the variable refraction index and in the field of gravitation with variable waves length of photons due to the change of their kinetic energy $h\nu$, caused by the proportional change of the field potential. In the case, we form the instantaneous angle of incidence ε by the speed of light *vector* $\mathbf{c}$ and the radius-vector $\mathbf{r}$ (directed as the field intensity $g_{(f)}$). The angle is ε if ε is acute, and it is $\pi - \varepsilon$ if ε is obtuse. The oscillations frequency of photons ν increases in the 1-st part of the trajectory and decreases in its 2-nd part due to (213A). By analogy with the W. Snellius Law (1626), this may be interpreted as *additional* bend of a light ray towards the barycenter of M: $\sin\varepsilon/\sin(\varepsilon - d\delta_{II}) = \frac{c+dc}{c}$, $\varepsilon < \pi/2$; $\sin(\pi-\varepsilon)/\sin(\pi-\varepsilon+d\delta_{II}) = \frac{c-dc}{c}$, $\varepsilon > \pi/2$; $\rightarrow$
$\rightarrow d\delta_{II} = \tan\varepsilon\,dc/c = \overset{\perp}{dc}/c > 0$ for both the cases; dc is a parallel projection of $d\mathbf{c}$.
$d\delta_{II} = \overset{\perp}{dc}/c = \overset{\perp}{g}\,d\tau/c = \overset{\perp}{g}\,dl/c^2 \equiv \overset{\perp}{g}_{(f)}\,dl/c^2 = \tan\varepsilon\,g_{(f)}\,dr/c^2 = \tan\varepsilon\,d[-P(\varepsilon)/c^2] \approx$
$\approx \tan\varepsilon\,d[(fM\cdot\sin\varepsilon)/(bc^2)] = [fM/(bc^2)]\cdot\sin\varepsilon\,d\varepsilon = [-P(\varepsilon)/c^2]d\varepsilon \equiv d\delta_I$.
We obtain the twice deviation of a photon only from the Sun scalar potential change iff $c = \nu\cdot\lambda = \text{const}$, but no for vector $\mathbf{c}$ (!), with a photon's full energy conservation:
$\delta = \delta_I + \delta_{II} = 4fM/(bc^2) = 4(-P_{min}/c^2) = 4(-P_S/c^2)\cdot(r/b)$ – under $c = \text{const}$.
Under the conception used above, in both these cases, $P(\varepsilon) = P_{min}\cdot\sin\varepsilon$. We have:
$\overset{\perp}{dc}(\varepsilon) = 2\tan\varepsilon\,d[-P(\varepsilon)/c] = 2\sin\varepsilon(-P_{min}/c)\,d\varepsilon = -2P(\varepsilon)/c\,d\varepsilon \rightarrow dc = 2d(-P/c)$.
The second part $d(-P) = c\,dc$ at $c = \text{const}$ is not transformed here into $d(c^2/2)$, then a gravitational potential at any world point can not be determined by the value of the *local speed of light* measured at this point by some manner. This corresponds to the *General Principle of Relativity in the case of conserving basis space-time* $\langle\mathcal{P}^{3+1}\rangle$. Scalar speed of light c in the cosmic vacuum is equal to the Poincaré scale coefficient!
$dE = -P_{min}\cdot\sin\varepsilon\,d\varepsilon\cdot m_0$, $E_{max} - (-P_{min}\cdot m_0) = E_0$, $h\nu_{max} = h\nu_0\cdot[1 + (-P_{min}/c^2)]$.
Finally, the photon's energy is preserved, momenta vectors $\mathbf{P}_0, \mathbf{p}$ change only direction.

For a ray of light along the central axis from the Star to the center of mass M, there is no similar gravitational deviation at all (as for an optical spherical lens too). The physical reason for this is that the normal deviating projection of the gravitation force is zero. The work of parallel positive or negative projection, equal to the force itself, turns into a positive or negative change of the kinetic energy of photons as $\pm\Delta h\nu$. The variant is realizing for another extreme general relativistic effect. It is the well-known "red shift" of the Sun radiation spectrum, predicted in 1913 by Albert Einstein too. It is caused by slowing-down of all electromagnetic oscillations from the Sun surface due to its very strong negative potential [84, p. 346-347]. Let us pay special attention to the fact that the assessment of this effect is confirmed precisely on the Earth, that is, by an external Observer in a weak gravitational field! Due to (212A, 213A), it is

$$\frac{\dot{\lambda}}{\lambda} = \frac{\nu}{\dot{\nu}} = \frac{d(c\,\dot{\tau})}{d(ct)} = \frac{d(\dot{\tau})}{d(t)} \approx 1 - \frac{fM}{r}/c^2 < 1 \Rightarrow \lambda > \dot{\lambda}\ ,\ (\nu\lambda = \text{const} = c). \quad (215A)$$

This "red shift" was precisely affirmed directly on the Earth in 1959 by R. Pound and Jt. Rebka with the use of Mössbauer's effect [77]. They confirms with high precision the Inertial and Gravitational Mass Identity Law too with gamma-particles. (In this experiment, the difference of potentials was very small.) This "red shift" effect was predicted first in 1783 by John Michell in his letter to the London Royal Society [58]. The letter contains also the first prediction of "black holes" in the Universe with evaluations of their parameters based on Newton's corpuscular and gravitation theories!

We can interpret this "red shift" with an *energetic part* of the used conception. Photons or other massless particles under negatively acting gravitation get decreasing kinetic energy $E = h\nu$ with increasing the wave length $\lambda = c/\nu = h/(E/c) = h/p$ for an Observer on the Earth. (For massless particles, there holds $E = pc$ as $E_0 = 0$.) For a body, we have *equivalent* decreasing total energy E and $pc = mvc$ as $E_0 = \text{const}$ with increasing De Broglie wave length $\lambda = h/p = h/(mv)$. Then this effect is explained clear by the Law of Energy Conservation, *if it is compatible with gravitation theory.* Evaluate the effect for the Sun radiation with the use of quantum-mechanical approach:

$$E_L = \dot{E}_L - (-P_S)\cdot m_L = h\,\dot{\nu} - (-P_S)\cdot m_L = h\nu < h\,\dot{\nu}\ ,\ \Rightarrow \quad \nu < \dot{\nu}\ , \qquad (216A)$$

where $m_L = h\nu/c^2$ is the Planck–Einstein formula for the mass of a moving photon, $\dot{\nu}$, $\dot{\lambda}$ are the local values on the Sun surface; ν, λ are the values on the Earth. Energetic approach (216A) were first noted by Max Born [73]. He did not develop this idea and rested the local GTR interpretation of the effect. Indeed, due to this Law of Nature, while photons get farer from the Sun to the Earth, its initial local kinetic energy and frequency must decrease due to overcoming the negative Sun potential in direction to the Earth. Without the *Doppler effect*, the speeds of light near both these objects are: $\dot{c} = \dot{\nu} \cdot \dot{\lambda}$ – on the Sun, and $c = \nu \cdot \lambda =$ on the Earth at $h = \text{const}$.

From (216A), we obtain: $\nu = c/\lambda \ < \ \overset{\bullet}{c} \,/\, \overset{\bullet}{\lambda} = \overset{\bullet}{\nu}$. Further we have only two variants:
(1) $\lambda > \overset{\bullet}{\lambda} \Rightarrow \overset{\bullet}{c} = c$ – it is correct variant, the effect "red shift" is fixed on the Earth;
(2) $\lambda = \overset{\bullet}{\lambda} \Rightarrow \overset{\bullet}{c} > c$ – it is uncorrect variant. The case $\overset{\bullet}{c} < c$ is absent in (216A) at all!

One must choose either the correct variant (1), where it is $\overset{\bullet}{c} = c =$ const, or choose even the non-existing here variant $\overset{\bullet}{c} < c$ and *refuse in general relativistic theory of the Law of Energy Conservation.* We choose variant (1). It corresponds to strictly inferred relation (215A). The radiation on the Sun in its strong gravitational field has the original frequency (as radiated on the Earth or without gravitation at all). Hence, this general effect is *observational* one, and it is perceived on the Earth through the gravitational lens $G^{\pm}$ as non-local one, due to this *external gravitational time dilation.*

Interchange the source of radiation and Observer. Then, according to the Principle of Relativity, Observer in the strong gravitational field will see inverse "blue shift" of the Earth source radiation spectrum with the equivalent increase of photons energy.

Due to the Quantum Mechanics approach, a red slowing-down of De Broglie wave length must take place too, these waves are generated by any moving massive or massless particle. Their length λ must increase, due to overcoming the Sun potential.

This energetic approach with presence or absence of refractional curving a light ray in these two extreme cases shows wonderful compatience of STR, BMT, Quantum Mechanics, and Law of Energy Conservation in interpretations of the GR-effects!

It is usually believed that the third GR-effect "perihelion shift of Mercury" can be explained exclusively by GTR, but with interpretation of type: *it is equations solution.* Specific interpretation of this effect with introduction of all possible dilations of time is simplest and clear. So, in $\langle \mathcal{P}^{3+1} \rangle$ it may be caused by three equivalent time dilations.
1. The relativistic kinematic factor $v^*/c = \cosh\gamma \cdot v/c$, which dilates the observational time of the Mercury motion on the Earth proportionally to $\cosh\gamma$.
2. The Einsteinian factor $[1 + fM_0/(rc^2)] = \cosh\gamma$ as the *gravitational time dilation.*
3. The relativistic dynamical factor $m \cdot fM_0/r = \cosh\gamma \cdot m_0 \cdot fM_0/r$, with the additionally increased time dilation by the same coefficient $\cosh\gamma$.
The Mercury proper time $d\,\overset{\bullet}{\tau} \approx dt/cosh^3\gamma$ is decreased relatively to the Earth time dt on the value $3(\cosh\gamma - 1)dt$. As a result, its perihelion is shifting with the real coefficient $3 \times (2\pi R)$, that now nobody physically understands. Karl Schwarzschild (1916) in the frame of GTR introduced similar changes of coordinate time [84, p. 326, 348], and, in his new coordinates, realized the so called *exact solution* for this bias, predicted without inferring in first Einsteinian article on GTR [51]. In Chs. 5A we showed: even change $dt \to d\tau$ leads to the loss of two- and multi-step principal operations $\langle roth\ \Gamma_k \rangle$. Then in the homogeneous Lorentzian group only the orthospherical rotations remain.

It is interesting that with such simplest approach to interpretations of GR-effects by different kinds of proper time dilations, it was not necessary to reduce the local speed of light c and to curve the space-time $\langle \mathcal{P}^{3+1} \rangle$, but only additionally curve a world-line!

There is an undeniable fact: some GR-effects in the Solar system are fixed by Observers on the Earth in a weak field of gravitation, *but occur locally in a strong field of gravitation near the Sun.* Therefore, their real description must have corresponding dualism from these two points of view including bimetric variant. GTR, according to its equations, gives only single physical interpretation of these effects [84, p. 346-356], i. e., as seen by Earth Observers without taking into account that local information about them must reach him through the decreasing field of gravitation. Above we have shown, with the simplest descriptive approach to the question, that single interpretation inevitably leads *to the violation of the Law of Energy Conservation.* Similar inference may be suggested also by the works in publications of some very eminent authors.

The historical statement by David Hilbert, as first author of the movement equations in GTR [52] (1917), becomes clear: "I assert... that for the general theory of relativity, i. e., in the case of general invariance of the Hamiltonian function, energy equations... corresponding to the energy equations in orthogonally invariant theories do not exist at all. I could even take this circumstance as the characteristic feature of the general theory of relativity." [59]. However, it has not been recognized by the physical community for a long time. This violation has a simple mathematical explanation. Bases of GTR do not contain the ten-parametric group of motions presenting in the Minkowskian space-time. This takes place due to transformation of movement in a gravitational field in bent pseudo-Riemannian space-time not connected with $\langle \mathcal{P}^{3+1} \rangle$ (see [61, p. 163]).

That is why, D. Hilbert, yet in the beginning of 1915, set the task for his famous colleague Emmy Noether: to find full conditions for fulfilling this Law of Nature. As a result of it solution, in the same 1915, she proved the fundamental theorem of mathematical physics, connected the Integral Law of Conservation of Energy and Momentum for material motions with parameters of a space-time symmetry (published in 1918). The *general* pseudo-Riemannian space is nonhomogeneous and anisotropic. Hence the Law cannot hold in it. The curved space-time cannot have even constant curvature, as it depends on hierarchical casual mass distribution.

As well-known, in addition, V. Fok proved that predictions of GTR concerning general relativistic effects in the Solar system are ambiguous [56]. They depend on coordinate conditions. Change of the initial base leads to non-covariant change of these effects. In order to make estimations more definite, Einstein considered the general relativistic effects as they are in a weak stationary gravitational field as if embedded into the Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$ [60, p. 156–165]. Such an artificial approach did not fix the problem.

A. Logunov with colleagues from MSU had created RTG kind of BMT [60], where $\langle \mathcal{P}^{3+1} \rangle$ was preserved, but the equations of motions are formulated in the so-called *effective Riemannian space-time*, identical in fact to our observable space-time, i. e., they developed the main idea of N. Rosen of two metrical tensors.

In 2004 with publication in Russia of the 1-st edition of the Tensor Trigonometry [17] at the same time, eminent English scientist Roger Penrose, professor of Mathematics at the University of Oxford, wrote in [81]: "We seem to have lost those most critical conservation laws of physics, the laws of conservation of energy and momentum! In fact, there is a more satisfactory perspective on energy-momentum conservation, which refers also to certain curved space-times M as well as to Minkowski space ... These conservation laws hold only in a space-time for which there is the appropriate symmetry, given by the Killing vector $\mathbf{k}$. Nevertheless, they do not really help us in understanding what the fate of the conservation laws will be when gravity itself becomes an active player. We still have not regained our missing conservation laws of energy and momentum, when gravity enters the picture." Anything to add to this clear unambiguous conclusion is not required!

Moreover, numerous attempts to combine GTR with the laws of Quantum Mechanics, have not yielded significant results. These attempts are hindered by the curvature of a real space-time in it. But if in some way to return in theory of gravitation the flat basis space-time with homogeneity and isotropy (variants of this were considered above), then this problem can be solved quite naturally using the Dirac approach [55].

The main conception of Einsteinian GTR is expressed by his General Principle of Relativity as the Postulate: *All physical laws in free arbitrary moving frames of reference must have locally standard forms determined by metric tensor* $I^{\pm}$ (as if in STR). Strictly speaking, this Postulate is a hypothesis, while it is not confirmed experimentally convincing enough, for example, by experiment with a free horoscope in a space orbit. We discussed some contradictions in GTR. Thus, STR is valid in GTR only infinitesimally, and hence the certain Mach's base $\tilde{E}_0$ [87] is refused. Frames of reference free-moving in presence of gravitation became equivalent. This was expressed in well-known extreme, but scientifically honest Einstein's statement on equal rights of Kopernik's and Ptolemy's systems. Since he abandoned the world material ether, starting at construction of STR, then the answer to the cardinal question was remained dark: what is curved in GTR – as if *non-material space* or as supposed its *more convenient coordinates* chosen for describing motions?

We can classify *general relativistic effects in the Solar system*, taking for illustration 3 well-known ones. The *single effect* of the one-time "red shift" of the Sun radiation as a result of the photons overcoming the strong negative gravitational potential of the Sun on the way to Observer, fixing this shift on the Earth. The *double effect* of the twice "deviation of a light ray" after passing near the Sun as a result of summing deviations by gravity and refraction of a ray in the changing strong field, and fixing by Observer on the Earth. The *triple effect* of the "perihelion shift of Mercury" as a result of three different reasons but leading to the triple equal time dilations, and fixing also by Observer on the Earth.

Wee see, that GTR does not have any other tools for revealing these general effects, but only by using Minkowskian space-time with its Euclidean subspace both for mathematical comparison and experimental observation, which it itself and denies!? Therefore, $\langle \mathcal{P}^{3+1} \rangle$ exists really and together with the Mach Principle!

Nathan Rosen's BMT [75] (*he was Einsteinian colleague!*), with metric tensors $I^{\pm}$ and $G^{\pm}$ of basis $\langle \mathcal{P}^{3+1} \rangle$ and observable $\langle \mathcal{R}^{3+1} \rangle$, is in compatibility with the *Principles of correspondence, causality, uniqueness*, with the Law of Energy-Momentum Conservation, and with the Quantum mechanics (as STR). BMT may interpret simply the second metric tensor $G^{\pm}$ as the gravitational lens creating $\langle \mathcal{R}^{3+1} \rangle$.

Such dualism of BMT approach may be used in explicit descriptions of relativistic motions in the real space-time with a field of gravitation: firstly, as local ones, in 4-dimensional Minkowskian space-time, and secondly, as observable ones, in $\langle \mathcal{R}^{3+1} \rangle$ or principally in $10D$ flat space-time $\langle \mathcal{P}^{9+1} \rangle$, may be, with the use of the tensor trigonometry methods. (See about such idea in [80], [17] and [85]). It is quite logical that BMT varieties give the same estimates for GR-effects as GTR of the first order in the gravitational constant f. Hence, the idea of Nathan Rosen [75] of two metric tensors may be very fruitful and closer to the reality.

BMT with basis $\langle \mathcal{P}^{3+1} \rangle$ leads to Euclidean topology of the observational space-time with properties of *endlessness and infinity.* Ones argue so: the infinite space-like part of this 4-dimensional world must have, according to H. Olbers' paradox (1826), the light night sky, contrary to the finite world of radius R. But the mathematical infinity of $\langle \mathcal{P}^{3+1} \rangle$ does not mean the infinity of the material world mass. It may be limited. Besides, the Hubble Law in its ancestral form $-\Delta\nu/\nu = -\Delta h\nu/h\nu = Hl/c = Ht$ with this interpretation of the constant H only connects the relative "red shift" and the distance or time till a galaxy. From discovery in 1929, it is interpreted and used for inferring Theory of Expanding Universe with some acceleration. However, the Law may have other interpretations too. So, this "red shift" may express the lack of the photons energy proportionally to their long way of the motion from a galaxy to the Earth due to some cosmic friction, may be very small, – why not? Then the photons lose energy with decreasing frequency and increasing wave length. As a result of similar interpretation, the need for the so called *dark energy* to justify the hypothesis of the universe expansion with acceleration is absent! On the other hand, how can apologists of a finite space-time place in it the endless time-arrow without violating the Principle of determinism?

A priori a certain geometry of the real space-time in the large was not discussed here. For our opinion, the complete knowledge of its global structure in principle cannot be achieved. Illusions of complete knowledges in mathematics were broken by the Gödel's Theorems. In the theoretical physics, the idea about transcendent nature of the Universe is not yet understood. Albert Einstein has suggested the whole Universe homogeneity and isotropy, *but only in the mean*, although as a hypothesis! Alternative Rosen's point of view [75], *as it should be in free science*, continues to be developed by new independent researchers in [60], [85], [86], etc. We use Minkowskian space-time with tensor trigonometry in the small, in geometric sense without a distance.

As a result, it is now possible to adopt reasonably the following important inferences.

If we consider various relativistic motions exclusively locally as if in the real space-time including a gravitational field, but with the basis Minkowskian space-time, where $c = dx^{(k)}/dt^{(k)} = \text{const}$ (as in BMT), then it is possible, with fairly high degree of accuracy (as was shown above), to study and describe these motions with their kinematic and dynamic characteristics at a local level directly in this basis space-time.

All kinds of motions in $\langle \mathcal{P}^{3+1} \rangle$ are divided into absolute ones with absolute parameters and their relative projections with relative physical parameters. An absolute motion is mapped by a world line in $\langle \mathcal{P}^{3+1} \rangle$ in pseudo-Cartesian coordinates with admitted values of their slope to the isotropic cone. A world line has an important physical feature as its *dynamical* character. This enable one to determine absolute geometric and physical parameters of the motion along its world line. Relative physical movements are the pseudo-Euclidean projections of an absolute motion into the Euclidean subspace $\langle \mathcal{E}^3 \rangle$ and onto the time-arrow $\overrightarrow{ct}$.

Chapter 10A

Motions along world lines in $\langle \mathcal{P}^{3+1} \rangle$ and their geometry

Locally, each material point M, in particular, the barycenter of a material object, is *permanently absolutely moving* along its world line in the space-time $\langle \mathcal{P}^{3+1} \rangle$ due to Hermann Minkowski [49]. It is easiest to being analyze a curved world line with an increase in its complexity sequentially in pseudo-Cartesian bases of the space-time $\langle \mathcal{P}^{1+1} \rangle$, $\langle \mathcal{P}^{2+1} \rangle$, $\langle \mathcal{P}^{3+1} \rangle$ respectively for rectilinear, flat and space physical movements. A world line is *geometric invariant* of Lorentzian transformations of the base and a continuous regular curve, with 4×1 radius-vector $\mathbf{r}(c\tau)$, embedded in the 4-dimensional homogeneous and isotropic space-time. The inexorable absolute motion, limited by the slope of a world line to the time arrow, ensures its regularity. Physically its trajectory is the locally oriented proper time-arrow $\overrightarrow{c\tau}$ of M. The scalar integral value of proper time along a world line does not depend on a pseudo-Cartesian base too. By their slope $d\mathbf{r}$ – Figure 2A, the world lines relate only to the internal cavity of the light cone. For descriptivity and visuality, we analyze world lines with the initial pseudo-Cartesian base $\tilde{E}_1 = \langle \mathbf{x}, \overrightarrow{ct} \rangle$, where their slope corresponds, due to concrete tangent–tangent analogy (sect. 6.4), to the visual spherical angle φ_R : $\tanh\gamma \equiv \tan\varphi_R$. In a neighborhood of its point M, the world line with its orientation and configuration is completely determined by four current *scalar and 4-vector differential-geometric characteristics* corresponding to four dimensions of the space-time $\langle \mathcal{P}^{3+1} \rangle$. Scalar characteristics are invariant under homogeneous Lorentzian transformations. Such construction is based, mainly, on the Frenet–Serret absolute rotational approach to the differential theory of regular curves in $\langle \mathcal{E}^3 \rangle$ [16], when they are supposed to be embedded in homogeneous and isotropic space of a fixed dimension.

The hyperbolic angle γ of motion with directional cosines is defined in these relations

$$\left.\begin{array}{l}
\mathbf{r}^{(1)}(c\tau) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ ct \end{bmatrix} = \begin{bmatrix} \mathbf{x}(c\tau) \\ ct(c\tau) \end{bmatrix}, \quad d\mathbf{r} = \begin{bmatrix} d\mathbf{x}(c\tau) \\ dct(c\tau) \end{bmatrix}; \quad \mathbf{i}(\gamma, \mathbf{e}_\alpha) = \dfrac{d\mathbf{r}}{d(c\tau)} = \begin{bmatrix} \sinh\gamma \cdot \mathbf{e}_\alpha \\ \cosh\gamma) \end{bmatrix}(c\tau); \\
\mathbf{sinh}\,\gamma = \dfrac{d\mathbf{x}}{d(c\tau)} = \sinh\gamma\cdot\mathbf{e}_\alpha = \dfrac{\mathbf{v}^*}{c}, \quad \mathbf{tanh}\,\gamma = \dfrac{d\mathbf{x}}{d(ct)} = \tanh\gamma\cdot\mathbf{e}_\alpha = \dfrac{\mathbf{v}}{c} = \mathbf{sinh}\,\gamma/\cosh\gamma; \\
\gamma = \operatorname{arsinh}\dfrac{\sqrt{dx_1^2+dx_2^2+dx_3^2}}{d(c\tau)} = \operatorname{artanh}\dfrac{\sqrt{dx_1^2+dx_2^2+dx_3^2}}{d(ct)} > 0, \text{ as } d(ct) > 0; \\
\mathbf{e}_\alpha = \{\cos\alpha_k\},\ k = 1,2,3; \quad \cos\alpha_k = \dfrac{dx_k}{||d\mathbf{x}||_E}; \quad w_\alpha^* = d\alpha/d\tau,\ w_\alpha = d\alpha/dt.
\end{array}\right\} \quad (217A)$$

In particular, the so-called uniform absolute motions $\mathbf{r} = \mathbf{r}(c\tau)$ are of especial interest. Among them, the physically most important are the following three types:
the uniform rectilinear movement at $\gamma = \text{const}$, $\mathbf{e}_\alpha = \mathbf{const}$ (Chs. 1A–4A);
the uniformly accelerated rectilinear movement at $\eta_\gamma = \text{const}$, $\mathbf{e}_\alpha = \mathbf{const}$ (Ch. 5A);
the circular movement of a body with velocities $\mathbf{v}$ and w at $\gamma = \text{const}$, $w_\alpha = \text{const}$.

In the space-time $\langle\mathcal{P}^{3+1}\rangle$ with its metric tensor $\{I^{\pm}\}$, the 4×4 tensor of motion $roth\ \Gamma^{(m)} = F(\gamma, \mathbf{e}_\alpha)$, introduced in (100A), determines along a world line the absolute base $\tilde{E}_m^{(4)} = roth\ \Gamma^{(m)}\cdot\tilde{E}_1$, and also the local hyperbolic inclination Γ of a curve, with respect to the time arrow $\overrightarrow{ct}$, and the local Euclidean orientation $\mathbf{e}_\alpha$ of the curve, with respect to the 3 Cartesian axes in $\langle\mathcal{E}^3\rangle$. This tensor is defined at the current point M in the base $\tilde{E}_1 = \{I\}$ by canonical structure (362) or (363). The change $d\Gamma$ causes locally the change of inclination $d\gamma$ and the change of orientation $d\mathbf{e}_\alpha = \mathbf{d}\alpha$ for the curve in $\langle\mathcal{P}^{3+1}\rangle$. Such local changes can have different orders from 1 to 4. Generally, a world line in $\langle\mathcal{P}^{3+1}\rangle$ can have at the point M maximum four *absolute parameters* of orders up to 4, completely defined its local orientation and configuration. The pseudo-Euclidean integral length of a world line arc $\overrightarrow{c\tau}$ is counted from the reference point O, this length is an internal argument on the world line. In the theory of relativity (STR), speed of absolute motion of a material point M along its world line is defined in the vectorial form as so called 4-*velocity* introduced by H. Poincaré:

$$\left.\begin{array}{l}\mathbf{c}(c\tau) = c\cdot\dfrac{d\mathbf{r}}{d(c\tau)} = \dfrac{d\mathbf{r}}{d\tau} = \dfrac{d\overrightarrow{c\tau}}{d\tau} = c\cdot\mathbf{i}(c\tau),\\ \mathbf{c}'(c\tau)\cdot I^{\pm}\cdot\mathbf{c}(c\tau) = ||\mathbf{c}(c\tau)||_P^2 = -c^2 = \text{const.}\end{array}\right\}\quad (\,\overrightarrow{\mathbf{c}} = c\cdot\mathbf{i}\,) \qquad (218A)$$

It may be represented in $\langle\mathcal{P}^{3+1}\rangle$ as the time-like 4×1-radius-vector of the hyperboloid II (upper) with radius $R = ic$, see (146A). Its scalar value "c" is the constant normalizing scale multiplier to time, introduced by H. Poincaré too [47]. Isotropy and metric properties of $\langle\mathcal{P}^{3+1}\rangle$ take place due to it, see in Ch. 1A. Physically, c is the speed of light in the interstar vacuum. Other parameters of 1-st order mean the following:
$d\mathbf{r} = \mathbf{i}(c\tau)d(c\tau)$ is the 4-vector differential along the current proper time arrow $\overrightarrow{c\tau}$;
$\mathbf{i}(c\tau)$ may be as: 1) a time-like 4×1 radius-vector (146A) of the unity hyperboloid II, 2) a 4-vector of tangent to a world line in (217A), 3) a 4-th column of tensor $roth\ \Gamma^{(m)}$.

The velocity $\mathbf{v}$ is the *tangent cross 3-projection* of this local 4-velocity $\mathbf{c}$ into $\langle\mathcal{E}^3\rangle^{(1)}$. Its *sine 3-orthoprojection* is the proper velocity $\mathbf{v}^*$. These velocities have Euclidean direction $\mathbf{e}_\alpha$. The *scalar cosine orthoprojection* of 4-velocity $\mathbf{c}$ into $\overrightarrow{ct}$ is the proper velocity of the coordinate time t current stream as $c^* = \cosh\gamma\cdot c$. The 4-velocity $\mathbf{c}$ of a particle or a body can be changed only in its absolute directions: hyperbolic γ with respect to the time-arrow and/or spherical $\mathbf{e}_\alpha$ with respect to the Euclidean subspace. This takes place whenever a certain inner force $\overrightarrow{\mathbf{F}}$ acts on the particle or body. For any material objects (a photon, an electron, a down, a star, etc.) independently on its mass the pseudomodule of 4-velocity of their *absolute motion* in $\langle\mathcal{P}^{3+1}\rangle$ is the constant c.

All these arguments are summarized in the following assertion as the *Postulate.*
Any material body is permanently absolutely moving in the Minkowskian space-time $\langle\mathcal{P}^{3+1}\rangle$ *along its proper world line with vectorial* 4-*velocity* $\overrightarrow{\mathbf{c}} = c\cdot\mathbf{i}$ *having constant pseudomodule c. Its unit vector* $\mathbf{i}$ *as tangent to the curve is constant (as for uniform rectilinear physical movement) iff no any inner force* $\overrightarrow{\mathbf{F}}$ *is applicated to the object.*

In philosophy, such an assertion means the so called *perpetual matter movement.*

The Postulate is based on the original notions introduced by Poincaré and Minkowski, such as 4-velocity $\mathbf{c}$ and a world line in space-time as a trajectory of absolute motion of a particle M. With the latter we connect the main dynamical physical notions of STR: the own 4-momentum $\mathbf{P}_0 = m_0\mathbf{c} = m_0 c \cdot \mathbf{i}(c\tau)$, the real momentum $\mathbf{p} = m\mathbf{v}$ and the own Einsteinian energy $E_0 = m_0 c^2$ as scalar; both they are *external notions on a world line.* See them in Chs. 5A and 7A, where they were connected preliminary by the Absolute Pythagorean Theorem in its pseudo-Euclidean variant (in $\langle\mathcal{P}^{3+1}\rangle$). All measured physical values relate to their projections from the world line in $\langle\mathcal{P}^{3+1}\rangle$ onto $\overrightarrow{ct}$ and into $\langle\mathcal{E}^3\rangle$ expressed in a fixed frame of reference of $\langle\mathcal{P}^{3+1}\rangle$ in clear trigonometric forms! These values are changed iff the direction of $\mathbf{i}$ is changed.

Another remark: the scalar value of 4-velocity used in the postulate, identical to the scaling coefficient of Poincaré for the relativistic time coordinates (see in Ch. 1A), is equal here to the speed of light "c". However, the constancy of value c, as a result of observation in the nearest cosmos, is merely a hypothesis which cannot be inferred, and it may not spread strictly in the whole Universe and on all a world time.

The Postulate by Poincaré–Minkowski gives us some advantages.

1. It allows one to consider world lines not only geometrically, but and physically as time-like world trajectories with absolute current kinematic and dynamic parameters for a material point M in the metric space-time $\langle\mathcal{P}^{3+1}\rangle$, and, in particular, to evaluate additional such characteristics of order greater than 1 along a world line.

2. It gives simple explanation to the nature of permanent matter movement as stream of proper time $c\tau$ along a world line, and vice versa. And they both move with 4-velocity $\mathbf{c}$. What is more, the unit vector $\mathbf{i}$ is changed iff the passive force of matter inertia is overcome by an equivalent active force $\overrightarrow{\mathbf{F}}$. Hence for the complete geometric analysis of a world line it is enough of the sequential differentiations of its tangent $\mathbf{i}$.

3. It mathematically explains either hyperbolic, or spherical, or mixed trigonometric character of distortions of world lines in $\langle\mathcal{P}^{3+1}\rangle$ under factors acting onto a particle. Indeed, due to constant module of $\mathbf{c}$, its vector derivative along a world line is permanently pseudo-Euclidean orthogonal to $\mathbf{c}$:

$$\mathbf{c}'(c\tau) \cdot I^{\pm} \cdot \mathbf{c}(c\tau) = \text{const} \;\Rightarrow\; \mathbf{c}'(c\tau) \cdot I^{\pm} \cdot \left[c \cdot \frac{d\mathbf{c}(c\tau)}{d(c\tau)} \right] = 0. \qquad (219A)$$

Here at *pseudoorthogonal (generally) differentiation* along a world line in $\langle\mathcal{P}^{3+1}\rangle$, as the homogeneous and isotropic space-time, the gotten new 4 vector-derivative of the first order is always pseudoorthogonal to vector of 4-velocity $\mathbf{c}$. We got zero scalar product of a vector $\mathbf{c}$ with its first 4-vector-derivative along a world line, though the same holds for its derivative of higher orders – under here till fourth order! Hence, all such 4 vectors-derivatives are gotten in result of sequential rotational operations of initial 4-velocity along a world line. These operations are described by trigonometric 4×1 rotational tensors with their either hyperbolic, or spherical, or mixed tensor and scalar eigen angles as real, imaginary and complex ones (see it before in sect. 10.3).

Define the *order of embedding* ζ of a world line as the least dimension $\zeta = k+1$ of the pseudo-Euclidean subspace $\langle \mathcal{P}^{k+1} \rangle$ of the basis Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$ containing the whole curve. All possible values of this order are $\zeta \in \{1,2,3,4\}$ at $k = 0,1,2,3$. If $\zeta = 1$ ($k = 0$), then the enveloping subspace is only the straight time-arrow $\overrightarrow{ct}$ as itself. This is an abstract voyage in time along a straight world line. A flat world line has $\zeta = 2$ ($k = 1$). This corresponds, for example, to hyperbolic motion or another accelerated rectilinear movement. A twisted world line has order ζ as 3 or 4 corresponding to order of the line curvature 2 or 3. The order $k = \zeta - 1$ is the minimal dimension of the Euclidean subspace $\langle \mathcal{E}^k \rangle$, where a trajectory of physical movement is represented as Euclidean orthoprojection of absolute motion in $\langle \mathcal{P}^{k+1} \rangle$.

In the neighborhood of a point M, with exactness up to 2-nd order of differentiating $\mathbf{i}(c\tau)$ in $c\tau$ along a world line (not only in the osculating pseudoplane!) – see in (223A), we obtain the free *proportional total* 4-*vector characteristics* of the 2-nd order along $\mathbf{e}_\beta$: 4-pseudocurvature $\mathbf{k}$ (with radius $R_K = 1/\mathcal{K}$) and inner 4-acceleration $\mathbf{g}$, introduced in (79A), with their common unity vector of the *instantaneous pseudonormal* $\mathbf{p}$ as:

$$\mathcal{K}_\beta(c\tau) = 1/R_K^{(m)} = g_\beta(c\tau)/c^2 \quad (\lambda \geq 3,\ k \geq 2); \tag{220A}$$

$$\mathbf{k}_\beta(c\tau) = \mathcal{K}_\beta(c\tau) \cdot \mathbf{p}(c\tau) = \mathbf{g}_\beta(c\tau)/c^2 = [g_\beta(c\tau)/c^2] \cdot \mathbf{p}(c\tau). \tag{221A}$$

Natural change of the velocity of physical movement preserves smooth geometric form of a world line, that is why world lines are continuous regular curves in $\langle \mathcal{P}^{3+1} \rangle$. They as time-like curves have inclination to $\overrightarrow{ct}$, bounded in $\tilde{E}_1$ by angle $\varphi_R(\gamma) < \pi/4$.

Due to approach of the classic *differential–rotational theory* of regular curves by Frenet–Serret in the Euclidean space [16, p. 521–524], we can realize a similar theory in $\langle \mathcal{P}^{3+1} \rangle$ and $\langle \mathcal{Q}^{2+1} \rangle$ as also homogeneous and isotropic spaces with fixed dimension, but now in general tensor-vector-scalar trigonometric form! So, the tangent $\mathbf{i}$ in (217A) is 4-th vector-column of our trigonometric tensor of motion *roth* $\Gamma^{(m)} = F(\gamma, \mathbf{e}_\alpha)$, represented first in (100A). The tangent to a world line as a unity vector $\mathbf{i}_\alpha(c\tau)$ is its 1-st order differential characteristic in (218A). Then it is produced in the base $\tilde{E}_1$ with the 1-st differentiation in $c\tau$ along a world line in the space-time $\langle \mathcal{P}^{3+1} \rangle$ (at $\zeta \geq 2$):

$$\frac{d\mathbf{r}(c\tau)}{d(c\tau)} = \mathbf{i}_\alpha(c\tau) = \begin{bmatrix} \mathbf{sinh}\ \gamma_i \\ \cosh\gamma_i \end{bmatrix} = \begin{bmatrix} \sinh\gamma_i \cdot \mathbf{e}_\alpha \\ \cosh\gamma_i \end{bmatrix} = roth\Gamma_i \cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}. \tag{222A}$$

We have covered thoroughly in Ch. 5A the motions along $\mathbf{i}_\alpha(c\tau)$ at constant $\mathbf{e}_\alpha$. If to do this 1-st differentiation along a world line more free (as non-collinear), we receive in addition the spherical rotation, orthogonal to the first *hyperbolic* motion, and both ones must be in correspondence with the 1-st two-step metrical normal form on a hyperboloid I. What is more, the time-like tangent $\mathbf{i}_\alpha(c\tau)$ is simultaneously both a 4×1 radius-vector and pseudonormal (146A) of hyperboloid II and a 4-vector of a time-like tangent to the locally conjugated hyperboloid I, where only one geodesic hyperbola can pass through a point M. We leave locally hyperbolic motion for the final differentiations along a world line, when they will close all the cycle, and here's why.

Let's pre-attach to a world line with $d\gamma_i \neq 0$ at its point M the so-called *movable conjugate unity hyperboloids* I and II ($n = 3$) (see at Figure 4 in Ch. 12) so, that they can by determined locally by four current pseudoorthogonal each to other unity basis vectors of a world line (as tangent $\mathbf{i}$, pseudonormal $\mathbf{p}$, and so one). Our idea is to identify and connect one to one the 1-st local metrical forms along a world line with the analogous forms of these hyperboloids (see the end of Ch. 6A and of sect.12.1). We will calculate these metrical forms with finding their basis unity vectors in process of sequential differentiations along a world line. This will interrupt the process of differentiation in its final, as it should be in such a type of the theory. *In second*, we must connect this system of four basis vectors with the already existing pseudoorthogonal system of four basis vectors-columns in our trigonometric tensor of motion (100A).

The principal and free-valued characteristics $\mathbf{k}_\alpha$ and $\mathbf{k}_\beta$ are produced with the 2-nd differentiations in $c\tau$ along a world line with different orthogonality degree ($\zeta \geq 3$):

$$\left.\begin{aligned} \left\{\frac{d\mathbf{i}_\alpha(c\tau)}{d(c\tau)}\right\}_\alpha = \mathcal{K}_\alpha(c\tau)\cdot\begin{bmatrix}\cosh\gamma_i\cdot\mathbf{e}_\alpha\\ \sinh\gamma_i\end{bmatrix} = \mathcal{K}_\alpha(c\tau)\cdot\mathbf{p}_\alpha(c\tau) = \mathbf{k}_\alpha(c\tau),\\ \frac{d\mathbf{i}_\alpha(c\tau)}{d(c\tau)} = \mathcal{K}_\beta(c\tau)\cdot\begin{bmatrix}\cosh\gamma_p\cdot\mathbf{e}_\beta\\ \sinh\gamma_p\end{bmatrix} = \mathcal{K}_\beta(c\tau)\cdot\mathbf{p}_\beta(c\tau) = \mathbf{k}_\beta(c\tau).\end{aligned}\right\} \quad (223A)$$

Unity space-like 4×1 $\mathbf{p}(c\tau)$ are also pseudonormals to the hyperboloid I in (149A). Expression (223A) is the *pseudoanalog of the 1-st Frenet–Serret formula* [16, p. 522]. But $\mathbf{i}_\alpha$ and $\mathbf{p}$, under change of curve slope either converge or diverge! Any unity vector $\mathbf{p}$ in (223A) preserves *pseudoorthogonality* with $\mathbf{i}_\alpha$ in (222A). The *principal pseudonormal* $\mathbf{p}_\alpha$ is else *hyperbolically orthogonal* to it. It is obviously for collinear motions. In general, we have $\cos\varepsilon = \mathbf{e}'_\beta\mathbf{e}_\alpha = \mathbf{e}'_\alpha\mathbf{e}_\beta$. From the condition of pseudo-orthogonality for $\mathbf{i}_\alpha$ and $\mathbf{p}_\beta$, we obtain the connection between angles γ_p and γ_i:

$$\{\tanh\gamma_p = \cos\varepsilon\cdot\tanh\gamma_i \leftrightarrow \ \tanh\gamma_p = \cos\varepsilon\cdot v/c\} \ \rightarrow \ \gamma_p < \gamma_i, \ (\gamma \geq 0). \quad (224A)$$

If $\mathbf{e}_\beta = \mathbf{e}_\alpha$, then $\mathbf{i}$ and $\mathbf{j}_1 = \mathbf{p}_\alpha$ determine conjugate points on the hyperboloids I and II in (146A), (149A) and at Figure 4. If $\mathbf{e}_\nu \perp \mathbf{e}_\alpha$ in $\langle\mathcal{E}^2\rangle^{(m)} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle^{(m)} \equiv \langle\mathbf{v}, \mathbf{g}\rangle^{(m)}$, then $\mathbf{j}_2 = \mathbf{p}_\nu$ is here a *binormal* (a pseudonormal with its minimum *Euclidean projection* at $\cos\varepsilon = 0, \gamma_p = 0$, see bottom point on II). Recall relation (137A):

$$\mathbf{e}_\beta = \cos\varepsilon\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot\mathbf{e}_\nu, \text{where } \ \varepsilon\in[0;\pi], \quad (\mathbf{e}'_\nu\cdot\mathbf{e}_\alpha = 0, \quad \mathbf{e}'_\nu\cdot\mathbf{e}_\beta = \sin\varepsilon).$$

Here $\mathbf{e}_\nu$ is the vector of orthogonal increment; ε is the angle between $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ in $[0;\pi]$. The pseudocurvature $\mathbf{k}_\beta$ and inner 4-acceleration $\mathbf{g}_\beta$ as 4-vectors differ only by c^2. These proportional space-like vectors are directed inside region of concavity of a world line arc $d^2\mathbf{r}^{(m)}$ out center O of the osculating hyperbola – see at Figure 2A(3): $\cos\varepsilon > 0$ for acceleration (g>0), $\cos\varepsilon < 0$ for deceleration (g<0). If $\cos\varepsilon = \pm 1$, then the Euclidean projection of $\mathbf{g}$ is parallel to $\mathbf{v}$ (movement is rectilinear). If $\cos\varepsilon = 0$, then the Euclidean projection of $\mathbf{g}$ gives no increment to $||\mathbf{v}||$ and leads to world line bend towards $\mathbf{e}_\nu$, i. e., Euclidean orthogonally to the curve (movement is centripetal).

Now, at this stage, consider in details and completely orthoprojectional trigonometric representation of (223A), with the original base $\tilde{E}_1 = \{I\}$ and the current base $\tilde{E}_m^{(4)} = roth\ \Gamma^{(m)} \cdot \tilde{E}_1 = \{roth\ \Gamma^{(m)}\}$ (i. e., at instantaneous $v_i = c \cdot \tanh\gamma_i$) the current vector and scalar geometric and physical parameters of order 2 along a world line ($\mathbf{e}_\alpha \neq \mathbf{const}$, $\lambda = 3$). We realizes this description in the Euclidean *plane of curvature* $\langle\mathcal{E}^2\rangle_K^{(m)} \equiv \langle\mathcal{E}^2\rangle^{(m)} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle^{(m)} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\nu\rangle^{(m)} \equiv \langle\mathbf{v}, \mathbf{g}\rangle^{(m)}$ using further for evaluation of the total pseudocurvature $\mathbf{k}_\beta$ and its orthoprojections as these principal $\mathbf{k}_\alpha$ and normal $\mathbf{k}_\nu$ curvatures of a world line. Besides them, there are also different kinds of 3- and 4-accelerations proportional to these curvatures as in (220A), (221A). So, the total pseudocurvature $\mathbf{k}_\beta$ in (223A) is decomposed into these tangential and normal ones. These orthoprojections have unity vectors: a pseudonormal $\mathbf{p}_\alpha$ and a sine binormal $\mathbf{p}_\nu$.

In the Lagrangian space-time $\langle\mathcal{L}^{3+1}\rangle$ (see further the small illustration), the tangent and principal normal to a world line are applicated, as a pseudonormal does not exist in the space, see Ch. 1A. In the Minkowskian space-time $\langle\mathcal{P}^{3+1}\rangle$, we obtain these tangential and normal orthoprojections after decomposition of the vectorial pseudocurvature $\mathbf{k}_\beta$, with application of (137A) and according to (222A)–(224A), as follows:

$$\mathbf{k}_\beta(c\tau) = \frac{d\mathbf{i}_\alpha(c\tau)}{d(c\tau)} = \frac{d\gamma_p}{d(c\tau)} \cdot \mathbf{p}_\beta(c\tau) = \frac{\eta_p(c\tau)}{c} \cdot \mathbf{p}_\beta(c\tau) = \frac{d\gamma_p}{d(c\tau)} \cdot \begin{bmatrix} \cosh\gamma_p \cdot \mathbf{e}_\beta \\ \sinh\gamma_p \end{bmatrix} \equiv$$

$$\equiv \frac{d\gamma_i}{d(c\tau)} \cdot \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}_\alpha + \begin{bmatrix} \sinh\gamma_i \cdot \dfrac{d\mathbf{e}_\alpha}{d(c\tau)} \\ 0 \end{bmatrix}_\gamma^{(1)} = K_\alpha(c\tau) \cdot \mathbf{p}_\alpha(c\tau) + K_\nu(c\tau) \cdot \mathbf{p}_\nu(c\tau) =$$

$$= K_\alpha(c\tau) \cdot \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}_\alpha + K_\nu(c\tau) \cdot \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix}_\gamma^{(1)} \equiv \mathcal{K}_\beta(c\tau) \cdot \begin{bmatrix} \cosh\gamma_p \cdot \mathbf{e}_\beta \\ \sinh\gamma_p \end{bmatrix} =$$

$$= \mathcal{K}_\beta(c\tau) \cdot \mathbf{p}_\beta(c\tau) = \overline{\overline{\mathbf{k}}}_\beta(c\tau) + \overset{\perp}{\mathbf{k}}_\beta(c\tau) =$$

$$= \frac{d\gamma_p}{d(c\tau)} \cdot \left\{ \begin{bmatrix} \cos\varepsilon \cdot \cosh\gamma_p \cdot \mathbf{e}_\alpha \\ \sinh\gamma_p \end{bmatrix} + \begin{bmatrix} \sin\varepsilon \cdot \cosh\gamma_p \cdot \mathbf{e}_\nu \\ 0 \end{bmatrix}^{(1)} \right\}. \qquad (225A)$$

Here we use the intuitive notations of all figurated parameters in the original base $\tilde{E}_1$:

$$\mathcal{K}_\beta = \frac{d\gamma_p}{d(c\tau)} = \frac{g}{c^2}, \quad \overline{\overline{\mathcal{K}}}_\beta = \mathcal{K}_\alpha = \frac{\overline{\overline{d\gamma_p}}}{d(c\tau)} = \frac{\overline{\overline{g}}}{c^2} = \frac{d\gamma_i}{d(c\tau)}, \quad \overline{\overline{\mathcal{K}^*}}_\beta = \frac{\overline{\overline{g^*}}}{c^2} = \cosh\gamma_i \cdot \mathcal{K}_\alpha,$$

are the general and parallel principal pseudocurvatures with all projections into $\langle\mathcal{E}^3\rangle^{(1)}$;

$$\overset{\perp}{\mathcal{K}}_\beta = \overset{\perp}{\mathcal{K}^*}_\beta = \mathcal{K}_\nu = \frac{\overset{\perp}{d\gamma_p}}{d(c\tau)} = \frac{\overset{\perp}{g^*}}{c^2} = \frac{\overset{\perp}{g}}{c^2} = \sinh\gamma_i \cdot \frac{d\alpha_{(1)}}{d(c\tau)} = \frac{v^* \cdot w^*_{\alpha(1)}}{c^2}$$

is the *normal curvature* as this hyperbolic sine orthoprojections into $\langle\mathcal{E}^3\rangle^{(1)}$.
Here γ_i is a time-like hyperbolic angle in (146A) between $\mathbf{i}_\alpha$ and $\overrightarrow{ct}$ realificated by $I^\pm$.

$\mathbf{p}_\alpha = \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}$ is a principal pseudonormal, as a unity vector of the principal pseudocurvature $\mathcal{K}_\alpha$;

$\mathbf{p}_\nu = \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix}$ is a *sine binormal*, as the unity vector of the *normal curvature* $\mathcal{K}_\nu$ and $\mathbf{k}_\nu = \mathcal{K}_\nu \cdot \mathbf{p}_\nu$, situated contrary to the time-like angle γ_i as a space-like sine projection;

$\dfrac{d\mathbf{e}_\alpha}{d\tau} = \dfrac{d\mathbf{e}_\alpha}{||d\mathbf{e}_\alpha||_E} \cdot \dfrac{||d\mathbf{e}_\alpha||_E}{d\tau} = \dfrac{d\alpha_{(1)}}{d\tau} \cdot \mathbf{e}_\nu = w^*_{\alpha(1)} \cdot \mathbf{e}_\nu$ is the angular 3-velocity of $\mathbf{e}_{\alpha(1)}$.

The $3D$ *Relative Pythagorean theorem* follows **strictly** from Euclidean part of (225A), now in these complete $3D$-forms, and it acts in the movable subbase $\hat{E}^{(3)}$ for the 3-pseudocurvatures, 3-differentials and proper 3-accelerations as proportional ortho-projections into $\langle\mathcal{E}^3\rangle$ at $\gamma \in [0, \infty)$, $\varepsilon \in [0; \pi]$ [see before in (163A), (165A), (192A)]:

$$\left.\begin{array}{l} \mathcal{K}_\beta \cdot \cosh\gamma_p \cdot \mathbf{e}_\beta = \overline{\overline{\mathcal{K}}}_\beta \cdot \cosh\gamma_i \cdot \mathbf{e}_\alpha + \overset{\perp}{\mathcal{K}}_\beta \cdot \mathbf{e}_\nu, \\ (\mathcal{K}_\beta \cdot \cosh\gamma_p)^2 = (\overline{\overline{\mathcal{K}}}_\beta \cdot \cosh\gamma_i)^2 + \overset{\perp}{\mathcal{K}}{}^2_\beta, \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \mathbf{k}^*_\beta = \overline{\overline{\mathbf{k}^*_\beta}} + \overset{\perp}{\mathbf{k}}_\beta, \\ (\mathcal{K}^*_\beta)^2 = (\overline{\overline{\mathcal{K}^*_\beta}})^2 + (\overset{\perp}{\mathcal{K}}_\beta)^2, \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} \cosh\gamma_p\ d\gamma_p \cdot \mathbf{e}_\beta = \cosh\gamma_i\ d\gamma_i \cdot \mathbf{e}_\alpha + \sinh\gamma_i\ d\alpha_{(1)} \cdot \mathbf{e}_\nu, \\ \cosh^2\gamma_p\ d\gamma_p^2 = \cosh^2\gamma_i\ d\gamma_i^2 + \sinh^2\gamma_i\ d\alpha^2_{(1)} = \\ = \cosh^2\gamma_p \cdot [(\cos\varepsilon\ d\gamma_p)^2 + (\sin\varepsilon\ d\gamma_p)^2] = \cosh^2\gamma_p\ [(\overline{\overline{d\gamma_p}})^2_E + (\overset{\perp}{d\gamma_p})^2_E], \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} \mathbf{g}^*_\beta = \overline{\overline{\mathbf{g}^*_\beta}} + \overset{\perp}{\mathbf{g}}_\beta = \cosh\gamma_p \cdot \mathbf{g}_\beta = \cosh\gamma_i \cdot g_\alpha \cdot \mathbf{e}_\alpha + v^* \cdot w^*_{\alpha(1)} \cdot \mathbf{e}_\nu, \\ (g^*_\beta)^2 = (\overline{\overline{g^*_\beta}})^2 + (\overset{\perp}{g}_\beta)^2 = \cosh^2\gamma_p \cdot g^2_\beta = \cosh^2\gamma_i \cdot g^2_\alpha + (v^* \cdot w^*_{\alpha(1)})^2. \end{array}\right. \qquad (226A)$$

$$\mathcal{K}_\beta \cdot \sinh\gamma_p = \overline{\overline{\mathcal{K}_\beta}} \cdot \sinh\gamma_i \Leftrightarrow \sinh\gamma_p\ d\gamma_p = \sinh\gamma_i\ d\gamma_i\ \rightarrow\ d\gamma_p/d\gamma_i > 1. \qquad (227A)$$

From (225A) we obtain these additional hierarchical relations for all scalar parameters:

$\sinh\gamma_p\ d\gamma_p = \sinh\gamma_i\ d\gamma_i \Rightarrow d\gamma_p/d\gamma_i > 1, (\gamma_p/\gamma_i < 1, \text{see (224A)}),\ d\cosh\gamma_p = d\cosh\gamma_i$.

$\{\gamma_i = 0 \rightarrow \gamma_p = 0\}$; $\cosh\gamma_p\ \overline{\overline{d\gamma_p}} = \cosh\gamma_p \cdot \cos\varepsilon\ d\gamma_p = \cosh\gamma_i\ d\gamma_i\ \Rightarrow\ \overline{\overline{d\gamma_p}}/d\gamma_i > 1$.

(226A) and (227A) give us in $\hat{E}^{(4)}$ the $4D$ *Absolute Euclidean Pythagorean theorem*, identical to the Riemannian metrical form in $\langle\mathcal{P}^{2+1}\rangle$ (Ch. 6A) on the hyperboloid II, for the proportional 4-pseudocurvature, 4-differential and inner 4-acceleration:

$$\left.\begin{array}{l} \mathcal{K}_\beta\ \mathbf{p}_\beta = \overline{\overline{\mathcal{K}}}_\beta\ \mathbf{p}_\alpha + \overset{\perp}{\mathcal{K}}_\beta\ \mathbf{p}_\nu = \mathcal{K}_\alpha\ \mathbf{p}_\alpha + \mathcal{K}_\nu\ \mathbf{p}_\nu, \\ \mathcal{K}^2_\beta = (\overline{\overline{\mathcal{K}}}_\beta)^2 + (\overset{\perp}{\mathcal{K}}_\beta)^2 = (\mathcal{K}_\alpha)^2 + (\mathcal{K}_\nu)^2, \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \mathbf{g}_\beta = g_\alpha\ \mathbf{p}_\alpha + v^*\ w^*_{\alpha(1)}\ \mathbf{p}_\nu, \\ g^2_\beta = g^2_\alpha + (v^*\ w^*_{\alpha(1)})^2, \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} d\gamma_p \cdot \mathbf{p}_\beta = d\gamma_i \cdot \mathbf{p}_\alpha + \sinh\gamma_i\ d\alpha_{(1)} \cdot \mathbf{p}_\nu, \\ \{d\lambda/R\}^2 = d\gamma_p^2 = d\gamma_i^2 + \sinh^2\gamma_i\ d\alpha^2_{(1)} = \left(\overline{\overline{d\gamma_p}}\right)^2_P + \left(\overset{\perp}{d\gamma_p}\right)^2_E. \end{array}\right\} \qquad (228A)$$

Factor $\boxed{\sinh\gamma_p\ d\gamma_p = \sinh\gamma_i\ d\gamma_i = d\cosh\gamma_p = d\cosh\gamma_i}$ causes *dilation of an inner 4-acceleration.* Logically, that if $\gamma_i = 0$, then the second orthospherical part is absent!

From the point of view of the absolute $4D$ base $\hat{E}^{(4)}$ in $\langle\mathcal{P}^{3+1}\rangle$, the characteristic $\mathbf{g}_\beta$ is a 4-acceleration of a particle M motion along a world line.

Without a time part, (228A) gives the $3D$ *Absolute Euclidean Pythagorean theorem* as (145A), (198A). It acts in the current Cartesian subbase $\hat{E}^{(3)}$ at the condition $\gamma_p, v_p = 0$ $(g_p \neq 0) \to \overline{\overline{d\gamma_p}} = \cos\varepsilon\ d\gamma_p = \cosh\gamma_i\ d\gamma_i$, with curvature of the rotation

$$\text{in } \langle\mathcal{E}^2\rangle_K^{(m)} \subset \langle\mathcal{P}^{3+1}\rangle \quad \left\{\begin{array}{l} d\gamma_p \cdot \mathbf{e}_\beta = \cosh\gamma_i\ d\gamma_i \cdot \mathbf{e}_\alpha + \sinh\gamma_i\ d\alpha_{(1)} \cdot \mathbf{e}_\nu, \\ \{d\lambda/R\}^2 = d\gamma_p^2 = \cosh^2\gamma_i\ d\gamma_i^2 + \sinh^2\gamma_i\ d\alpha_{(1)}^2 = \\ = (\cos\varepsilon\ d\gamma_p)^2 + (\sin\varepsilon\ d\gamma_p)^2 = (\overline{\overline{d\gamma_p}})_E^2 + (\overset{\perp}{d}\gamma_p)_E^2, \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} \mathbf{g}_\beta = \overline{\overline{\mathbf{g}_\beta}} + \overset{\perp}{\mathbf{g}}_\beta = \cosh\gamma_i \cdot g_\alpha \cdot \mathbf{e}_\alpha + v^* \cdot w^*_{\alpha(1)} \cdot \mathbf{e}_\nu, \\ (g_\beta)^2 = (\overline{\overline{g_\beta}})^2 + (\overset{\perp}{g}_\beta)^2 = \cosh^2\gamma_i \cdot g_\alpha^2 + (v^* \cdot w^*_{\alpha(1)})^2. \end{array}\right. \tag{229A}$$

The $4D$ theorem is given with tensor $I^\pm$ (17A) and shows that at two-step differentiation in (225A), we got decomposition of the non-Euclidean segment $d\gamma_p \cdot \mathbf{p}_\beta$ with respect to direction $\mathbf{e}_\alpha$ (or velocity $\mathbf{v}$) into principal $d\gamma_i \cdot \mathbf{p}_\alpha$ (along the 4-pseudonormal) and secondary $\sinh\gamma_i\ d\alpha_{(1)} \cdot \mathbf{p}_\nu$ (along the sine 3-binormal) in the base $\hat{E}^{(2)}$ of the *whole* $\langle\mathcal{P}^{3+1}\rangle$. *The latter determines its absolute character.* Both parts of $d\gamma_p \cdot \mathbf{p}_\beta$ are pseudoorthogonal on the movable accompanying unity space-like hyperboloid II as its 1-st Riemannian metrical normal form. (See the end of Ch. 6A, of sect.12.1 and at Figure 4.) *Trigonometric expansion in (228A) corresponds now to this in the more general vector-scalar form.* Its two vectors are a first pair of the absolute base $\hat{E}_m^{(4)}$. A next pair will be realized at hyperboloid I. This two-step differentiation in (225A) corresponds to two types of the principal tangent $\mathbf{i}_\alpha$ rotations!

In (226A) as the Euclidean fragment of (225A), the relation of its right and left parts is $\tan\varepsilon$. Hence with (226A) for any similar two-step 1-st metrical forms, in particular, in (228A), we may introduce the characteristic ratio between two Euclidean orthogonal parts of a segment on the hyperboloid II and in its hyperbolic geometry at $(n = 2)$ as:

$$\tan\varepsilon = \sinh\gamma_i\ d\alpha_{(1)}(\sqrt{1 + \sinh^2\gamma_i}\ d\gamma_i). \tag{230A}$$

Connect $d\alpha_{(1)}$ with normal hyperbolic differential as $\sin\varepsilon \cdot \cosh\gamma_p\ d\gamma_p = \sinh\gamma_i\ d\alpha_{(1)}$.

Now, with the use of (244A), we can express, in addition, the Euclidean Thomas orthospherical shift and precession of the Euclidean plane of curvature $\langle\mathcal{E}^2\rangle_K^{(m)}$ with the base $\tilde{E}_m^{(2)}$ in result of summing two spherically orthogonal segments in (227A) – how velocity and acceleration in (172A) with the axis as next third basis vector $\mathbf{e}_\mu$:

$$\left.\begin{array}{l} -d\theta = \tanh(\gamma_i/2)\ \overset{\perp}{d}\gamma_p = \tanh(\gamma_i/2) \cdot \sinh\gamma_i\ d\alpha_{(1)}, \to \\ \to -w^*_\vartheta = \tanh(\gamma_i/2) \cdot \sinh\gamma_i \cdot w^*_{\alpha(1)} = \tanh(\gamma_i/2) \cdot v^* \cdot w^*_{\alpha(1)}/c. \end{array}\right\} \tag{231A}$$

All used parameters above and further (with R) have instantaneous values.

* * *

In decomposition (225A), $\mathbf{i}_\alpha(c\tau)$, $\mathbf{p}_\alpha(c\tau)$ and $\mathbf{p}_\nu(c\tau)$ are the basis vectors of the right mobile base $\hat{E}^{(3)}_m = \langle \mathbf{p}_\alpha(c\tau),\ \mathbf{p}_\nu(c\tau),\ \mathbf{i}_\alpha(c\tau) \rangle$ in $\langle \mathcal{P}^{2+1} \rangle$. But, in $\langle \mathcal{P}^{3+1} \rangle$, it is subbase of the *cardinal current pseudo-Cartesian base* $\hat{E}^{(3)}_m \subset \hat{E}_m = \langle \mathbf{j_1}(c\tau),\ \mathbf{j_2}(c\tau),\ \mathbf{j_3}(c\tau),\ \mathbf{i}(c\tau) \rangle$. Differentiating anyone of these four basis unity vectors $\mathbf{a}_1$ along a world line is reduced, *as a rule*, to its rotation around other second basis vector $\mathbf{a}_2$ with another third basis vector $\mathbf{a}_3$ in a certain pseudoplane or plane formed by $\mathbf{a}_1$ and $\mathbf{a}_3$. Then fourth rested basis unity vector $\mathbf{a}_4$ in the full space-time $\langle \mathcal{P}^{3+1} \rangle$ must be immobile. A result of this rotation is the unity vector $\mathbf{a}_3$, what is equivalent to the vector product of $\mathbf{a}_1$ and $\mathbf{a}_2$. Below, for illustration of this approach, we give these complete Tables of signs for such vectorial products in $\langle \mathcal{P}^{3+1} \rangle$ with a frame axis $\mathbf{i}$ and $\langle \mathcal{Q}^{2+1} \rangle$ with a frame axis $\mathbf{j}_1$.

	i	j_1	j_2	j_3
i	0	$-j_2$	$-j_3$	$-j_1$
j_1	$+j_2$	0	$+i$	$-$
j_2	$+j_3$	$+i$	0	$-$
j_3	$+j_1$	$-$	$-$	0

,

	i	j_1	j_2	j_3
i	0	$-$	$-$	$-$
j_1	$-$	0	$-j_3$	$+j_2$
j_2	$-$	$+j_3$	0	$-j_1$
j_3	$-$	$-j_2$	$+j_1$	0

;

	j_1	j_2	j_3
j_1	0	$+j_3$	$-j_2$
j_2	$-j_3$	0	$+j_1$
j_3	$+j_2$	$-j_1$	0

$(of Frenet - Serret)$,

where $j_1 = \mathbf{p}_\alpha,\ j_2 = \mathbf{p}_\nu,\ j_3 = \mathbf{p}_\mu$ are the axes of the cardinal Cartesian subbase $\tilde{E}^{(3)}_1$.

Due to these Tables of signs (in the left one for hyperbolic rotations, in the central one for orthospherical rotations), in the upper row we chose the rotated (differentiated) unity vector and in the left column we chose the axis of its rotation in the subspace. In the intersection, we get the sign of the vectorial product. So, for example, we obtain $\mathbf{i}_\alpha \times \mathbf{p}_\alpha = +\mathbf{p}_\nu$ as for any hyperbolic rotations, but $\mathbf{p}_\alpha \times \mathbf{p}_\nu = +\mathbf{p}_\mu$ and $\mathbf{p}_\nu \times \mathbf{p}_\alpha = -\mathbf{p}_\mu$ as for orthospherical rotations. The mathematical reason for this behavior of signs is that hyperbolic functions preserve their sign during differentiation, while spherical functions change it. The difference in signs of both theories in Euclidean space is eliminated by operation $\mathbf{p}_\nu \leftrightarrow \mathbf{p}_\mu$, due to our chosen strategic plan.

The hyperbolic rotations are described by the sine-cosine functions. Differentiations alonge the curve as a world-line lead here to the equivalent trigonometric processes $\sinh\gamma \to \cosh\gamma \to \sinh\gamma...$ and $\cosh\gamma \to \sinh\gamma \to \cosh\gamma...$ for radius-vectors of hyperboloids I and II (Figure 4), where we have sign "$+$" for both the concave arcs on hyperboloids I and II. For analogous trigonometric version of the Frenet–Serret theory, we obtain such processes for radius-vectors of *hyperspheroid* with signs variations: $\sin\varphi \to \cos\varphi \to -\sin\varphi...$ and $\cos\varphi \to -\sin\varphi \to -\cos\varphi....$ Such peculiarity, in hyperbolic description of rotations, may be valid, if these differentiations along a world line but in $d(ic\tau)$ to transfer into complex-valued space-time of Poincaré $\langle \mathcal{Q}^{3+1} \rangle_c$.

The partial pure hyperbolic rotation of the pseudonormal $\mathbf{p}_\alpha$, for example, around the sine binormal $\mathbf{p}_\nu$ (in general, nonuniform), is expressed at $\alpha = \text{const}$ as follows:

$$\frac{\eta_i{}^{(m)}}{c} = \mathcal{K}_\alpha = \frac{d\gamma_i^{(m)}}{d(c\tau)} = \cos\varepsilon \cdot \mathcal{K}_\beta = \overline{\overline{\mathcal{K}_\beta}} = \frac{\overline{\overline{\eta_p}}^{(m)}}{c} = \frac{\overline{\overline{d\gamma_p}}^{(m)}}{d(c\tau)} = \frac{\overline{\overline{g}}^{(m)}}{c^2}.$$

* * *

Non-relativistic decompositions of acceleration at the point M of a world line in the Euclidean plane $\langle\mathcal{E}^2\rangle_K^{(m)} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle^{(m)} \equiv \langle\mathbf{v}, \mathbf{g}\rangle^{(m)}$ in the base $\tilde{E}_1$ are performed in the Euclidean-affine Lagrangian space-time $\langle\mathcal{L}^{3+1}\rangle$ (see Ch. 1A), they are the following:

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}, \qquad \frac{d\mathbf{u}}{dt} = \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} v\cdot\mathbf{e}_\alpha \\ 1 \end{bmatrix},$$

$$\frac{d^2\mathbf{u}}{dt^2} = \begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix} = \begin{bmatrix} g\cdot\mathbf{e}_\beta \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{dv}{dt}\cdot\mathbf{e}_\alpha \\ 0 \end{bmatrix} + \begin{bmatrix} v\cdot\dfrac{d\mathbf{e}_\alpha}{dt} \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{\overline{g}}\cdot\mathbf{e}_\alpha \\ 0 \end{bmatrix} + \begin{bmatrix} \overset{\perp}{g}\cdot\mathbf{e}_\nu \\ 0 \end{bmatrix},$$

$$\text{where} \qquad \overline{\overline{g}} = \frac{dv}{dt} = g\cdot\cos\varepsilon, \quad \overset{\perp}{g} = v\cdot\frac{||d\mathbf{e}_\alpha||_E}{dt} = v\cdot w_\alpha = \frac{v^2}{r} = g\cdot\sin\varepsilon;$$

$$g^2 = (\overline{\overline{g}})^2 + (\overset{\perp}{g})^2, \quad \mathbf{g} = \overline{\overline{\mathbf{g}}} + \overset{\perp}{\mathbf{g}}, \quad \overline{\overline{\mathbf{g}}} \,||\, \mathbf{v}, \quad \overset{\perp}{\mathbf{g}} \perp \mathbf{v} \ \ (\overset{\perp}{\mathbf{g}} = \mathbf{g} - \overline{\overline{\mathbf{g}}}).$$

Here $\mathbf{g}(t)$ is decomposed along the direction $\mathbf{e}_\alpha$ of the velocity $\mathbf{v}$ and the orthogonal direction $\mathbf{e}_\eta$ of the principal normal to the curve in the constant Euclidean subspace $\langle\mathcal{E}^3\rangle$ of the Lagrangian space-time $\langle\mathcal{L}^{3+1}\rangle$, but with single Pythagorean Theorem!

* * *

For nonuniform rectilinear physical movement ($\mathbf{e}_\alpha = \mathbf{const}$), pseudoanalog (223A) of the 1-st Frenet–Serret formula may be inferred by simplest trigonometric way with the use the hyperbolic angle of motion γ as the argument of differentiation in the osculating pseudoplane with respect to the current base $\hat{E}_m$:

$$d\mathbf{i} = \mathbf{p}\,d\gamma \ \Leftrightarrow\ \frac{d\mathbf{i}}{d\gamma} = \mathbf{p} \ \Leftrightarrow\ \frac{d\mathbf{i}}{Rd\gamma} = \frac{d\mathbf{i}}{d(c\tau)} = \frac{\mathbf{p}}{R} = \mathcal{K}\cdot\mathbf{p}.$$

For the curves $\overrightarrow{l}$ with $\mathbf{e}_\alpha = \mathbf{const}$ in the quasi-Euclidean space $\langle\mathcal{Q}^{2+1}\rangle$ (Ch. 8A), in its *osculating quasiplane*, with respect to the current base $\hat{E}_m$ (but with reper axis $\overrightarrow{y}$ for the principal angle φ), the quasianalog of the 1-st Frenet–Serret formula holds:

$$d\mathbf{e} = \mathbf{n}\,d\varphi \ \Leftrightarrow\ \frac{d\mathbf{e}}{d\varphi} = \mathbf{n} \ \Leftrightarrow\ \frac{d\mathbf{e}}{Rd\varphi} = \frac{d\mathbf{e}}{dl} = \frac{\mathbf{n}}{R} = \mathcal{K}\cdot\mathbf{n}.$$

* * *

Continuing the previous process, we will take a next step of differentiation, but now for the principal pseudonormal, in order to find remaining motion parameters. Principal and free-valued parameters $\mathbf{q}_\alpha$ and $\mathbf{q}_\kappa$ are produced with the 3-rd differentiations in $c\tau$ along a world line after (222A), (223A) with different orthogonality degree

$$\left.\begin{aligned} &\left\{\frac{d\mathbf{p}_\alpha(c\tau)}{d(c\tau)}\right\}_\alpha = \mathcal{K}_\alpha(c\tau)\cdot\begin{bmatrix} \sinh\gamma_i\cdot\mathbf{e}_\alpha \\ \cosh\gamma_i \end{bmatrix}_\alpha = \mathcal{K}_\alpha(c\tau)\cdot\mathbf{i}_\alpha(c\tau) = \mathbf{q}_\alpha(c\tau), \\ &\frac{d\mathbf{p}_\alpha(c\tau)}{d(c\tau)} = \mathcal{Q}_\kappa(c\tau)\cdot\begin{bmatrix} \sinh\gamma_p\cdot\mathbf{e}_\kappa \\ \cosh\gamma_p \end{bmatrix} = \mathcal{Q}_\kappa(c\tau)\cdot\mathbf{t}_\kappa(c\tau) = \mathbf{q}_\kappa(c\tau). \end{aligned}\right\} \quad (232A)$$

Adopt the additional relation of type (137A) with the connection of type (224A) for an new directional angle "κ" with the angles $\mathbf{e}_\alpha$ and $\mathbf{e}_\mu$ in their Euclidean plane:

$$\mathbf{e}_\kappa = \cos\epsilon\cdot\mathbf{e}_\alpha + \sin\epsilon\cdot\mathbf{e}_\mu, \quad \epsilon\in[0;\pi], \quad (\mathbf{e}'_\alpha\cdot\mathbf{e}_\kappa = \cos\epsilon, \quad \mathbf{e}'_\mu\cdot\mathbf{e}_\alpha = 0, \quad \mathbf{e}'_\mu\cdot\mathbf{e}_\kappa = \sin\epsilon). \tag{233A}$$

$$\{\tanh\gamma_i = \cos\epsilon\cdot\tanh\gamma_q \rightarrow \ \gamma_i < \gamma_q\};\ \gamma\in[0,\infty). \tag{234A}$$

At the 3-rd free two-step differentiation in $c\tau$ along a world line after (222A) and (225A) at $\zeta = 4$, we obtain the tensor trigonometric analog of the 2-nd Frenet-Serret formula [16, p. 522] with identification of the torsion $\mathcal{T}_\mu$ with its cosine binormal $\mathbf{p}_\mu$:

$$\frac{d\mathbf{p}_\alpha(c\tau)}{d(c\tau)} = \mathcal{K}_\alpha(c\tau)\cdot\mathbf{i}_\alpha + \mathcal{T}_\mu(c\tau)\cdot\mathbf{p}_\mu = \mathbf{q}_\alpha(c\tau) + \mathbf{q}_\mu(c\tau) \equiv \mathcal{Q}_\kappa(c\tau)\cdot\mathbf{t}_\kappa(c\tau) \equiv \tag{235A}$$

$$\equiv \frac{d\gamma_i}{d(c\tau)}\cdot\mathbf{i}_\alpha + \cosh\gamma_i\cdot\frac{||d\mathbf{e}_\alpha||_E^{(m)}}{d(c\tau)}\cdot\mathbf{p}_\mu = \frac{d\gamma_i}{d(c\tau)}\cdot\mathbf{i}_\alpha + \cosh\gamma_i\cdot\frac{d\alpha_{(2)}}{d(c\tau)}\cdot\mathbf{p}_\mu =$$

$$= \frac{d\gamma_i}{d(c\tau)}\cdot\begin{bmatrix}\sinh\gamma_i\cdot\mathbf{e}_\alpha\\ \cosh\gamma_i\end{bmatrix}_\alpha + \cosh\gamma_i\cdot\frac{w^*_{\alpha(2)}}{c}\cdot\begin{bmatrix}\mathbf{e}_\mu\\ 0\end{bmatrix}_\gamma = \mathcal{K}_\alpha(c\tau)\cdot\mathbf{i}_\alpha + \mathcal{T}_\mu(c\tau)\cdot\mathbf{p}_\mu \equiv$$

$$\equiv \mathcal{Q}_\kappa(c\tau)\cdot\mathbf{t}_\kappa(c\tau) = \overline{\overline{\mathcal{Q}}}_\kappa\cdot\mathbf{i}_\alpha + \overset{\perp}{\mathcal{Q}}_\kappa\cdot\mathbf{p}_\mu = \mathbf{q}_\kappa(c\tau) = \overline{\overline{\mathbf{q}}}_\kappa(c\tau) + \overset{\perp}{\mathbf{q}}_\kappa(c\tau) =$$

$$= \frac{d\gamma_q}{d(c\tau)}\cdot\begin{bmatrix}\sinh\gamma_q\cdot\mathbf{e}_\kappa\\ \cosh\gamma_q\end{bmatrix} = \frac{d\gamma_q}{d(c\tau)}\cdot\left\{\begin{bmatrix}\cos\epsilon\cdot\sinh\gamma_q\cdot\mathbf{e}_\alpha\\ \cosh\gamma_q\end{bmatrix} + \begin{bmatrix}\sin\epsilon\cdot\sinh\gamma_q\cdot\mathbf{e}_\mu\\ 0\end{bmatrix}\right\}.$$

The final 4-vector $\mathbf{q}_\kappa$ may be time-like or space-like, since it acts in the pseudoplane $\langle\mathcal{P}^{(1+1)}\rangle^{(m)} \equiv \langle\mathbf{p}_\mu, \mathbf{i}_\alpha\rangle^{(m)}$. For unambiguity, we have chosen so far the first option.

The 3D *Relative Pythagorean theorem* follows from the Euclidean part:

$$\left.\begin{matrix}\mathcal{Q}_\kappa\cdot\sinh\gamma_q\cdot\mathbf{e}_\kappa = \overline{\overline{\mathcal{Q}}}_\kappa\cdot\sinh\gamma_i\cdot\mathbf{e}_\alpha + \overset{\perp}{\mathcal{Q}}_\kappa\cdot\mathbf{e}_\mu,\\ (\mathcal{Q}_\kappa\cdot\sinh\gamma_q)^2 = (\mathcal{K}_\alpha\cdot\sinh\gamma_i)^2 + (\mathcal{T}_\mu)^2,\end{matrix}\right\} \Rightarrow \left\{\begin{matrix}\mathbf{q}^*_\kappa = \overline{\overline{\mathbf{q}^*_\kappa}} + \overset{\perp}{\mathbf{q}}_\kappa,\\ (\mathcal{Q}^*_\kappa)^2 = (\overline{\overline{\mathcal{Q}^*_\kappa}})^2 + (\overset{\perp}{\mathcal{Q}}_\kappa)^2,\end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l}\sinh\gamma_q\ d\gamma_q\cdot\mathbf{e}_\kappa = \sinh\gamma_i\ d\gamma_i\cdot\mathbf{e}_\alpha + \cosh\gamma_i\ d\alpha_{(2)}\cdot\mathbf{e}_\mu;\\ \sinh^2\gamma_q\ d\gamma_q^2 = \sinh^2\gamma_i\ d\gamma_i^2 + \cosh^2\gamma_i\ d\alpha_{(2)}^2 =\\ = \sinh^2\gamma_q\cdot[(\cos\epsilon\ d\gamma_q)^2 + (\sin\epsilon\ d\gamma_q)^2] =\\ = \sinh^2\gamma_q\ [(\overline{\overline{d\gamma_q}})^2 + (d\overset{\perp}{\gamma}_q)^2],\end{array}\right. \tag{236A}$$

$$\mathcal{Q}_\kappa\cdot\cosh\gamma_q = \overline{\overline{\mathcal{Q}_\kappa}}\cdot\cosh\gamma_i \Leftrightarrow \cosh\gamma_q\ d\gamma_q = \cosh\gamma_i\ d\gamma_i. \tag{237A}$$

Differentiation in (235A) corresponds also to rotations: $\mathbf{p}_\alpha\times\mathbf{p}_\nu = +\mathbf{i}_\alpha, (\mathbf{p}_\mu = \text{const})$ and $\mathbf{p}_\alpha\times\mathbf{p}_\nu = +\mathbf{p}_\mu, (\mathbf{i}_\alpha = \text{const})$ in $\langle\mathcal{P}^{(3+1)}\rangle$. In result, we obtain the *space-like cosine torsion* $\mathcal{T}_\mu = \overset{\perp}{\mathcal{K}}_\kappa$ in the *plane of torsion* $\langle\mathcal{E}^{(2)}\rangle \equiv \langle\mathbf{p}_\alpha, \mathbf{p}_\mu\rangle$, with its unity cosine binormal $\mathbf{p}_\mu$. Here γ_i is *a space-like hyperbolic angle in* (149A) between $\mathbf{p}_\alpha$ and $\langle\mathcal{E}^3\rangle^{(1)}$ indifferent to tensor $I^\pm$. For a sine binormal $\mathbf{p}_\nu$ in (235A) γ_i is *a time-like hyperbolic angle in* (146A) between $\mathbf{i}_\alpha$ and $\overrightarrow{ct}$ realificated by $I^\pm$. To now we got all unity vectors of the cardinal pseudo-Cartesian base as $\hat{E}_m^{(4)} = \langle\mathbf{p}_\alpha(c\tau),\ \mathbf{p}_\nu(c\tau),\ \mathbf{p}_\mu(c\tau),\ \mathbf{i}(c\tau)\rangle$!

From (236A), (237A), we get the 4*D Absolute pseudo-Euclidean Pythagorean theorem.* This theorem for rotation of $\mathbf{p}_\alpha$ corresponds to the pseudo-Riemannian metric form on the movable hyperboloid I with $I^\pm$ and $I^\mp$ (see Ch. 6A) in these *vector-scalar forms*:

$$\left\{ \begin{array}{l} \mathcal{Q}_\kappa \cdot \mathbf{t}_\kappa = \overline{\overline{\mathcal{Q}}}_\kappa \cdot \mathbf{i}_\alpha + \overset{\perp}{\mathcal{Q}}_\kappa \cdot \mathbf{p}_\mu = \mathcal{K}_\alpha \cdot \mathbf{i}_\alpha + \mathcal{T}_\mu \cdot \mathbf{p}_\mu, \\ \mp \mathcal{Q}_\kappa^2 = (\overline{\overline{\mathcal{Q}}}_\kappa)^2 - (\overset{\perp}{\mathcal{Q}}_\kappa)^2 = (\mathcal{K}_\alpha)^2 - (\mathcal{T}_\mu)^2 = -[-(\mathcal{K}_\alpha)^2 + (\mathcal{T}_\mu)^2]; \end{array} \right\} \Rightarrow \tag{238A}$$

$$\Rightarrow \left\{ \begin{array}{l} d\gamma_q \cdot \mathbf{t}_\kappa = d\gamma_i \cdot \mathbf{i}_\alpha + \cosh\gamma_i \; d\alpha_{(2)} \cdot \mathbf{p}_\mu, \\ \mp\{d\lambda/R\}^2 = \mp d\gamma_q^2 = d\gamma_i^2 - \cosh^2\gamma_i \; d\alpha_{(2)}^2 = \left(\overline{\overline{d\gamma_q}}\right)_P^2 - \left(\overset{\perp}{d\gamma_q}\right)_E^2. \end{array} \right\}$$

For final quadric characteristics we use the signs $\pm$ or $\mp$ accordingly to their mupping with metric tensor $I^\pm$ (47A), adopted by us, or $I^\mp$, used sometimes in literature. The tensor $I^\pm$ conserves Euclidean spaces, but $I^\mp$ transforms them into anti-Euclidean analogues, which contradicts classic geometry and physics and violates the Principle of Correspondence. In (238A) and metric form of the hyperboloid I (Ch. 6A), the quadric pseudocurvature $(\pm\mathcal{K}_\alpha)^2 = \mathcal{K}_\alpha^2$ of the tangent $\mathbf{i}_\alpha$ is contrary by signs to the quadric torsion $(\pm\mathcal{T}_\mu)^2 = \mathcal{T}_\mu^2$ of the cosine binormal $\mathbf{p}_\mu$. *We will take this into account at calculating quadric curvature of a world line with (228A), (238) and metric tensors.* We have two extreme cases at the point M: a local hyperbolic shift if $d\alpha_{(2)} = 0$ and a local spherical shift if $d\gamma_i = 0$ at the necessary local slope of a world line $\tanh\gamma_i < 1$. So, in the 1-st case, we have a local hyperbolic increment in some pseudoplane. And in the 2-nd case, we have a local spherical increment in some cylindrical pseudosurface.

Above we considered the variant of space-like torsion. Though it may be time-like precession with its time-like cosine binormal too and with a pseudo-hyperbolic angle of time-like motion (rotation) as $\gamma = i\alpha_{(2)}$ (see in sections 6.1 and 10.3) in a pseudoplane. But here the tensor $I^\mp$ negates this imaginary factor i. (So, see further the complete description of the pseudoscrewed motion *with as if torsion* time-like precession.)

The pseudoorthogonality of (228A) and (238A) is reduced to $\mathbf{p}'_\beta \cdot \{I^\pm\} \cdot \mathbf{t}_\kappa = 0$, which exists iff $\tanh\gamma_p / \tanh\gamma_q = \mathbf{e}'_\beta \cdot \mathbf{e}_\kappa$. Due to this and (224A), (233A), (234A), $\tanh\gamma_p/\tanh\gamma_q = \mathbf{e}'_\beta\ \mathbf{e}_\kappa = \cos\epsilon \cdot \cos\varepsilon = \mathbf{e}'_\beta \cdot \overleftarrow{\mathbf{e}_\alpha\ \mathbf{e}_\alpha{}'} \cdot \mathbf{e}_\kappa = (\overleftarrow{\mathbf{e}_\alpha\ \mathbf{e}_\alpha{}'} \cdot \mathbf{e}_\beta)' \cdot (\overleftarrow{\mathbf{e}_\alpha\ \mathbf{e}_\alpha{}'} \cdot \mathbf{e}_\kappa)$. From this we have the orthogonality full condition: $\mathbf{e}_\beta = \mathbf{e}_\nu, \mathbf{e}_\kappa = \mathbf{e}_\mu$, and opportunity of curvature independent decompositions with these three orthogonal basis vectors.

Following our scheme along a world line, we get the two last formulae as results of 4-th differentiation of a cosine binormal $\mathbf{p}_\mu$ in the normal plane $\langle\mathcal{E}^2\rangle_N^{(m)} \equiv \langle\mathbf{p}_\alpha, \mathbf{p}_\mu\rangle$

$$\left\{\frac{d\mathbf{p}_\mu(c\tau)}{d(c\tau)}\right\}_{\mathbf{i}_\alpha, \mathbf{p}_\nu} = -\mathcal{T}_\mu(c\tau) \cdot \mathbf{p}_\alpha = -\frac{w^*_{\alpha(2)}}{c} \cdot \left[\begin{array}{c} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{array}\right], \; \mathbf{p}_\mu \times \mathbf{i}_\alpha = -\mathbf{p}_\alpha, \tag{239A}$$

and of 5-th differentiation of a sine binormal in the *binormal* plane $\langle\mathcal{E}^2\rangle_B^{(m)} \equiv \langle\mathbf{p}_\nu, \mathbf{p}_\mu\rangle$

$$\left\{\frac{d\mathbf{p}_\nu(c\tau)}{d(c\tau)}\right\}_{\mathbf{i}_\alpha, \mathbf{p}_\mu} = -\mathcal{T}_\nu(c\tau) \cdot \mathbf{p}_\mu(c\tau) = -\frac{w^*_{\alpha(1)}}{c} \cdot \left[\begin{array}{c} \mathbf{e}_\mu \\ 0 \end{array}\right], \; \mathbf{p}_\nu \times \mathbf{p}_\alpha = -\mathbf{p}_\mu. \tag{240A}$$

If a world line has the order of embedding $\zeta = 3$, there hold two variants: either (1), when $\mathbf{p}_\nu$ is absent, it corresponds to above stated complex analogue of *Frenet–Serret formulae* with third formula (239A) and *Frenet movable trihedron* $\hat{E}_m^{(3)} \equiv \{\mathbf{p}_\alpha, \mathbf{p}_\mu, \mathbf{i}_\alpha\}$; or (2), when $\mathbf{p}_\mu$ is absent, it is the real-valued variant with a right part of (225A) as its third formula and trihedron $\hat{E}_m^{(3)} \equiv \{\mathbf{p}_\alpha, \mathbf{p}_\nu, \mathbf{i}_\alpha\}$ as alternative to the former case.

The quadruple $\hat{E}_m^{(4)} = \{\mathbf{p}_\alpha(c\tau), \mathbf{p}_\nu(c\tau), \mathbf{p}_\mu(c\tau), \mathbf{i}_\alpha(c\tau)\}$ is a *movable tetrahedron* to a world line in $\langle\mathcal{P}^{3+1}\rangle$. It complements the accompanying movable hyperboloids and gives the asymmetric *pseudoorthogonal tensor* $U(\gamma_i, \mathbf{e}_\alpha, \mathbf{e}_\nu, \mathbf{e}_\mu)$, only at $n = 3$. It is useful addition to 4 basis vectors of symmetric tensor of motion *roth* $\Gamma_i = F(\gamma_i, \mathbf{e}_\alpha)$. $U(\gamma_i, \mathbf{e}_\alpha, \mathbf{e}_\nu, \mathbf{e}_\mu)$ defines completely orientation and configuration of a world line at M.

We have all three independent orthoprojections of a world line *general curvature* $\mathcal{C}_\mathcal{R}$. In order to combine metric forms (228A) and (238A), we take into account the fact that in these two-step forms the hyperbolic angle γ_i accordingly is time-like and space-like! Therefore for descriptiveness to different readers, we apply below, for example, metric tensor $I^\mp$ (see above). In result, we obtain the final $4D$ pseudo-Euclidean Absolute Pythagorean theorem for the full quadric curvature of a world line in a such form:

$$\left.\begin{aligned} & d\gamma \cdot \mathbf{d}_\rho = d\gamma_i \cdot \mathbf{p}_\alpha + \sinh\gamma_i \; d\alpha_{(1)} \cdot \mathbf{p}_\nu + \cosh\gamma_i \; d\alpha_{(2)} \cdot \mathbf{p}_\mu = \\ & = \mathcal{K}_\alpha \cdot \mathbf{p}_\alpha + \mathcal{K}_\nu \cdot \mathbf{p}_\nu + \mathcal{T}_\mu \cdot \mathbf{p}_\mu; \\ & \pm\{d\lambda/R\}^2 = \pm d\gamma^2 = d\gamma_i^2 + \sinh^2\gamma_i \; d\alpha_{(1)}^2 - \cosh^2\gamma_i \; d\alpha_{(2)}^2, \\ & \pm{\mathcal{C}_\mathcal{R}}^2 = \frac{\eta_i^2}{c^2} + \sinh^2\gamma_i \cdot \frac{w^{*\,2}_{\alpha(1)}}{c^2} - \cosh^2\gamma_i \cdot \frac{w^{*\,2}_{\alpha(2)}}{c^2} = \mathcal{K}_\alpha^2 + \mathcal{K}_\nu^2 - \mathcal{T}_\mu^2. \end{aligned}\right\} \quad (241A)$$

$\mathcal{T}_\mu = 0$ *is full condition of a flat curve.* $\mathcal{K}_\nu = \mathcal{T}_\mu = 0$ *is full condition of a straight line.*

The movable tetrahedron is rotated in $\langle\mathcal{P}^{3+1}\rangle$ with the general angular velocity $\mathcal{C}_\mathcal{R} \cdot c$ around the special instantaneous 4-vector-axis $\{\mathcal{T}_\mu \cdot \mathbf{i}_\alpha + \mathcal{K}_\alpha \cdot \mathbf{p}_\mu + \mathcal{K}_\nu \cdot \mathbf{p}_\alpha\}$.

In the real-valued quasi-Euclidean space $\langle\mathcal{Q}^{3+1}\rangle$ (see in Chs. 5, 6, 8A), for Euclidean regular curves at $v = \text{const}$, we have the quasi-Euclidean analogue using $I^\pm$ and (322):

$$\left.\begin{aligned} & \{dl/R\}^2 = d\varphi^2 = d\varphi_i^2 + (\sin^2\varphi_i \; d\alpha_{(1)}^2 + \cos^2\varphi_i \; d\alpha_{(2)}^2), \\ & {\mathcal{C}_\mathcal{R}}^2 = {w_{\varphi_i}}^2/v^2 + \sin^2\varphi_i \cdot w_{\alpha(1)}^2/v^2 + \cos^2\varphi_i \cdot w_{\alpha(2)}^2/v^2 = \mathcal{K}_\alpha^2 + \mathcal{K}_\nu^2 + \mathcal{T}_\mu^2. \end{aligned}\right\} \quad (242A)$$

In $\langle\mathcal{Q}^{2+1}\rangle$ for the Frenet–Serret $3D$ theory, but with our frame axis $\mathbf{z}$, we have these three variants of $3D$ curves:

(1) if $w_{\alpha(1)} = 0 \rightarrow \mathcal{K}_\nu = 0$, then ${\mathcal{C}_\mathcal{R}}^2 = [w_{\varphi_i}^2 + \cos^2\varphi_i \cdot w_{\alpha(2)}^2]/v^2 = \mathcal{K}_\alpha^2 + \mathcal{T}_\mu^2$.
(2) if $w_{\alpha(2)} = 0 \rightarrow \mathcal{T}_\mu = 0$, then ${\mathcal{C}_\mathcal{R}}^2 = [w_{\varphi_i}^2 + \sin^2\varphi_i \cdot w_{\alpha(1)}^2]/v^2 = \mathcal{K}_\alpha^2 + \mathcal{K}_\nu^2$.
(3) if $w_{\varphi_i} = 0 \rightarrow \varphi_i = \text{const} \rightarrow \mathcal{K}_\alpha = 0$, then
${\mathcal{C}_\mathcal{R}}^2 = [\sin^2\varphi_i \cdot w_{\alpha(1)}^2 + \cos^2\varphi_i \cdot w_{\alpha(2)}^2]/v^2 = \mathcal{K}_\nu^2 + \mathcal{T}_\mu^2$.

For curves in $3D$ quasi-Euclidean spaces, only $\mathcal{K}_\nu = \mathcal{T}_\mu = 0$ may be full condition of a flat curve! In case (3), for cylindrical enveloping form of a curve, we may adopt $w_{\alpha(1)} = w_{\alpha(2)}$ and obtain a spherical screw, with trigonometric description of its two variants: $\tan\varphi_i \leq 1, \tan\varphi_i > 1$. But in $\langle\mathcal{P}^{2+1}\rangle$, we have only one variant $\tanh\gamma_i \leq 1$!

Above in (235A) we chosen $\mathbf{p}_\mu$ as the unity vector of a space-like torsion in a pseudoplane of torsion. But, if $\mathcal{K}_\alpha(c\tau) = 0$ ($\mathbf{p}_\alpha$ is the same), we have an opportunity to choose the unity vector $\mathbf{i}_1$ of the time-arrow $\overrightarrow{ct}^{(1)}$ as the basis vector of a time-like general spherical torsion $\mathbf{y}_1 = \mathcal{Y} \cdot \mathbf{i}_1$. Note (see in sect. 10.3), that for acceptable pseudoized spherical angles, as differentials too, we have in a plane and a pseudoplane (if $d\gamma = 0$) accordingly the two clockwise forms as space-like $d\alpha$ and time-like $d(i\alpha)$ ones, pseudo-Euclidean orthogonal each to other! The rotation of a principal time arrow $\mathbf{i}_\alpha$ in the pseudoplane $\langle \mathcal{P}^{(1+1)} \rangle^{(m)} \equiv \langle \mathbf{p}_\nu, \mathbf{i}_1 \rangle$ produces the time-like 3-pseudoscrew $\mathbf{h}_\nu$ in $\langle \mathcal{P}^{2+1} \rangle$. Its unity vector is formed as a result of orthogonal spherical rotation $rot\ \Pi/2 \cdot \mathbf{i}_\alpha = \mathbf{i}_\nu$.

To illustrate this, let us consider a $3D$-pseudoscrewed motion in space-time $\langle \mathcal{P}^{2+1} \rangle$. Its world line has constant inclination γ_i and angular speed w^*_α with *pseudoized* angle α. We get so the 3-rd *kind of uniform motion* as the *pseudoscrew* map of a planetary movement of a body or point M at $v = c{\cdot}\tanh\gamma_i$, $d\gamma_i = 0$, $d\alpha/dt = w_\alpha$, ($d\alpha/d\tau = w^*_\alpha$):

$$\left.\begin{aligned} &\left\{\frac{d\mathbf{i}_\alpha(c\tau)}{d(c\tau)}\right\}_\gamma = \sinh\gamma_i \cdot \frac{w^*_\alpha}{c} \cdot \mathbf{p}_\nu + \cosh\gamma_i \cdot \frac{w^*_\alpha}{c} \cdot \mathbf{i}_1 = \mathbf{k}_\nu + \mathbf{y}_1 = \mathbf{h}_\nu = \\ &= \mathcal{K}_\nu \cdot \mathbf{p}_\nu + \mathcal{Y} \cdot \mathbf{i}_1 = \sinh\gamma_i \cdot \frac{w^*_\alpha}{c} \cdot \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix} + \cosh\gamma_i \cdot \frac{w^*_\alpha}{c} \cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \\ &= \frac{w^*_\alpha}{c} \cdot \begin{bmatrix} \sinh\gamma_i \cdot \mathbf{e}_\nu \\ \cosh\gamma_i \end{bmatrix} = \mathcal{C}_\nu \cdot \mathbf{i}_\nu \Rightarrow \boxed{\mp {\mathcal{C}_\mathcal{R}}^2 = \mathcal{C}_\nu^2 = \mathcal{Y}^2 - \mathcal{K}_\nu^2 = [w^*_\alpha/c]^2}\ . \end{aligned}\right\} \quad (243A)$$

Here $\mathbf{i}_1$ is a unity vector of *rotated* $\overrightarrow{ct}^{(1)} = y \cdot \alpha \mathbf{i}_1$, $y = \cosh\gamma_i \cdot c/w^*_\alpha = c^*/w^*_\alpha = c/w_\alpha$ is a step at $\alpha = 1$rad of the world line *progressive precession* at speed c^* along $\overrightarrow{ct}^{(1)}$;
$\mathbf{h}_\nu = \mathcal{C}_\nu \cdot \mathbf{i}_\nu$ is a *time-like 3-pseudoscrew* along $\mathbf{i}_\nu$ as a normal tangent with direction $\mathbf{e}_\nu$;
$\mathcal{C}_\nu = w^*_\alpha/c$ is, on the first point of view, a general negative scalar time-like pseudocurvature (see above) in $\langle \mathcal{P}^{2+1} \rangle$ of the screwed world line at its point M, and it is, on the second point of view, a scalar combined pseudoscrew parameter (for a right screw);
$\mathcal{Y} = \cosh\gamma_i \cdot w^*_\alpha/c$ is a progressive time-like precession of the world line with its current point M along $\mathbf{i}_1$ and time arrow $\overrightarrow{ct}^{(1)}$ as a time-like part of this pseudoscrew;
$\mathcal{K}_\nu = \sinh\gamma_i \cdot w^*_\alpha/c$ is a normal curvature as a space-like part of this pseudoscrew.

Moreover, γ_i affects on r/y, but w^*_α affects on y. So, this screw may be as a model of a physical accelerator with these parameters. More generally, a planet (or sputnik) is rotated around of a star (or planet) along orbit of the Euclidean radius $r = v/w_\alpha$.

As the extreme example of such motions, we may give a pseudoscrewed world line of a photon circular movement, realized on the isotropic cone with velocities $\mathbf{c} = c \cdot \mathbf{e}_\alpha$ and w_α of the radius $r = y = c/w_\alpha$, when in the limit $r/y = \mathcal{K}_\nu/\mathcal{Y} = \tanh\gamma_i = \pm 1$. We see that $w_\alpha = c/r$ is determined only by given r. Einsteinian photon is rotated at velocities $\mathbf{c}$ and w_α around some *black hole* of radius "r" with the time step "$y = r$" (at $\Delta ct = r$), predicted in 1783 by John Michell [58] only with the Newton Theory.

Integral pseudoscrewed motion with its differential form (243A) is realized on the lateral cylindrical surface. A natural question arises: what relation does this surface have to the used moving hyperboloid I, besides their common topology?

The point is that we use hyperboloids not to describe integral motion, but to map their metric forms. These two kinds of applications can only coincide in particular cases as such motion. In Figure 4, we can see that the increment of this motion is mapped into the region of the central circular zone ("equator") of the movable hyperboloid I. The applied cylinder of radius r is tangent surface to the equator of this hyperboloid. Pseudo-Riemannian differential form (238A) gives us wide possibilities for application of various motion angles as hyperbolic, spherical, real-valued, imaginary and complex.

For consistent differential–geometric description of such rotational-progressive motion leading to this pseudoscrew, we introduced the spherical rotation of $\overrightarrow{ct}$ *and* $\mathbf{h}_\nu$ *with their common subvector* $\mathbf{i}_1$ *synchronized with the rotation of a principal pseudonormal in the pseudoplane* $\langle\mathcal{P}^{(1+1)}\rangle^{(m)} \equiv \langle\mathbf{p}_\nu, \mathbf{i}_1\rangle$. *The world line precession along* $\overrightarrow{ct}$ *at speed* $c^* = c \cdot \cosh\gamma_i$ *arises as a result of the spherical rotation of the screwed curve with a point* M *at angular velocity* w^*_α, *caused by this synchronous rotation of a principal pseudonormal* $\mathbf{p}_\alpha$. *On the other hand, this precession is the coordinate time* t *stream at velocity* $c^* = \cosh\gamma \cdot c$ *as a usual scalar cosine orthoprojection of any absolute motion's 4-velocity* $\mathbf{c}$ *from (218A) onto* $\overrightarrow{ct}$. *Such progressive motion influences too on the inner geometry of a world line, because it strains the curve along time. Compatibility of rotations and precession in* $\langle\mathcal{P}^{2+1}\rangle$ *is mathematically explained by the fact that* $\mathbf{i}_1$ *is as if common subvector for* $\mathbf{i}_\alpha$ *and* $\mathbf{i}_\nu$, *which will exclude mixing of basis vectors of the given motion in its base* $\hat{E}_{(m)} \equiv \langle\mathbf{p}_\alpha, \mathbf{p}_\nu, \mathbf{i}_\alpha\rangle$*:*

$$\mathcal{K}_\nu = 1/R_K = \sinh\gamma_i \cdot\ w^*_\alpha/c = \overset{\perp}{g}/c^2,\ (\overset{\perp}{g} = g), \quad \mathbf{k}_\nu = \mathcal{K}_\nu \cdot \mathbf{p}_\nu; \tag{244A}$$

$$\mathcal{Y} = 1/R_Y = \cosh\gamma_i \cdot\ w^*_\alpha/c, \quad \mathbf{y}_1 = \mathcal{Y} \cdot \mathbf{i}_1. \tag{245A}$$

The properties follow: $\mathbf{p}_\nu(c\tau) \perp \mathbf{i}_\alpha(c\tau),\ \mathbf{e}_\nu \perp \mathbf{e}_\alpha \leftrightarrow \mathbf{g} \perp \mathbf{v}$. Three of basis vectors determine the movable base $\hat{E}_{(m)} \equiv \langle\mathbf{p}_\alpha, \mathbf{p}_\nu, \mathbf{i}_\alpha\rangle$ of cardinality 3. The triple $\mathcal{K}_\nu$, $\mathcal{Y}$ (legs), $\mathcal{C}_\nu$ (hypotenuse) forms *interior right triangle of pseudoscrew* $\mathbf{P}$ in (243A).

On the cylindrical surface, we get the interior right triangle **A1** with legs r, y and hypotenuse $R_C = 1/\mathcal{C}_\nu$, where $r = \sinh\gamma{\cdot}R_C$, $y = \cosh\gamma{\cdot}R_C$ (in **A1** y is coaxial to $\overrightarrow{ct}$). And it is $y^2 - r^2 = R_C^2$. Here R_C expresses the pseudo-Euclidean length of the world line arc at $\alpha = 1$ rad. The *identical* right triangle **A2** is realized in the pseudoplane of precession $\langle\mathcal{P}^{1+1}\rangle_\mathcal{Y} \equiv \{\mathbf{p}_\nu, \mathbf{i}_1\}$. Their common leg is $y > r$ (in **A2** r is coaxial to $\mathbf{p}_\nu$).

As a **geometric paradox of this screw**, we obtain two wonderful right triangles: **P** of pseudoscrew and **A** in the two variants. Their legs are proportional with coefficient $y/\mathcal{Y} = r/\mathcal{K}_\nu$, they have equal adjacent angles. Hence, these triangle are homothetic. But their hypotenuses are inverse each another as $\mathcal{C}_\nu = 1/R_C$! From here the world line generates, in addition, two pseudo-Euclidean right triangles: they are the interior right triangle **B** and the exterior right triangle **C**, with also their Pythagorean theorems. This paradox extends to similar screwed lines in the quasi-Euclidean space $\langle\mathcal{Q}^{2+1}\rangle$ too!.

The triangle **B** has hypotenuse $R_K = 1/\mathcal{K}_\nu$ (radius of the spherical curvature under inclination γ to $\overrightarrow{ct}$), leg $R_C = \sinh\gamma \cdot R_K$ is opposite to γ, leg $b_K = \cosh\gamma \cdot R_K$ is adjacent to γ. This metric triangle lies in the osculating pseudoplane $\langle\mathbf{p}_\alpha(c\tau), \mathbf{i}_\alpha(c\tau)\rangle$. From triangles **B** and **A** we have $r = R_K \cdot \sinh^2\gamma, y = R_Y \cdot \cosh^2\gamma$ and $y^2 - r^2 = R_C^2$.

The triangle **C** has hypotenuse $R_Y = 1/\mathcal{Y}$ ("radius" of the time precession under inclination γ to $\langle\mathcal{E}^2\rangle$, leg $R_C = \cosh\gamma \cdot R_Y$ is adjacent to γ, leg $b_Y = \sinh\gamma \cdot R_Y$ is opposite to γ. This metric triangle lies in the rectifying pseudoplane $\langle\mathbf{p}_\nu(c\tau), \mathbf{i}_\alpha(c\tau)\rangle$.

With first triangle **P**, *such screwed world line has 5 characteristic right triangles*!

If the enveloping tangent cylinder with this screwed curve is cutted along the central axis $\overrightarrow{ct}$, further to develop it into fragments of the pseudoplane and finally to add these fragments so to coincide windings of this screw, then we obtain the same but straight continuous world line in the flat pseudoplane. Such flat mapping of the original pseudo-screwed world line is determined geometrically only by the one constant parameter γ_i.

This convincing example demonstrates very clearly how *minimal* curving the basis space $\langle\mathcal{P}^{2+1}\rangle$ into the cylindrical space $\langle\mathcal{C}^{2+1}\rangle$ complicates in a *large* extent description of the simplest straight world line with introducing a lot of additional parameters!!!

Therefore, before curving some simplest unique space, we need to realistically assess: which is more expedient − curving space or additionally curving the trajectories of motions in the original space due to the appearance of additional influencing factors.

In the same pseudo-Cartesian basis, we can also transform this cylinder (together with the screwed line) with its round base into another cylinder with its ellipsoidal base, applying small dividing again into maximal $w^*_{\alpha(1)}$ and minimal $w^*_{\alpha(2)}$. Then with this ellipsoidal cylinder it is possible to describe more real planetary movements.

Besides, this unique motion is not purely rotational in nature. Due to the coordinate time stream, it contains a progressive part along time-arrow $\overrightarrow{ct}$. Hence, this precession should be considered as geometrically and physically independent on rotations, as *rotational-progressive motion* beyond the scope of the theory by Frenet–Serret.

Conclusion. Due to the presence of progressive motion along y-axis, a full geometric theory of curves on the base only rotations of basis unity vectors cannot be built.

In cylindrical coordinates, we display descriptively all parameters of the motion. $x_1 = r\cdot\cos\alpha,\ \ x_2 = r\cdot\sin\alpha,\ \ ct = y\cdot\alpha\ \ (r = v/w_\alpha = \text{const},\ y = c/w_\alpha = \text{const})$.

$$\left.\begin{aligned}\sinh\gamma = r/R_C = R_C/R_K \to \sinh^2\gamma = r/R_K = r\cdot\mathcal{K}_\nu, \to r = v/w,\\ \cosh\gamma = y/R_C = R_C/R_Y \to \cosh^2\gamma = y/R_Y = y\cdot\mathcal{Y}, \to y = c/w,\end{aligned}\right\}\to \tag{246A}$$

$$\to b_K = R_K\cdot\cosh\gamma,\ b_Y = R_Y\cdot\sinh\gamma, b_Y/b_K = \tanh^2\gamma, \mathcal{Y}/\mathcal{K}_\nu = \coth\gamma, \tag{247A}$$

$$\to \mathcal{Y}^2 - \mathcal{K}_\nu^2 = \mathcal{C}_\nu^2 = 1/R_C^2 = 1/R_Y^2 - 1/R_K^2 > 0\ \to\ R_C = 1/\mathcal{C}_\nu. \tag{248A}$$

Note: If $\overline{\overline{\mathbf{k}}} = 0$, then $\overline{\overline{k}} = 0$, $\mathbf{p}_\alpha$ is the same. If $\overset{\perp}{\mathbf{k}} = 0$, then $\overset{\perp}{k} = 0$, $\mathbf{p}_\nu$ is the same.

For the pseudoscrewed motion as circular physical movement, the normal hyperbolic and spherical angular velocities with inner accelerations are the following ($\overline{\overline{g}} = 0$):
$v^* = c\cdot\sinh\gamma,\ w^*_\alpha = c\cdot\sinh\gamma/(R_K\sinh^2\gamma) = v^*/r,\ v = c\cdot\tanh\gamma,\ w_\alpha = v/r;$
$\overset{\perp}{g} = c\cdot\sinh\gamma\cdot w^*_\alpha = v^*\cdot w^*_\alpha = (v^*)^2/r = c^2K_\nu = c^2/R_K = \overset{\perp}{g}{}^*$;
$\overset{\perp}{g}{}^{(1)} = \overset{\perp}{g}\cdot\text{sech}\,\gamma = v\cdot w^*_\alpha$. See also in (165A)–(168A).

For the time part of (243A) there hold:
$g_b = c^2/R_Y = c^2\mathcal{Y} = c^2\mathcal{C}_\nu\cosh\gamma = c\cdot\cosh\gamma\cdot w^*_\alpha = c^*\cdot w^*_\alpha = (c^*)^2/y$, where again $c^* = \cosh\gamma\cdot c$ is the proper velocity of the coordinate time t stream for any world line.

* * *

Let's go back to general motions with variable two parameters of *roth* $\Gamma_i = F(\gamma_i, \mathbf{e}_\alpha)$. Pay attention to the fact that simultaneously with a point M of a world line and the accompanied movable unity hyperboloid I with their common time-like tangent $\mathbf{i}_\alpha{}^{(I)}$ and space-like pseudonormal $\mathbf{p}_\alpha{}^{(I)}$, moving all at 4-velocity $\mathbf{c}$, there is the instantaneous point N on the conjugate hyperboloid II with its also conjugate space-like tangent $\mathbf{i}_\alpha{}^{(II)}$ and time-like pseudonormal $\mathbf{p}_\alpha{}^{(II)}$. In Ch. 12 we denoted such conjugate points of two hyperboloids as $\mathbf{v}$ and $\mathbf{u}$ in a textual part and at the Figure 4. Between all our four unity basis vector on these hyperboloids there is one-to-one correspondence as follows:

$$\left.\begin{array}{l} \mathbf{i}_\alpha{}^{(I)} \equiv \ \mathbf{p}_\alpha{}^{(II)} = \ \mathbf{r}^{(II)}_{(N)}, \ \text{(as of 4-velocities for I and II)}; \\ \mathbf{p}_\alpha{}^{(I)} \equiv \ \mathbf{i}_\alpha{}^{(II)} = \ \mathbf{r}^{(I)}_{(M)}, \ \text{(as of 4-accelerations for I and II)}; \\ \mathbf{p}_\nu{}^{(I)} = \mathbf{p}_\nu{}^{(II)} = \mathbf{n}_\nu^{(I)}{}_{(M)} = \mathbf{n}_\nu^{(II)}{}_{(N)}, \ \text{(as of normal 3-shifts for I and II)}; \\ \mathbf{p}_\mu{}^{(I)} = \mathbf{p}_\mu{}^{(II)} = \mathbf{n}_\mu^{(I)}{}_{(M)} = \mathbf{n}_\mu^{(II)}{}_{(N)}, \ \text{(as of torsion 3-shifts for I and II)}. \end{array}\right\} \quad (249A)$$

where $\mathbf{n}$ are binormals to these hyperboloids too in these point $\mathbf{v}$ and $\mathbf{u}$ (Figure 4).

Formally here, this dependent space-like pure geometric motions of the point N on the movable hyperboloid II (connected with the point M on the movable hyperboloid I) are realized by our tensor of motion *roth* $\Gamma_i = F(\gamma_i, \mathbf{e}_\alpha)$ (100A) under connected changes in two of its parameters as the hyperbolic angle of motion (velocity) and its direction (see in details in Ch. 7A).

The correspondences (249A) make it possible to better see the reason why motions along a world-line are displayed on both unity accompanying hyperboloids I and II. This can be presented more clearly in Figure 4 with two these hyperboloids.

Indeed, the first two-step rotation-differentiation of the tangent $\mathbf{i}_\alpha{}^{(I)}$ in (225A) is executed with the geometric space-like principal motions in the Euclidean plane of curvature $\langle\mathcal{E}^2\rangle_K^{(m)} \equiv \langle\mathbf{p}_\alpha, \mathbf{p}_\nu\rangle^{(m)}$. But this plane is tangent namely to the hyperboloid II at the point M'! Hence the motion is transferred mathematically (no physically) from the hyperboloid I into a surface of the hyperboloid II according to the first and third relations in (249A). However the second two-step rotation-differentiation of the pseudo-normal $\mathbf{p}_\alpha{}^{(I)}$ in (235A) is executed with the physical time-like principal motions in the pseudo-Euclidean plane of torsion $\langle\mathcal{P}^{(1+1)}\rangle_T^{(m)} \equiv \langle\mathbf{p}_\mu, \mathbf{i}_\alpha\rangle^{(m)}$, which is tangent namely to the hyperboloid I at the point M!

As final result, we obtain all absolute parameters of a world line in $\tilde{E}_1 = \{I\}$ and $\hat{E}_m$ under permanent action on a particle of the current motion tensor *roth* $\Gamma_i = F(\gamma_i, \mathbf{e}_\alpha)$:

$$\mathbf{p}_\alpha = \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}, \quad \mathbf{p}_\nu = \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix}, \quad \mathbf{p}_\mu = \begin{bmatrix} \mathbf{e}_\mu \\ 0 \end{bmatrix}, \quad \mathbf{i}_\alpha = T \cdot \mathbf{i}_1 = \begin{bmatrix} \sinh\gamma_i \cdot \mathbf{e}_\alpha \\ \cosh\gamma_i \end{bmatrix}; \ \mathbf{i}_1 = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}.$$

$$\mathcal{K}_\alpha = \eta_i/c, \ \mathbf{k}_\alpha = \mathcal{K}_\alpha \, \mathbf{p}_\alpha; \ \mathcal{K}_\nu = \sinh\gamma_i \cdot w^*_\alpha/c, \ \mathbf{k}_\nu = \mathcal{K}_\nu \, \mathbf{p}_\nu; \ \mathcal{Q}_\mu = \cosh\gamma_i \cdot w^*_\alpha/c, \ \ \mathbf{q}_\mu = \mathcal{Q}_\mu \, \mathbf{p}_\mu; \ \mathbf{q}_\alpha = \mathcal{K}_\alpha \, \mathbf{i}_\alpha.$$

In $\hat{E}_m \ (\gamma_i = 0):$ $\mathbf{p}_\alpha = \mathbf{j}_1 = \begin{bmatrix} \mathbf{e}_\alpha \\ 0 \end{bmatrix}$ –see (149A), $\mathbf{p}_\nu = \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix}$, $\mathbf{p}_\mu = \begin{bmatrix} \mathbf{e}_\mu \\ 0 \end{bmatrix}$, $\mathbf{i}_\alpha = \mathbf{i}_1 = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$ –see (146A).

In the basis Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$, we obtained exactly the maximal order of curvature $\zeta_{max} - 1 = 3$ and all one-valued results. In the curving pseudo-Riemannian space-time, it may be that $\zeta_{max} = 4$, but due to its anisotropy and non-homogeneity, corresponding formulae cannot give results undependent on the initial base. Moreover, analogous situation takes place, starting with the characteristics of 1-st order of motion along a world line as 4×1-momentum, real 3-momentum and energy, with the Integral Laws of their conservation. First this was showed by D. Hilbert [52], see in Ch. 9A.

These Laws in STR may be inferred in $\langle \mathcal{P}^{3+1}$ by the *Absolute Pythagorean Theorem* with the use of *own* 4×1-*momentum* $\mathbf{P}_0 = P_0 \cdot \mathbf{i}_\alpha$ as a hypotenuse and right column of the tensor of impulse $\mathcal{T}_P = P_0 \cdot roth\Gamma_i$, i. e., produced by the symmetric tensor of motion (100A) *with four independent elements.* In any physical pseudo-Cartesian base $\tilde{E} = \langle \mathbf{x}, \overrightarrow{ct} \rangle$ with inertial Observer N_1, the tensor $\mathcal{T}_P$ has physical structure (101A):

$$\mathcal{T}_P = P_0 \cdot roth\Gamma = m_0 c \cdot roth\Gamma = P_0 \cdot \begin{array}{|c|c|} \hline \cosh\gamma \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & \sinh\gamma \cdot \mathbf{e}_\alpha \\ \hline \sinh\gamma \cdot \mathbf{e}'_\alpha & \cosh\gamma \\ \hline \end{array} =$$

$$= \begin{array}{|c|c|} \hline P \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} + P_0 \cdot \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & \mathbf{p} \\ \hline \mathbf{p}' & P \\ \hline \end{array} = \begin{array}{|c|c|} \hline mc \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} + m_0 c \cdot \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & m\mathbf{v} \\ \hline m\mathbf{v}' & mc \\ \hline \end{array}, \quad (\mathbf{P}'_0 \cdot I^{\pm} \cdot \mathbf{P_0}) = P_0^2.$$

Then we have own 4×1-momentum as $\mathbf{P}_0 = P_0 \cdot \mathbf{i}_\alpha = m_0 \cdot \mathbf{c}$, where the notion $m_0 \neq 0$ is used only for massive material objects; $\mathbf{i}_\alpha$ is a time-like unity 4×1-vector of the world line inclination (222A) and for Poincaré 4-velocity $\mathbf{c} = c \cdot \mathbf{i}_\alpha$ ($\gamma > 0$ as $\Delta ct > 0$). $\mathbf{P}_0$ and $\mathcal{T}_P$ are the geometric invariants conservative under Lorentzian transformations in any insulated physical system: $\{\mathbf{F} = \mathbf{0} \leftrightarrow \mathbf{P}_0 = \mathbf{const},\ \mathcal{T}_P = \mathrm{CONST}\}$. For the progressive moving material body M, geometric invariant $\mathbf{P}_0$ has scalar invariant P_0 and gives two trigonometric projections of $\mathbf{P}_0$ as *relative cosine and sine parameters*: $\mathbf{P} = P_0 \cdot \cosh\gamma \cdot \mathbf{i}_1 = P \cdot \mathbf{i}_1 = mc \cdot \mathbf{i}_1$ as 4×1 cosine one onto the time-arrow $\overrightarrow{ct^{(1)}}$, and $\mathbf{p} = P_0 \cdot \sinh\gamma \cdot \mathbf{e}_\alpha = m_0 v^* \cdot \mathbf{e}_\alpha = m\mathbf{v}$ as sine one into the Euclidean subspace $\langle \mathcal{E}^3 \rangle^{(1)}$. In right triangle, P_0, P, p satisfy the Absolute pseudo-Euclidean Pythagorean Theorem. In an insulated for the body M system, there is the principal preserving characteristic under Lorentzian transformations: $\mathbf{P}_0 = P_0 \cdot \mathbf{i}_\alpha = \mathbf{const}$ as the *invariant hypotenuse.* Other *dynamical cathetuses* are preserved under conditions (γ and $\mathbf{e}_\alpha$ are constant) and expressed by projecting $\mathbf{P}_0$ and constant $c:\ E = c \cdot P_0 \cdot \cosh\gamma = cP,\ \mathbf{p} = P_0 \cdot \sinh\gamma \cdot \mathbf{e}_\alpha$, where $P \sim E \sim m$ are expressed proportionally, because $E = P \cdot c,\ \ m = P/c$. Such approach is applicable at arbitrary quantity of moving *massive* material points in an insulated for them system, for example, in the same original base $\tilde{E}_1$ with N_1: $\Sigma[P_{0(k)} \cdot \mathbf{i}_{(k)}] = c \cdot \Sigma[m_{0(k)} \cdot \mathbf{i}_{(k)}] = \Sigma\mathbf{P}_{0(k)} = \Sigma[P_{0(k')} \cdot \mathbf{i}_{(k')}] = c \cdot \Sigma[m_{0(k')} \cdot \mathbf{i}_{(k')}] = \mathbf{const}$.

$\Sigma\mathbf{P}_{0(k)}$ has cosine projection onto $\overrightarrow{ct^{(1)}}$ as sum of P and sine projection into $\langle \mathcal{E}^3 \rangle^{(1)}$ as sum of $\mathbf{p}$. Trigonometric projecting for inferring in such insulated system of these two relative Laws of Energy and Momentum Conservation may be realized one-to-one only in isotropic and homogeneous spaces as the Minkowskian space-time. The inference is in correspondence with the fundamental E. Noether's Theorem (1915 – Göttingen).

As we have seen repeatedly, consideration of relativistic problems in Minkowskian space-time gives us *their clear interpretation*, especially using the tensor trigonometry tools with the connected Poincaré Relativity Principle and Mach Principle. Kinematic effects of STR are realized exclusively under the action of these Principles of the Nature. So, the "twins paradox" does not have any physical meaning without the operation of the Mach Principle with its Star system of the Universe $\tilde{E}_0$. We accompany both these Principles with our 4 primary dimensionless trigonometric tensors of absolute motion:

$$\mathit{rot}\ \Phi_i = F_s(\varphi_i, \mathbf{e}_\alpha),\ \langle \mathcal{Q}^{n+1} \rangle \qquad\qquad V(\varphi_i, \mathbf{e}_\alpha, \mathbf{e}_\nu),\ (U \neq U'),\ \langle \mathcal{Q}^{2+1} \rangle$$

$$\begin{array}{|c|c|}\hline \cos\varphi_i \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' & -\sin\varphi_i \cdot \mathbf{e}_\alpha \\ \hline +\sin\varphi_i \cdot \mathbf{e}'_\alpha & \cos\varphi_i \\ \hline \end{array} \cdots\cdots\cdots\cdots \begin{array}{|c|c|c|}\hline \cos\varphi_i \cdot \mathbf{e}_\alpha & \mathbf{e}_\mu & -\sin\varphi_i \cdot \mathbf{e}_\alpha \\ \hline +\sin\varphi_i & 0 & \cos\varphi_i \\ \hline \end{array} \ \text{for Frenet–Serret theory.}$$

$$\mathit{roth}\ \Gamma_i = F_h(\gamma_i, \mathbf{e}_\alpha),\ \ (F = F'),\ \langle \mathcal{P}^{n+1} \rangle \qquad\qquad U(\gamma_i, \mathbf{e}_\alpha, \mathbf{e}_\nu, \mathbf{e}_\mu),\ \ (U \neq U'),\ \langle \mathcal{P}^{3+1} \rangle$$

$$\begin{array}{|c|c|}\hline \cosh\gamma_i \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' & \sinh\gamma_i \cdot \mathbf{e}_\alpha \\ \hline \sinh\gamma_i \cdot \mathbf{e}'_\alpha & \cosh\gamma_i \\ \hline \end{array} \cdots\cdots\cdots\cdots \begin{array}{|c|c|c|c|}\hline \cosh\gamma_i \cdot \mathbf{e}_\alpha & \mathbf{e}_\nu & \mathbf{e}_\mu & \sinh\gamma_i \cdot \mathbf{e}_\alpha \\ \hline \sinh\gamma_i & 0 & 0 & \cosh\gamma_i \\ \hline \end{array}\ . \qquad (250A)$$

U, inferred above, is a movable trigonometric tetrahedron along a world line (as a pseudoanalogue of the classic Frenet trihedron in $\mathcal{E}^3$), and it is an asymmetric tensor of motion with its orthospherical part *rot* Θ leading to combined normal Euclidean motion in $\mathcal{E}^{3(m)} \subset \langle \mathcal{P}^{3+1} \rangle$ as in (241A). The polar decompositions of U and V (inferred in Chs. 11, 7A, 8A) lead again to these principal symmetric tensors of motion with the tensor of secondary normal rotation *rot* Θ. For example, with (474), (475) we have $U = \mathit{roth}\ \Gamma_i \cdot \mathit{rot}\ \Theta_i \ \Rightarrow\ \mathit{roth}\ \Gamma_i = \sqrt{UU'},\ \mathit{rot}\ \Theta = \sqrt{UU'}^{-1} \cdot U = \mathit{roth}\ (-\Gamma_i) \cdot U.$

We see that the asymmetric tensor U contains *in an explicit form* the possibilities for 4 independent motions of Lorentz. Two of them is purely hyperbolic with synchronous rotations of the tangent and the principal pseudonormal, and else two are purely spherical with Euclidean rotations of the sine and cosine binormals. In an insulated system, these 4 motions are absent. Then we have 4 conserved physical characteristics: real momentum, total energy (see above) and two angular momenta $\mathcal{M}_1$ and $\mathcal{M}_2$ as a consequence of normal curvature and torsion. If we add to them 2 more independent Thomas purely orthospherical rotations and 4 independent translations, then we obtain a complete 10-parametric group of Poincaré with all conserved physical characteristics. For a particle or a body M, this corresponds to the Noether's Theorem in $\langle \mathcal{P}^{3+1} \rangle$!

These hyperbolic F_h and spherical F_s tensors with the orthospherical tensor $\langle \mathit{rot}\ \Theta \rangle$ produce all set of homogeneous Lorentzian and Special quasi-Euclidean transformations in clear trigonometric forms due to unambiguous polar decompositions of the latters. In (154A), (202A) we expressed such mixed tensors by canonical forms in the base $\tilde{E}_1$.

In the end of this Appendix and all the book, we would like to emphasize once again that both these concomitant hyperbolic geometries with their Euclidean and cylindrical topologies, which are mapped isometrically on the hyperboloids II and I in the common enveloping pseudo-Euclidean space $\langle \mathcal{P}^{n+1} \rangle$, must be considered as a three sheets unified hyperbolic geometry with the one Lorentzian group of transformations. *In this chapter, we have actually applied the unified three-sheets hyperbolic geometry.*

Mathematical–Physical Kunstkammer

The following questions and problems may be solved with the use of the book material and basic knowledges. As seem to the author, they are enough interesting for inquisitive readers.

1. Consider an algebraic equation of power n with real positive coefficients in its alternating-sign form. Represent Cardano's ($n = 3$) and Ferrari's ($n = 4$) formulae in terms of small and large medians.

Prove that, if the roots of the equation in such form are real-valued numbers, then at any "n" there hold:

$$0 < k_2 < [(n-1)/2n]k_1^2.$$

Give the similar chain for all the coefficients.

2. Explain why each of the following equations has complex conjugate roots with positive real parts.

$$y(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 90x - 64 = 0,$$

$$y(x) = x^5 - 10x^4 + 40x^3 - 70x^2 + 80x - 64 = 0,$$

$$y(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 75x - 60 = 0,$$

$$y(x) = x^5 - 25x^4 + 90x^3 - 640x^2 + 80x - 1 = 0,$$

$$y(x) = x^5 - 25x^4 + 160x^3 + 80x - 1 = 0.$$

General conditions to coefficients of an algebraic equation for its roots to be real see in our monograph [18].

3. Equation $y = ||\mathbf{z}(\mathbf{x})|| = ||A\mathbf{x} - \mathbf{a}|| = \min$, where A is a $m \times n$-matrix, $\mathbf{a}$ is a n-vector, has a unique solution $\mathbf{x} = \mathbf{b}$. Express $\mathbf{b}$, $\mathbf{z}(\mathbf{b})$, and $y(\mathbf{b})$ as formulae only with A and $\mathbf{a}$. Find the spherical angle between the vector $\mathbf{b}$ and the plane $\langle im\ A\rangle$. Find condition for it be zero, be right. What is the geometric nature of the vector $\mathbf{z}(\mathbf{b})$ in the m-dimensional Euclidean space? How does geometry of solutions depend on relations between m and n?

4. For a pair of conjugate complex numbers and operations over them, give the real-number representations without the imaginary unit. What is the main distinction between complex-valued representations of the numbers and the operations and these real-valued ones?

Prove that a *real-valued* algebraic equation of power n has a complete *real-valued* general solution unique up to admitted permutations.

5. In the first half of the 19-th century Urbain Le Verrier "discovered on tip of a pen" (by the words of F.-J. Arago) the new planet Neptune (1846). He used his own algorithm for inverting a square matrix B with evaluating scalar characteristic coefficients of the matrix B in terms of traces of powers B^t. Prove the following statements for these characteristic coefficients of a $n \times n$-matrix B and its powers B^t, $1 \le t \le n$.

a. If $tr\ B = tr\ B^2 = \cdots = tr\ B^j = \cdots = tr\ B^t = +1$, then $k(B,t) = 0$. In particular, $det\ B = 0$ if $t = n$.

b. If $tr\ B = tr\ B^2 = \cdots = tr\ B^j = \cdots = tr\ B^t = -1$, then $k(B,t) = (-1)^t$. In particular, $det\ B = (-1)^n$.

c. If $tr\ B = tr\ B^2 = \cdots = tr\ B^j = \cdots = tr\ B^t = +t$, then $k(B,t) = +1$.

d. If $-tr\ B = +tr\ B^2 = \cdots = (-1)^j tr\ B^j = \cdots = (-1)^t tr\ B^t = t$, then $k(B,t) = (-1)^t$.

e. If $tr\ B = tr\ B^2 = \cdots = tr\ B^j = \cdots = trB^t = +n$, then $k(B,t) = +C_n^t$.

f. If $-tr\ B = +tr\ B^2 = \cdots = (-1)^j tr\ B^j = \cdots = (-1)^t tr\ B^t = n$, then $k(B,t) = (-1)^n C_n^t$.

6. Integer-number $n \times n$-matrices generalize the notion of number. They keep also a lot of mysteries and phenomena. Prove the following formulae (they are connected with these characteristic coefficients too).

$$det \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 2 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & t-2 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & t-1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} = 0. \tag{1}$$

$$det \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & -(t-2) & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & -(t-1) \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} = t!. \tag{2}$$

$$det \begin{bmatrix} t & 1 & 0 & \cdots & 0 & 0 & 0 \\ t & t & 2 & \cdots & 0 & 0 & 0 \\ t & t & t & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ t & t & t & \cdots & t & t-2 & 0 \\ t & t & t & \cdots & t & t & t-1 \\ t & t & t & \cdots & t & t & t \end{bmatrix} = t!. \tag{3}$$

$$det \begin{bmatrix} -t & 1 & 0 & \cdots & 0 & 0 & 0 \\ +t & -t & 2 & \cdots & 0 & 0 & 0 \\ -t & +t & -t & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (-1)^{t-2}t & (-1)^{t-3}t & (-1)^{t-4}t & \cdots & -t & t-2 & 0 \\ (-1)^{t-1}t & (-1)^{t-2}t & (-1)^{t-3}t & \cdots & +t & -t & t-1 \\ (-1)^{t}t & (-1)^{t-1}t & (-1)^{t-2}t & \cdots & -t & +t & -t \end{bmatrix} = (-1)^t t!. \tag{4}$$

$$det \begin{bmatrix} n & 1 & 0 & \cdots & 0 & 0 & 0 \\ n & n & 2 & \cdots & 0 & 0 & 0 \\ n & n & n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & n & n & \cdots & n & t-2 & 0 \\ n & n & n & \cdots & n & n & t-1 \\ n & n & n & \cdots & n & n & n \end{bmatrix} = t!C_n^t. \tag{5}$$

$$det \begin{bmatrix} -n & 1 & 0 & \cdots & 0 & 0 & 0 \\ +n & -n & 2 & \cdots & 0 & 0 & 0 \\ -n & +n & -n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (-1)^{t-2}n & (-1)^{t-3}n & (-1)^{t-4}n & \cdots & -n & t-2 & 0 \\ (-1)^{t-1}n & (-1)^{t-2}n & (-1)^{t-3}n & \cdots & +n & -n & t-1 \\ (-1)^{t}n & (-1)^{t-1}n & (-1)^{t-2}n & \cdots & -n & +n & -n \end{bmatrix} = (-1)^t t!C_n^t. \tag{6}$$

Note. For (5) and (6) there holds, if $t > n$, then the determinant is 0.

7. For $n \times m$-matrices A_1 and A_2, prove equalities for the following scalar coefficients of any order t:

$$k(A_1 \cdot A_2', t) = k(A_1' \cdot A_2, t) = k(A_2 \cdot A_1', t) = k(A_2' \cdot A_1, t).$$

8. For $r \times r$-matrices B and C of rank r, give the matrix interpretation of the scalar relation:

$$\frac{det\ B_{11}}{det\ B_{21}} = \frac{det\ B_{11}}{det\ (C_{21} \cdot B_{11})} = \frac{det\ (B_{11} \cdot C_{12})}{det\ (C_{21} \cdot B_{11} \cdot C_{12})} = \frac{det\ B_{12}}{det\ B_{22}} \Rightarrow \frac{det\ B_{11}}{det\ B_{21}} = \frac{det\ B_{12}}{det\ B_{22}} \Leftrightarrow$$

$$\Leftrightarrow\ det\ B_{11} \cdot det\ B_{22} = det\ B_{12} \cdot det\ B_{21},$$

where B and C – $r \times r$-matrices of rank r. For example, with the use of the relation, infer exact formula for the orthogonal quasi-inverse matrix A^+ from Ch. 2.

9. For singular matrices determining *planars* or *lineors*, write down in the unified notation all characteristic eigenprojectors, *orthogonal* and *oblique* ones. Their quantities are:

- 8 and 12 for real-number and complex-number square matrices,
- 4 and 6 for real-number and complex-number rectangular matrices,
- 8 for a pair of real-number rectangular matrices,
- 12 for a pair of complex-number rectangular matrices.

Compose the *multiplication table* for these eigenprojectors.

Why paired orthogonal and oblique eigenprojectors mutually change their nature under translations from quasi-Euclidean space into pseudo-Euclidean one and vice versa?

Are there any geometric distinctions between orthogonal and symmetric eigenprojectors, oblique and nonsymmetric ones in the spaces with quadratic metrics?

10. In a geometry with its binary space and quadratic metric, a reflector tensor and the mid-reflector of the tensor angle have similar expressions. What is the principal distinction between these notions?

11. For generalized circles and hyperbolae in a real affine plane, draw graphs of the following functions $y(x)$:

$$|y|^n + |x|^n = |R|^n, \quad |y|^n - |x|^n = |R|^n, \quad n = 0, 1/4, 1/3, 1/2, 1, 3/2, 2, 3, 4, \infty.$$

Why the value $n = 2$ is chosen just for Euclidean, quasi- and pseudo-Euclidean spaces (for the spaces with quadratic metrics)? Does the parameter n have any geometric sense for affine planes and spaces?

These questions are connected with justification of the Pythagorean Theorem, as well as the quadratic metrics in Euclidean, quasi-Euclidean and non-Euclidean geometries and in the theory of relativity (and also of the Gaussian method of least squares and quadratic regression).

Whether it is possible to consider that the mathematical condition $n = 2$ follows from the nature of our real space and space-time or it is used as an axiom for them?

Give comparative analysis of the following *generalized* trigonometric functions for integer $n \geq 1$:

$$y/R = Sin\ \varphi, \ x/R = Cos\ \varphi; \quad y/R = Sinh\ \gamma, \ \ x/R = Cos\ \gamma;$$

If $n = 2$, then *in the universal bases $\tilde{E}_{1u}$ the covariant concrete spherical–hyperbolic analogy* takes place:

$$y/R = \sin\varphi \equiv \tanh\gamma, \ \ x/R = \cos\ \varphi \equiv \text{sech}\ \gamma \ \Leftrightarrow \ \sinh\ \gamma \equiv \tan\varphi, \ \ \cosh\ \gamma \equiv \sec\ \varphi.$$

Why angles in quadratic geometries (i. e., Euclidean, quasi-Euclidean, and pseudo-Euclidean), as well as their trigonometric functions have the nature of bivalent tensors?

When the tensors are orthogonal, either spherically, or quasi-Euclidean, or hyperbolically, or pseudo-Euclidean, and when they are affine ones?

What kinds of invariants and quasi-invariants take place for functions of spherical and hyperbolic angles?

What distinction in the kinds of invariants is there between spherical and hyperbolic angles?

Why a choice of $n = 2$ for the relativistic space-time according to Poincaré is equivalent to Einsteinian physical definition of simultaneity?

12. The concrete *sine-tangent analogy* leads to the hyperbolic analog ω of the spherical number $\pi/4$:

$$\sinh \omega = 1 = \tan \pi/4 \ \Rightarrow \ \omega = \text{arsinh } 1 \approx 0.881 \text{ rad}, \quad \pi/4 = \arctan 1 \approx 0.785 \text{ rad}.$$

Both these constants are represented with similar number series:

$$\pi/4 = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots \quad \text{(the Leibnitz series)},$$

$$\omega = \text{arsinh } 1 = 1 - \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1\cdot 3}{2\cdot 4} - \frac{1}{7}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \cdots + \frac{(-1)^n}{2n+1}\cdot\frac{(2n-1)!!}{(2n)!!} + \cdots .$$

Why ω as well as $\pi/4$ is a transcendental number? What is the geometric sense of ω? Why $\omega > \pi/4$?

13. What common trigonometric property do flat curves tractrix and cycloid have?

What common geometric feature do all these objects have: a circle and a sphere, an equilateral hyperbola and hyperboloids, a catenary and catenoids, a tractrix and tractricoids including a pseudosphere?

How does a quadrohyperbola in a pseudoplane lead isomorphically to the emergence of four catenaries and tractrices in quasiplanes (two time-like and two space-like) with a common determining parameter R?

Describe geometrically tractricoids obtained with parametric rotation of these time-like and space-like tractrices around the single ordinate axis.

What is a difference in the first metric forms of a Beltrami pseudosphere, hyperboloids I and II and a hyperspheroid given in their quasi- and pseudo-Euclidean spaces? Which kinds of transformations are invariant for a hyperspheroid and a Beltrami pseudosphere of the parameter R in an Euclidean space?

14. Which kinds of angles do measure segments and angles in both hyperbolic geometries on concomitant Minkowskian hyperboloids and in the spherical geometry on the hyperspheroid? What is the main difference of measuring segments and angles for figures in a pseudoplane and a quasiplane?

15. What roles do play the angles γ and υ in the pseudo-Euclidean geometry and in the theory of relativity? How are they connected to each other and correspond to the contravariant Lobachevskian parallel angle Π and the covariant parallel angle of motion γ? What types of the bases may be used for these bonds?

How does the angle of orthospherical rotation θ (scalar) or Θ (tensor) appear in non-geodesic (or non-collinear) motions (1), in metric forms (2), in angular deviations inside figures from geodesic segments (3)?

What tensor trigonometric distinction does exist in the mathematical description and interpretation of these well-known relativistic effects: Einsteinian dilation of time and Lorentzian contraction of extent? Describe concomitant to them other relativistic effects.

16. What does the *mathematical principle of relativity* in geometries consist in? How does it correspond to the *physical principle of relativity* in nature?

Do there exist chemical–mathematical isomorphism, biological-mathematical isomorphism etc.?

17. How curvature of a world line is related with the 2-nd Newtonian Law? Which kinds of curvatures and their orders do take place for world lines in the Minkowskian space–time? How they correspond to types of physical movement of a particle and its kinematical and dynamical characteristics?

18. Describe the symmetric trigonometric tensor of motion and its connection with relativistic physical tensors of momentum and energy. How it leads to the pseudo-Euclidean Pythagorean Theorem of three momenta and to the Law of momentum–energy conservation in an insulated system? What additional analogous Laws of conservation does give the asymmetric trigonometric tensor of motion?

19. Give interpretation of approximately uniform relativistic pseudoscrewed motion of the Earth in $\langle \mathcal{P}^{3+1} \rangle$ using the approximately inertial base $\tilde{E}_0$ connected with a barycenter of the Sun and results of Chapter 10A. Reveal the vectors of a tangent, a pseudonormal with $\mathbf{e}_\alpha$, and a sine binormal with $\mathbf{e}_\nu$ with all accompanied differential characteristics. Why in the pseudoplane of this pseudoscrewed motion precession we deal only with the rotational angle of spherical type?

20. Construct the trigonometric kind of the Frenet–Serret theory in $\langle \mathcal{E}^3 \rangle$. Why it may be realized in two different variants? How the order of differentiations–rotations must be changed? Give correct criterium of a flat curve in the Euclidean $3D$ space.

21. A sign-indefinite $(n+1)\times(n+1)$ symmetric tensor as a function in $(n+1)$ curvilinear coordinates determines a variable quadratic metric in a certain $(n+1)$-dimensional binary space, either quasi-Riemannian or pseudo-Riemannian. What is necessary for unambiguously determine of a *concrete non-Euclidean metric* from the variants of Riemannian or pseudo-Riemannian metrics? What tensor is true in both these variants?

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For main Parts I and II

For Appendix

ABSTRACT

Ninúl A. S. Tensor Trigonometry.

Appendix. Trigonometric models of motions in non-Euclidean Geometries and in Theory of Relativity.
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The main objectives of this book are to develop a number of geometric notions of the theory of exact matrices and on this platform work out the fundamentals of the tensor trigonometry with its applications. It operates with binary tensor angles formed by linear subspaces or in accordance to their rotations.

In the first part (chs. 1÷4), for the sequential, a number of questions in the theory of exact matrices are considered. The general inequality for all means is inferred, and hierarchical invariants of a spectrally positive matrix are installed. For an $n \times n$-matrix, its eigenprojectors, quasi-inverse matrices, and minimal annulling polynomial are expressed in explicit form, in terms of its characteristic coefficients. Primary parameters of matrices singularity and fundamental inequalities, connected with them, are revealed. Singular null-prime and null-normal matrices are defined and used. The sine and cosine relations for matrices were introduced.

In the second part (chs. 5÷12), the tensor trigonometry (generally as a multi-dimensional trigonometry) in affine and metric forms is developed. For the binary tensor angles between linear objects or in accordance to their rotations, the full set of tensor functions and reflectors is defined. Quasi-Euclidean and pseudo-Euclidean tensor trigonometries are constructed in these three kinds: projective, reflective and motive, the last term relates to rotations and deformations. For two- and multistep rotational transformations, as pseudo-Euclidean (Lorentzian) and quasi-Euclidean (new) ones, polar decomposition into principal and secondary rotations is installed and widely applicated. The covariant abstract and specific spherical-hyperbolic analogies are introduced and used. All quadratic norms of matrices are defined. The matrices trigonometric spectra are established, which serve as a basis for obtaining general cosine and sine normalizing inequalities. A number of the various applications are given in passing. So, the full solution of the pseudo-Euclidean right triangles is given. The new hyperbolic (primary) and spherical equations for a tractrix with a Beltrami pseudosphere defined by only parameter R are gotten. The trigonometric models of two complementary each to another hyperbolic geometries in the large are constructed and interpreted on the Minkowskian hyperboloids II (as a flat disc of tangents similar to Klein's one) and I (as a flat ring of cotangents or a cylinder of tangents). And especial *n-dimensional* isometry of the hyperboloid I and the Beltrami pseudosphere is installed.

In Appendix, the tensor trigonometry in elementary forms is used for studying motions in quasi-Euclidean and pseudo-Euclidean geometries with index $q = 1$, in non-Euclidean geometries of hyperbolic and spherical types, and in the theory of relativity. The contravariant Lobachevskian parallel angle is supplemented by the covariant universal parallel angle for these geometries (as the angle of principal motions). The general law of summing principal motions or velocities is established in the trigonometric matrix, vector and scalar forms with the orthospherical rotation. The tensor trigonometric model with its vector and scalar orthoprojections for kinematics and dynamics of a material point in the $(3 + 1)D$ Minkowskian space-time is proposed. It is shown that the mathematical source for complete physical description of motions in the homogeneous and isotropic space-time is the fundamental dimensionless trigonometric motion tensor producing proportionally the momentum tensor and its scalar and vector orthoprojections into the time-arrow and the Euclidean space, with the rectangular pseudo-Euclidean triangle and the pseudo-Pythagorean Theorem for three momenta. In addition, the hyperbolic formalizations of Einsteinian dilation of time and Lorentzian seeming contraction of extent with supplementary to them effects are realized as consequences of the rotational and deformational trigonometric transformations of coordinates. The four absolute vector and scalar differentially-geometric and physical characteristics of a world line, completely defining its orientation and configuration in the vicinity of its each world points in the $(3 + 1)D$ Minkowskian pseudo-Euclidean space-time, are established as complete $4D$ trigonometric pseudoanalog of the Theory by Frenet–Serret for the usual $3D$ Euclidean space (or for the $(2 + 1)D$ quasi-Euclidean space), with a detailed consideration of some particular motions.

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"...The manuscript contains many of fresh ideas, which open new posibilities! ..." - Professor of Moscow St. University M.M. Postnikov

general inequality for all average positive values (means)
minimal annulling polynomial of a square matrix in its explicit form
orthogonal and oblique projectors and reflectors for a singular matrix
tensorial octahedron from all these eigenprojectors of a singular matrix
quasi- and pseudo-Euclidean spaces and their tensor trigonometries
projective and motive tensor mutual angles and their functions
spherical, hyperbolic and orthospherical tensors of motion (rotation)
polar representation of mixed and multi-step tensors of motion
trigonometric nature of matrices commutativity and anticommutativity
null-prime singular matrix and its general cosine inequality and norm
nxr-lineors and their pair cosine and sine inequalities and norms
quadratic norms of a matrix from Frobenius up to general ones
elementary tensors of two types rotations and deformations
trigonometric vectorial models of non-Euclidean geometries
geometry of the two concomitant Minkowskian hyperboloids
3-sheets hyperbolic geometry with Lorentzian group of motions
group of quasi-Euclidean motions and the oriented hyperspheroid
covariant angle of parallelism in both types non-Euclidean geometries
Big and Small Pythagorean Theorems in non-Euclidean geometries
hyperbolic equations of a tractrix in only R-factor as principal ones
trigonometric equations of the Beltrami pseudosphere in only R-factor
general trigonometric Laws of summing motions and velocities in STR
differential geometry of world lines in the space-time of Minkowski with Riemannian metric forms and 4D pseudoanalog of 3D Frenet – Serret theory

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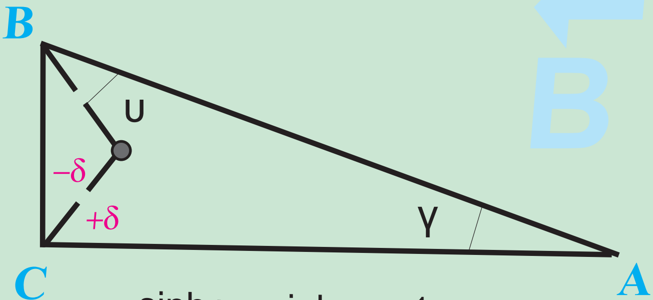

$$\sinh \gamma \cdot \sinh u = 1$$